

Setting. G loc cpt group,

fix left Haar meas. on G .

$$\int f(yx) dx = \int f(x) dx$$

$$\int \Delta(x^{-1}) f(x^{-1}) dx \\ = \int f(x) dx.$$

$L^1(G)$, convolution:

$$(f * g)(x) = \int f(y) g(y^{-1}x) dy$$

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

$L^1(G)$, involution:

$$f^*(x) = \Delta(x^{-1}) \overline{f(x^{-1})}.$$

Then $L^1(G)$ Banach $*$ -algebra.

Unitary representation of G .

$\pi: G \rightarrow U(\mathcal{H}_\pi)$ (unitary ops on a
grp homom. and Hilb. sp $\mathcal{H}_\pi$).

$x \mapsto \pi(x)u: G \rightarrow \mathcal{H}_\pi$ contin $\forall u \in \mathcal{H}_\pi$.

Associated representation of $L'(G)$:

$$\pi: L'(G) \rightarrow \mathcal{B}(\mathcal{H}_\pi).$$

$$"\pi(f) = \int f(x) \pi(x) dx"$$

$\forall f \in L'(G) \exists$ l.b.d lin op. $\pi(f)$ on $\mathcal{H}_\pi$

$$\langle \pi(f)u, v \rangle = \int f(x) \langle \pi(x)u, v \rangle dx.$$

Th 2.5: $f \mapsto \pi(f)$ $L'(G) \rightarrow \mathcal{B}(\mathcal{H}_\pi)$
is a non-degen $*$ -representation.

- $\pi: G \rightarrow \mathcal{U}(\mathcal{H}_\pi)$ is reducible if $\exists$ closed lin. subsp M of $\mathcal{H}_\pi$, $M \neq \{0\}$, $M \neq \mathcal{H}_\pi$,
 $\forall x \in G \quad \pi(x)(M) \subseteq M$
 $(\Rightarrow \pi(x)(M^\perp) \subseteq M^\perp)$.
 π is irreducible if not reducible.

- If $u \in \mathcal{H}_\pi$ st. $\overline{\text{span}} \{ \pi(x)u : x \in G \} = \mathcal{H}_\pi$
then u is a cyclic vector for π .
 π is cyclic if there exists a cyclic vector for π .

§3.3

Def. $\phi \in L^\infty(G)$ is a function of positive type on G if

$\forall f \in L^1(G)$:

$$\int (f^* * f) \phi \geq 0.$$

Note:

$$\int (f^* * f) \phi = \int (f^* * f) |x| |x| dx$$

$$= \int \int \overline{f^*(y)} f(y^{-1}x) dy \phi(x) dx$$

$$= \int \int \overline{\Delta(y^{-1}) f(y^{-1})} f(y^{-1}x) dy \phi(x) dx$$

$$= \int \int \overline{f(y)} f(yx) dy \phi(x) dx$$

$$= \int \overline{f(y)} \int f(yx) \phi(x) dx dy$$

$$= \int \overline{f(y)} \int f(x) \phi(y^{-1}x) dx dy$$

$$= \iint f(x) \overline{f(y)} \phi(y^{-1}x) dy dx.$$

So

So ϕ is of pos. type $\iff$ (3.13)
 $\forall f \in L^1(G): \iint f(x) \overline{f(y)} \phi(y^{-1}x) dy dx \geq 0$

Prop. 14. If $\phi \in L^\infty(G)$ is of pos. type, then $\bar{\phi}$ is of pos. type.

Prf. Let $f \in L^1(G)$. Then

$$\iint \overline{f(x)} \overline{f(y)} \overline{\phi(y^{-1}x)} dy dx$$

$$= \iint \overline{f(x)} \overline{f(y)} \phi(y^{-1}x) dy dx$$

Apply (3.13) to $\bar{f}$ instead of f . $\hookrightarrow \geq 0$.

