

G : loc. cpt. Abelian G .

$$x, y \in G. \quad x \cdot y = y \cdot x$$

The Dual group $\hat{G}$.

$$\text{If } \pi: G \rightarrow H_\pi = \mathbb{C} \text{ irred. rep.}$$

Corollary 3.6

$$\pi(x)z = \chi(x)z$$

Character: a conti. homo of G into

$$T = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$$

$$\hat{G} = \{\chi : \chi \text{ is a character of } G\}$$

$$\hat{G} = \{(P_1(G))\}$$

$$\chi(x) = \langle \pi(x), 1 \rangle \quad \text{Pro 3.15}$$

$$\|\chi\|_{\infty} = \langle \pi(1), 1 \rangle = 1 \quad \text{Corollary 3.22}$$

Theorem 3.25

$$f \in L_1(G)$$

$$\chi(f) = \int \langle x, \chi \rangle f(x) dx \in \mathbb{C} = \mathbb{C}$$

$$\langle x, \chi \rangle = \chi(x)$$

$$\zeta: L_1(G) \rightarrow L(\mathbb{C}) = \mathbb{C}$$

$$\hat{G} \subset \mathcal{B}(L_1(G)) = \{ \text{all nonzero homo. from } L_1(G) \text{ to } \mathbb{C} \}$$

4.2 Theorem $\hat{G} = \mathcal{B}(L_1(G))$

Proof $\Phi \in \mathcal{B}(L_1(G)) \exists \phi \in L^\infty(G), \phi' \in L^1(G)$

$$\Phi(\phi') = \int \phi \phi' d\mu.$$

$\exists t \in L_1(G)$ s.t. $\Phi(t) \neq 0$.

$$\begin{aligned} \underbrace{\Phi(t) \int \phi(y) g(y) dy}_{\text{wavy line}} &= \Phi(t) \Phi(g) = \Phi(\underbrace{t * g}_{\text{wavy line}}) \\ &= \Phi\left(\int \int t(xy^{-1}) g(y) dy dx\right) \\ &= \int \int \phi(x) t(xy^{-1}) g(y) dy dx \\ &= \int \Phi(L_y t) g(y) dy \\ &\quad \uparrow \frac{\Phi(L_y t)}{\Phi(t)} g(y) dy \\ \int \phi(y) g(y) dy &= \int \frac{\Phi(L_y t)}{\Phi(t)} g(y) dy \end{aligned}$$

Redefine $\phi(y) = \Phi(L_y t) / \Phi(t)$

$\phi(xy) = \phi(x)\phi(y), \quad x, y \in G.$
 $\phi(xy)\overline{\phi(y)} = \overline{\phi(y)}\phi(xy) = \overline{\phi(y)}\phi(x)\phi(y)$
 $= \phi(x)\overline{\phi(y)}\phi(y)$
 $= \phi(x)\phi(y)\overline{\phi(y)}$

$|\phi(x)| = 1 \quad \forall x \in G.$
 $\phi(x^n) = (\phi(x))^n, \quad \phi(x) \leq M \quad \forall x \in G,$
 $\Rightarrow \phi: G \rightarrow \mathbb{T}, \quad \phi \in \hat{G}$

□

$\hat{G}$: group operation: pointwise multiplication

τ : topology of cpt. convergence (weak* topology)

$\hat{G}$: loc. cpt. Abelian group
dual group of G

G is cpt.

$\hat{G} \subset L^\infty(G) \subset L^p(G) \quad 1 \leq p < \infty.$

Prop. 4.3

If G is cpt. $|G|=1$, then $\hat{G}$ is orthonormal set

of $L^2(G)$

Proof. ① $\xi \in \hat{G}$, $\|\xi\|_2 = 1$, $|\xi|^2 = 1$

② $\xi, \eta \in \hat{G}$ $\xi \eta^{-1} \neq 1$. $\exists x_0 \in G$ s.t.
 $\langle x_0, \xi \eta^{-1} \rangle \neq 1$

$$\int \xi \bar{\eta} = \int \langle \underline{x}, \xi \eta^{-1} \rangle dx$$

Abc.

$$= \int \langle \underline{x_0^{-1} x}, \xi \eta^{-1} \rangle dx$$

$$= \langle x_0, \xi \eta^{-1} \rangle \int \langle \underline{x_0^{-1} x}, \xi \eta^{-1} \rangle dx$$

$$= \langle x_0, \xi \eta^{-1} \rangle \int \xi \bar{\eta}$$

$$\Rightarrow \int \xi \bar{\eta} = 0$$

□

Prop. 4.4 ① G is discrete $\Rightarrow \hat{G}$ is cpt.

② G is cpt $\Rightarrow \hat{G}$ is discrete.

Proof: ①. $\delta(x) = \begin{cases} 1 & x = 1 \in G \\ 0 & x \neq 1 \end{cases}$

δ is identity of $L'(G)$

$GL'(G) = \hat{G}$ is cpt

② $1 \in L'(G)$

$\{f \in L^\infty, |f| > \frac{1}{2}\}$ is weak*-open

By Prop 4.3. $\int \delta \cdot 1 = \int \delta = \begin{cases} 1 & \delta = 1 \\ 0 & \delta \neq 1 \end{cases}$

$\{1\}$ is open in $\hat{G}$ $\{x\}$ is also open

□

4.5. a. $\hat{K} \cong \mathbb{R}$ with pairing $\langle x, \xi \rangle = e^{2\pi i x \xi}$

b. $\hat{T} \cong \mathbb{Z}$

c. $\hat{\mathbb{Z}} = \mathbb{T}$

d. $\mathbb{Z}_k$ = additive group of integers mod k .