Prop 15. Let $(\pi, \mathcal{H}_\pi)$ be a unitary repr of G . Let $u \in \mathcal{H}_\pi$. Define:

$$\phi(x) := \langle \pi(x)u, u \rangle, \quad x \in G.$$

Then ϕ is of positive type and ϕ is continuous.

Proof. $x \mapsto \pi(x)u : G \rightarrow \mathcal{H}_\pi$ is contin., so ϕ is contin. and $|\phi(x)| \leq \|\pi(x)u\| \|u\| = \|u\|^2$, so $\phi \in L^\infty(G)$. For $f \in L^1(G)$:

$$\iint f(x) \overline{f(y)} \underbrace{\phi(y^{-1}x)} dy dx$$

$$= \langle \pi(y^{-1}x) u, u \rangle$$

$$= \langle \pi(y^{-1}) \pi(x) u, u \rangle$$

$$= \langle \pi(x) u, \pi(y) u \rangle$$

$\nwarrow$ $\pi(y)$ is unitary.

$$= \iint f(x) \overline{f(y)} \langle \pi(x) u, \pi(y) u \rangle dy dx$$

$$= \iint \langle f(x) \pi(x) u, f(y) \pi(y) u \rangle dy dx$$

$$= (*)$$

$\exists$ bad lin $\pi(f)$ on H_π st $\forall u, v \in H_\pi$:

$$\langle \pi(f) u, v \rangle = \int f(x) \langle \pi(x) u, v \rangle dx$$

$$\|\pi(f) u\|^2 = \langle \pi(f) u, \pi(f) u \rangle$$

$$= \int f(x) \langle \pi(x) u, \pi(f) u \rangle dx$$

$$= \int f(x) \int \overline{f(y)} \langle \pi(y) u, \pi(x) u \rangle dy dx$$

$$= \iint \langle f(x) \pi(x) u, f(y) \pi(y) u \rangle dy dx$$

so

$$(*) = \|\pi(f) u\|^2 \geq 0.$$

so ϕ not pos type. □

Cor.¹⁶ Let $f \in L^2(G)$, $\tilde{f}(x) := \overline{f(x^{-1})}$, $x \in G$.
 Then $f * \tilde{f} \in L^\infty(G)$ is contin. and of pos type.

Pf. Let $\pi_L: G \rightarrow U(L^2(G))$ be the left regular representation.

$$(\pi_L(x)f)(y) = (L_x f)(y) = f(x^{-1}y).$$

Then π_L is a unitary repr. Moreover,

$$\begin{aligned} \langle \pi_L(x)f, f \rangle &= \int (\pi_L(x)f)(y) \overline{f(y)} dy \\ &= \int f(x^{-1}y) \overline{f(y)} dy = \int f((y^{-1}x)^{-1}) \overline{f(y)} dy \end{aligned}$$

$$= \int \overline{\tilde{f}(y^{-1}x)} \overline{f(y)} dy = \overline{(f * \tilde{f})(x)}$$

By Prop 15, $f * \tilde{f}$ is contin. and of pos type.

By Prop 14, $f * \tilde{f} = \overline{\overline{f * \tilde{f}}}$ is of pos type, and contin.

Now: aim at a converse:

$\forall \phi \neq 0$ of positive type $\exists$ unitary repr.

$\pi: G \rightarrow U(\mathcal{H}_\pi)$ s.t. $\phi(x) = \langle \pi(x)u, u \rangle$ for every $x \in G$.

Let $\phi \in L^\infty(G)$ of positive type,
 $\phi \neq 0$. For $f, g \in L^1(G)$:

$$\langle f, g \rangle_\phi := \int (g^* * f) \phi.$$

Note: $\langle f, f \rangle_\phi = \int (f^* * f) \phi \geq 0$.
 We have

$$\begin{aligned} \langle f, g \rangle_\phi &= \int (g^* * f) \phi \\ &= \int (g^* * f)(x) \phi(x) dx \\ &= \int \int g^*(y) f(y^{-1}x) dy \phi(x) dx \end{aligned}$$

$$\begin{aligned} &= \iint \Delta(y^{-1}) \overline{g(y^{-1})} f(y^{-1}x) dy \phi(x) dx \\ &= \iint \overline{g(y)} f(yx) dy \phi(x) dx \\ &= \iint \overline{g(y)} f(x) \phi(y^{-1}x) dy dx \\ &= \iint f(x) \overline{g(y)} \phi(y^{-1}x) dx dy. \end{aligned}$$

observe:

$$|\langle f, g \rangle_\phi| \leq \|\phi\|_\infty \|f\|_1 \|g\|_1.$$

Clearly: $f \mapsto \langle f, g \rangle_\phi$ is linear,
 $g \mapsto \langle f, g \rangle_\phi$ is conjugate lin.

i.e. $(f, g) \mapsto \langle f, g \rangle_\phi$ is sesquilinear.

Since $\langle f, f \rangle_\phi \geq 0 \quad \forall f \in L'(G)$,

we have: $\overline{\langle f, g \rangle_\phi} = \langle g, f \rangle_\phi$

(follow by polarization ^{Hermitian} formula)

Cauchy-Schwarz inequality:

If $\langle \cdot, \cdot \rangle$ is sesquilinear and $\langle f, f \rangle \geq 0 \quad \forall f$

then $|\langle f, g \rangle| \leq |\langle f, f \rangle|^{1/2} |\langle g, g \rangle|^{1/2}$

$$N := \{ f \in L'(G) : \langle f, f \rangle_\phi = 0 \}$$

Then $f \in N \iff \forall g \in L'(G) : \langle f, g \rangle_\phi = 0$

($\Leftarrow$ clear, $g = f$
 $\Rightarrow$ Cauchy-Schw.)

N is a lin subsp.

$L'(G)/N$ vector space, $\langle [f], [g] \rangle_\phi$
 $= \langle f, g \rangle_\phi$.



Let $\phi \in L^\infty(G)$ pos. (y.p.s. $\neq 0$).

$$\langle f, g \rangle_\phi = \int (g^* * f) \phi$$
$$= \iint f(x) \overline{g(y)} \phi(y^{-1}x) dy dx$$

$N = \{ f \in L^1(G) : \langle f, f \rangle_\phi = 0 \}$ l.h. subsp.

$L^1(G)/N$, $\langle [f], [g] \rangle_\phi = \langle f, g \rangle_\phi$
is an inner product on $L^1(G)/N$.

$\mathcal{H}_\phi$ = norm completion of $L^1(G)/N$

For $f \in L^1(G)$: $\tilde{f} = [f] \in \mathcal{H}_\phi$ w.r.t. $\langle, \rangle_\phi$.

For $f, g \in L^1(G)$, $z \in G$:

$$\langle L_z f, L_z g \rangle_\phi = \iint (L_z f)(x) \overline{(L_z g)(y)} \phi(y^{-1}x) dx dy$$
$$= \dots = \iint f(x) \overline{g(y)} \phi(y^{-1}zx) dx dy$$
$$= \langle f, g \rangle_\phi.$$

In partic: $L_z N \subseteq N$, so if
 $f, g \in L^1(G)$ with $\tilde{f} = \tilde{g}$ then $\widetilde{L_z f} = \widetilde{L_z g}$.

So $L_z \tilde{f} = \widetilde{L_z f}$ is well-defined.

Define $\pi_\phi(x) \tilde{f} := \widetilde{L_x f}$, $f \in L^1(G)$, $x \in G$.

Then: $\pi_\phi(x): L^1(G) \subseteq \mathcal{H}_\phi \rightarrow \mathcal{H}_\phi$ is lin.
and $\|\pi_\phi(x) \tilde{f}\|_{\mathcal{H}_\phi}^2 = \|\widetilde{L_x f}\|_{\mathcal{H}_\phi}^2 = \|\tilde{f}\|_{\mathcal{H}_\phi}^2$
so $\pi_\phi(x)$ is an isometry. Hence $\pi_\phi(x)$ extends
uniquely to a lin. isom. $\pi_\phi(x): \mathcal{H}_\phi \rightarrow \mathcal{H}_\phi$.
 $\pi_\phi(x)$ is unitary.

Also, $\pi_\phi(xy) \tilde{f} = \widetilde{L_{xy} f} = \widetilde{L_x L_y f} = \pi_\phi(x) \pi_\phi(y) \tilde{f}$
• $\pi_\phi(y^{-1}) \tilde{f} = \widetilde{L_{y^{-1}} f} = \pi_\phi(y)^{-1} \tilde{f}$

So $\pi_\phi: G \rightarrow U(\mathcal{H}_\phi)$ is a homom.

Associate representation of $L^1(G)$:

$$\begin{aligned} \pi_\phi(f) \tilde{g} &= \int f(x) \pi_\phi(x) \tilde{g} dx \\ &= \int f(x) \widetilde{L_x g} dx \\ &= \int f(x) g(x^{-1} \cdot) dx = (\widetilde{f * g})(\cdot) \end{aligned}$$

$$\text{So } \pi_\phi(f) \tilde{g} = \widetilde{f * g}, \quad f, g \in L^1(G).$$

T20: Let $f \in L^\infty(G)$ be of pos type,
 $f \neq 0$, $\mathcal{H}_f \ni f \mapsto \tilde{f}: L(G) \rightarrow \mathcal{H}_f$
 $\pi_f: G \rightarrow U(\mathcal{H}_f)$ as above.

There exists a cyclic vector ε
for π_f st $\pi_f(f)\varepsilon = \tilde{f} \forall f \in L(G)$
and

$$f(x) = \langle \pi(x)\varepsilon, \varepsilon \rangle \text{ locally a.e.}$$

P.f. Let $\{\psi_u\}_u$ be an approx. identity,

$$\begin{aligned} \psi_u^*(x) &:= \Delta(x^{-1}) \overline{\psi_u(x^{-1})} \\ &= \Delta(x^{-1}) \psi_u(x^{-1}) \end{aligned}$$

Then: (i) $\text{supp } \psi_u^* \subseteq U$.

$\text{supp } \psi_u^*$ is compact

$$(ii) \quad \psi_u^*(x) = \underbrace{\Delta(x^{-1})}_{\geq 0} \underbrace{\psi_u(x^{-1})}_{\geq 0} \geq 0$$

$$\begin{aligned} \int \psi_u^*(x) dx &= \int \underbrace{\Delta(x^{-1})}_{\geq 0} \underbrace{\psi_u(x^{-1})}_{\geq 0} dx \\ &= \int \psi_u(x) dx = 1. \end{aligned}$$

By Prop. 2.44: $\forall f \in L^1(G)$.

$$\| \gamma_u^* * f - f \|_1 \rightarrow 0. \quad \text{So}$$

$$\langle \tilde{f}, \tilde{\gamma}_u \rangle_\phi = \int (\gamma_u^* * f) \phi \rightarrow \int f \phi.$$

Also:

$$\| \tilde{\gamma}_u \|_\phi \leq \| \phi \|_\infty^{1/2} \| \gamma_u \|_1 = \| \phi \|_\infty^{1/2}$$

Claim: $\tilde{\gamma}_u$ converges weakly in $\mathcal{H}_\phi$ to some

$\varepsilon \in \mathcal{H}_\phi$.

Pf. • $\langle v, \tilde{\gamma}_u \rangle$ Cauchy for every $v \in$ dense subspace of $\mathcal{H}_0$.

$$\bullet \| \tilde{\gamma}_u \| \leq C.$$

Then $\langle v, \tilde{\gamma}_u \rangle$ Cauchy for every $v \in \mathcal{H}_0$ in $\mathbb{C}$.

Hence $\lim \langle v, \tilde{\gamma}_u \rangle$ exists for every $v \in \mathcal{H}_\phi$
 the $v \mapsto \lim \langle v, \tilde{\gamma}_u \rangle : \mathcal{H}_0 \rightarrow \mathbb{C}$ bdd lin map
 hence represented by some $\varepsilon \in \mathcal{H}_\phi$:

$$\forall v \in \mathcal{H}_\phi: \quad \boxed{\langle v, \tilde{\gamma}_u \rangle_\phi \rightarrow \langle v, \varepsilon \rangle_\phi.}$$

For $f \in L^1(G)$: $\langle \tilde{f}, \varepsilon \rangle_\phi = \lim \langle \tilde{f}, \tilde{\gamma}_u \rangle_\phi = \int f \phi.$

Claim. $\forall f, g \in L^1(G)$: $\langle \tilde{g}, \tilde{f} \rangle_\phi = \dots = \langle \tilde{g}, \pi_f(f) \varepsilon \rangle_\phi.$

$$\bullet \langle \tilde{g}, \pi_f(f) \varepsilon \rangle_\phi = \dots = \int g(x) f(y^{-1}x) dx.$$

So

$$\begin{aligned}
 \langle \tilde{g}, f \rangle_\phi &= \iint \underbrace{g(x)} \underbrace{\overline{f(y)} \phi(y-x)} dx dy \\
 &= \int \langle \tilde{g}, \pi_\phi(y) \varepsilon \rangle_\phi \overline{f(y)} dy \\
 &= \langle \tilde{g}, \underbrace{\int \pi_\phi(y) \varepsilon \overline{f(y)} dy}_{\pi_\phi(f) \varepsilon} \rangle \\
 &= \langle \tilde{g}, \pi_\phi(f) \varepsilon \rangle_\phi.
 \end{aligned}$$

Hence: • $\forall f \in L^1(G): \tilde{f} = \pi_\phi(f) \varepsilon$.

• $\text{span} \{ \pi_\phi(y) \varepsilon : y \in G \}$ dense in $\mathcal{H}_\phi$
 (as $g \perp \underbrace{\pi_\phi(y) \varepsilon}_{\forall y \in G} \Rightarrow \langle \tilde{g}, \tilde{f} \rangle = 0 \quad \forall f \in L^1(G) \Rightarrow \tilde{g} = 0$)

Hence ε is a cyclic vector for π_ϕ
 so π_ϕ is cyclic.

Finally, $\langle \pi_\phi(y) \varepsilon, \varepsilon \rangle = \overline{\langle \varepsilon, \pi_\phi(y) \varepsilon \rangle}$
 $= \phi(y) \text{ l.o.c. a.e.}$

$1/\varepsilon$
 χ_f
 $=0$
 $\forall f \in L^1(\mathbb{R})$

Indeed, for $f \in L^1(\mathbb{R})$

$$\int \langle \varepsilon, \pi_f(y) \rangle \overline{f(y)} dy$$

$$= \int \lim_{n \rightarrow \infty} \langle \tilde{f}_n, \pi_f(y) \varepsilon \rangle \overline{f(y)} dy \quad \leftarrow L^1(\mathbb{R})$$

$$\stackrel{\text{domin. conv.}}{=} \lim_{n \rightarrow \infty} \int \langle \tilde{f}_n, \pi_f(y) \varepsilon \rangle \overline{f(y)} dy$$

$$= \lim_{n \rightarrow \infty} \langle \tilde{f}_n, \int \overline{f(y)} \pi_f(y) \varepsilon dy \rangle$$

$$= \lim_{n \rightarrow \infty} \langle \tilde{f}_n, \tilde{f} \rangle$$

$$= \langle \varepsilon, \tilde{f} \rangle_f = \int \overline{f(y)} \overline{f(y)} dy$$

$$\text{So } \langle \varepsilon, \pi_f(y) \varepsilon \rangle = \overline{f(y)} \text{ for a.e.} \quad \square$$

Cor 21. If $\phi \in L^\infty(\mathbb{R})$ is of positive type then there is a continuous function $\gamma \in L^\infty(\mathbb{R})$ of positive type such that $\phi = \gamma$ loc. a.e.

Cor 22. If $\phi \in L^\infty(\mathbb{R})$ of positive type and continuous, then $\|\phi\|_\infty = \phi(1)$ and $\phi(x^{-1}) = \overline{\phi(x)}$.

Pf. By 20. $\phi(x) = \langle \pi(x) \varepsilon, \varepsilon \rangle$.
 then $|\phi(x)| \leq \|\pi(x)\| \|\varepsilon\|^2 = \|\varepsilon\|^2 = \langle \pi(1) \varepsilon, \varepsilon \rangle = \phi(1)$
 so $\|\phi\|_\infty \leq \phi(1)$ so $\|\phi\|_\infty = \phi(1)$.

Also, $\phi(x^{-1}) = \langle \pi(x^{-1}) \varepsilon, \varepsilon \rangle = \langle \varepsilon, \pi(x) \varepsilon \rangle = \overline{\langle \pi(x) \varepsilon, \varepsilon \rangle} = \overline{\phi(x)}$. $\square$

Prop. 23: Let π, ρ cyclic
unitary repr. of G with
cyclic vectors u and v resp.
If $\langle \pi(x)u, u \rangle = \langle \rho(x)v, v \rangle$
for every $x \in G$

then π and ρ unitarily equivalent.
P.f. $\pi: G \rightarrow U(H_\pi)$, $\rho: G \rightarrow U(H_\rho)$
 $\text{span}\{\pi(x)u: x \in G\}$ dense in H_π
 $\text{span}\{\rho(x)v: x \in G\}$ dense in H_ρ

Define: $T: \text{span}\{\pi(x)u: x \in G\} \rightarrow \text{span}\{\rho(x)v: x \in G\}$
by:

$$T\left(\sum_{j=1}^n \alpha_j \pi(x_j)u\right) := \sum_{j=1}^n \alpha_j \rho(x_j)v.$$

well defined: for $x, y \in G$

$$\langle \pi(x)u, \pi(y)u \rangle = \langle \pi(y^{-1}x)u, u \rangle$$

$$\text{Hence } = \langle \rho(y^{-1}x)v, \rho(y)v \rangle = \langle \rho(x)v, \rho(y)v \rangle.$$

$$\left\langle \sum \alpha_j \pi(x_j)u, \pi(y)v \right\rangle = \left\langle \sum \alpha_j \rho(x_j)v, \rho(y)v \right\rangle$$

the also with lin combination y .

T is lin, $\langle Tw, Tv \rangle = \langle w, v \rangle \rightarrow T$ is unitary map $H_\pi \rightarrow H_\rho$

$\mathcal{P} := \{ \phi \in l^\infty(G) : \phi \text{ contin. and of pos type} \}$
is a cone!

$$\mathcal{P}_1 := \{ \phi \in \mathcal{P} : \|\phi\|_\infty = 1 \}$$

$$= \{ \phi \in \mathcal{P} : \phi(1) = 1 \}$$

$$\mathcal{P}_2 := \{ \phi \in \mathcal{P} : 0 \leq \|\phi\|_\infty \leq 1 \}$$

$$= \{ \phi \in \mathcal{P} : 0 \leq \phi(1) \leq 1 \}$$

$\mathcal{P}_1$ and $\mathcal{P}_2$ convex sets.



T 25. Let $\phi \in \mathcal{P}_1$. Then ϕ is an extreme pt of $\mathcal{P}_1 \iff$ the repr. π_ϕ as in T 20 is irreducible.

Pf $\Rightarrow$) Suppose π_ϕ is reducible:

$$\mathcal{H}_\phi = \mathcal{M} \oplus \mathcal{M}^\perp \quad \mathcal{M} \text{ non-trivial cl. subspace,}$$

$$\pi_\phi(x)\mathcal{M} \subseteq \mathcal{M} \quad \forall x \in G$$

As ξ is cyclic, $\xi \notin \mathcal{M}$ and $\xi \notin \mathcal{M}^\perp$

$$\text{so } \xi = \alpha u + \beta v, \quad u \in \mathcal{M}, v \in \mathcal{M}^\perp, \alpha, \beta > 0$$

$$\|u\| = \|v\| = 1.$$

$$\psi_1(x) := \langle \pi_\phi(x)u, u \rangle, \quad \psi_2(x) := \langle \pi_\phi(x)v, v \rangle$$

Then $\gamma_1, \gamma_2 \in \mathcal{P}$. Also $\gamma_1(1) = \langle u, u \rangle = 1 = \gamma_2(1)$

So $\gamma_1, \gamma_2 \in \mathcal{P}_1$. We have

$$\gamma_1(x) - \gamma_2(x) = \langle \pi(x)u, u \rangle - \langle \pi(x)u, v \rangle$$

$$= \frac{1}{\alpha} \langle \pi(x) \alpha u, u \rangle - \frac{1}{\beta} \langle \pi(x) \beta v, v \rangle$$

$$= \frac{1}{\alpha} \langle \pi(x)(\alpha u + \beta v), u \rangle - \frac{1}{\beta} \langle \pi(x)(\alpha u + \beta v), v \rangle$$

$$(v \in \mathcal{M}^\perp \Rightarrow \pi(x)\beta v \in \mathcal{M}^\perp$$

$$\Rightarrow \langle \pi(x)\beta v, u \rangle = 0)$$

$$= \langle \pi(x)\varepsilon, \frac{1}{\alpha}u - \frac{1}{\beta}v \rangle_\phi$$

Since ε is cyclic $\exists x \in \mathcal{G}$ st $\langle \pi(x)\varepsilon, \frac{1}{\alpha}u - \frac{1}{\beta}v \rangle_\phi \neq 0$.

So $\gamma_1(x) \neq \gamma_2(x)$. Since $\gamma_1(1) = \gamma_2(1)$, so γ_1, γ_2 lin indep.

We have:

$$\phi(x) = \langle \pi_\phi(x)\varepsilon, \varepsilon \rangle_\phi = \langle \pi_\phi(x)(\alpha u + \beta v), \alpha u + \beta v \rangle_\phi$$

$$= \langle \pi_\phi(x)\alpha u, \alpha u \rangle_\phi + \langle \pi_\phi(x)\beta v, \beta v \rangle_\phi$$

$$= \alpha^2 \langle \pi_\phi(x)u, u \rangle_\phi + \beta^2 \langle \pi_\phi(x)v, v \rangle_\phi$$

$$= \alpha^2 \gamma_1(x) + \beta^2 \gamma_2(x)$$

$\alpha^2 + \beta^2 = \phi(1) = 1$. Thus ϕ is a nontrivial convex combination of $\gamma_1, \gamma_2 \in \mathcal{P}_1$. So ϕ not extreme in $\mathcal{P}_1$.

$\Leftarrow$) Suppose π_ϕ irreducible. Let

$$\phi = \psi + \psi' \text{ with } \psi, \psi' \in \mathcal{P}.$$

For $f \in L^1(G)$.

$$\langle f, f \rangle_\phi = \iint f(x) \overline{f(y)} \phi(y^{-1}x) dx dy$$

$$= \langle f, f \rangle_\psi + \langle f, f \rangle_{\psi'}.$$

$$\text{So } \langle f, f \rangle_\psi \leq \langle f, f \rangle_\phi.$$

Hence (P.2) $\langle f, g \rangle_\psi$ bdd Hermitian form on $\mathcal{H}_\psi$. Then $\exists T: \mathcal{H}_\psi \rightarrow \mathcal{H}_\phi$ self adj. bdd lin op s.t.

$$\langle f, g \rangle_\psi = \langle Tf, g \rangle_\phi.$$

For $f, g \in L^1(G)$.

$$\langle T\pi_\phi(x)\tilde{f}, \tilde{g} \rangle_\phi = \langle T\widetilde{x f}, \tilde{g} \rangle_\phi$$

$$= \langle L_x f, g \rangle_\psi = \langle f, L_{x^{-1}} g \rangle_\psi$$

$$= \langle T\tilde{f}, \widetilde{L_{x^{-1}} g} \rangle_\phi$$

$$= \langle T\tilde{f}, \pi_\phi(x^{-1})\tilde{g} \rangle_\phi$$

$$= \langle \pi_\phi(x)T\tilde{f}, \tilde{g} \rangle_\phi$$

So $T\pi_\phi(x) = \pi_\phi(x)T$
So T intertwining op of π_ϕ with itself.

Since π_ϕ irreducible, by Schur's lemma:

dim of set of all intertwining ops of π_ϕ with itself is at most 1.

Then $T = cI$ for some $c \in \mathbb{R}$.

Then: $\langle f, g \rangle_\psi = c \langle f, g \rangle_\phi$ (all f, g)

so $\psi = c\phi$ and $\psi' = (1-c)\phi$.

Then ϕ is extremal in $\mathcal{P}$.

□

T 31. On $\mathcal{P}_1 := \{f \in L^\infty(G) : f \text{ contin. of positive type, } f(1) = 1\}$ the weak* topology from $L^\infty(G)$ (viewed as dual of $L^1(G)$) and the topology of unif convergence on compact subsets of G coincide.

Prop 33. $\forall f, g \in C_c(G)$ we have

$$f * g \in \text{span} \{C_c(G) \cap \mathcal{P}(G)\} =: S.$$

Moreover, S is dense in $C_c(G)$ w.r.t $\|\cdot\|_\infty$

and S is dense in $L^p(G)$ w.r.t $\|\cdot\|_p$.

Prf. By Cor. 16, $f * \tilde{f} \in \mathcal{P}(G)$ (where $\tilde{f}(x) = \overline{f(x^{-1})}$).
So $\forall f \in C_c(G) : f * \tilde{f} \in S$.

By polarization: $\forall f, g \in C_c(G) : f * \tilde{g} \in S$.
so $\forall f, h \in C_c(G) : f * h \in S$ (take $h := \tilde{g}$)

$\{f * g : f, g \in C_c(G)\}$ dense in $C_c(G)$ w.r.t $\|\cdot\|_\infty$ or $\|\cdot\|_p$
(since g could be an approx identity). □



Theorem (Gelfand-Raikov)

Let G be loc cpt group.

The ^{irred} unitary repr. of G
separate the pts of G .

$\forall x, y \in G, x \neq y, \exists$ irreducible
unitary repr. π st

$$\pi(x) \neq \pi(y).$$

Pf. Take $f \in C_c(G)$: $f(x) \neq f(y)$.
By P33: may assume that f is a
lin combin of functions of pos t ype.

By T27, 31: we can approximate f on the
compact set $\{x, y\}$ uniformly by lin combin.
of extrem pts of P_f .

Then $\exists$ lin comb. g of extrem pts of P_f
st. $g(x) \neq g(y)$. Then $\exists$ extrem pt ϕ of P_f
st. $\phi(x) \neq \phi(y)$.

T20: corresponding irreducible unit repr.
 π_ϕ satisfies:

$$\langle \pi_\phi(x) \varepsilon, \varepsilon \rangle = \phi(x) \neq \phi(y) = \langle \pi_\phi(y) \varepsilon, \varepsilon \rangle$$

$$\text{So } \pi_\phi(x) \neq \pi_\phi(y).$$

□

f on the
lin comb.

pt of P_1

pt of P_1

unit repr.

$\epsilon > 0$

$\square$

Th 21: convex hull of $\mathcal{E}(P_1)$ is weak* dense
in P_1 .

T 31. On $P_1 := \{ f \in L^\infty(G) : f \text{ contin. of positive type, } f(1) = 1 \}$ the weak* topology from $L^\infty(G)$ (viewed as dual of $L^1(G)$) and the topology of unif convergence on compact subsets of G coincide.

Prop 33. $\forall f, g \in P_1$

$f * g \in P_1$

Moreover,

and

Prf. By Cor

So $\forall f \in C_c(G)$

By polarization

$\{ f * g : f, g \in C_c(G) \}$

(since