$\hat{\mathbb{Z}}_k = \mathbb{Z}_k$

Proof (a): If $\phi \in \hat{K}$, $\phi(0) = 1$. $\exists a > 0$ st.

$A = \int_0^a \phi(t) dt \neq 0$

4.6. If $G_1, \dots, G_n$ are loc. cpe Abelian g

$$(G_1 \times \dots \times G_n)^\wedge = \hat{G}_1 \times \dots \times \hat{G}_n$$

In Particular:

4.7 $(\mathbb{R}^n)^\wedge = \mathbb{R}^n$, $(\mathbb{T}^n)^\wedge = \mathbb{Z}^n$, $(\mathbb{Z}^n)^\wedge = \mathbb{T}^n$

$$\hat{G} \cong G \text{ if } G \text{ is finite.}$$

$$\bigoplus_{\alpha \in A} H_\alpha = \{ (h_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} H_\alpha, \text{ and } h_\alpha = 1 \text{ for all but finitely } \alpha \}$$

4.8 If $G = \prod_{\alpha \in A} H_\alpha$, G_α are cpe A.G.
 $\hat{G} \cong \bigoplus_{\alpha \in A} \hat{G}_\alpha$

Example $G = (\mathbb{Z}_2)^\omega$

For each n , φ_n .

$$\langle (a_i)_{i=1}^\infty, \varphi_n \rangle = (-1)^{a_n}$$

$$\hat{G} = \{ \text{finite product of } \varphi_n \} \cup \{ 1 \}$$

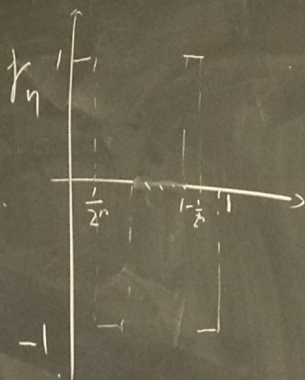
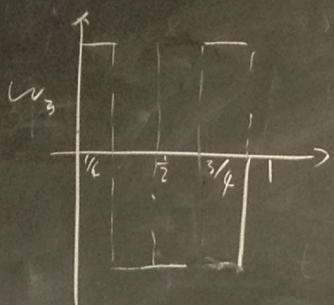
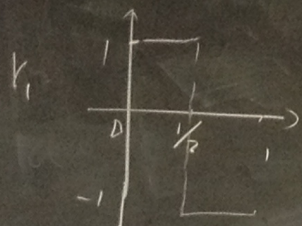
Section 2.2. $\Phi: (\mathbb{Z}_2)^\omega \rightarrow [0, 1]$
 almost bijective

$$G = [0, 1]$$

Rademacher function $\{r_n = r_n\}$

$$r_n = \text{sgn} \sin(2^n \pi t)$$

$$r_0 = 1$$



Walsh function

$$n = \sum_{k=1}^K b_k 2^{k-1} = b_K \dots b_1$$

$$w_n = r_{b_1} \dots r_{b_K}$$

$$w_{0,1,2} = r_{0,1,2} \quad w_3 = r_2 \cdot r_1$$

$$\{r_n\}_0^\infty \subset \{w_n\}_0^\infty$$

$$\hat{G} = ([0, 1])^\wedge = \{w_n\}_0^\infty$$

$\hat{G}$ is orthonormal set

indeed orthonormal basis §4.2



p -adic number

$0 \neq r \in \mathbb{Q}$

$$r = \frac{b}{a} = \frac{p^i p_1 \cdots p_s}{p^j q_1 \cdots q_t} = p^m \frac{b'}{a'}$$

p is prime

$$p\text{-adic norm: } \begin{cases} |r|_p = p^{-m} & 0 \neq r \in \mathbb{Q} \\ |0|_p = 0 \end{cases}$$

$$(2.7) \quad |r_1 + r_2|_p \leq \max\{|r_1|_p, |r_2|_p\}$$

$$|r_1 r_2|_p = |r_1|_p |r_2|_p$$

$$\text{Indeed, } r_1 = p^m \frac{a_1}{b_1} \quad r_2 = p^n \frac{a_2}{b_2} \quad m \geq n$$

$$|r_1 + r_2|_p = \left| p^n \frac{a_1 b_2 p^{m-n} + a_2 b_1}{b_1 b_2} \right|_p \leq p^{-n}$$

$$= \max\{|r_1|_p, |r_2|_p\}$$

$$|r_1 r_2|_p = \left| p^{m+n} \frac{a_1 a_2}{b_1 b_2} \right|_p = p^{-m-n} = |r_1|_p |r_2|_p$$

p-adic numbers.

$$\mathbb{Q}_p = \overline{\mathbb{Q}}^{1.p}$$

Pr. 2.8 1) $\sum_{j=m}^{\infty} c_j p^j$ converges in $\mathbb{Q}_p$ $c_j \in \{0, 1, \dots, p-1\}$

$$2) \forall x \in \mathbb{Q}_p \quad x = \sum_{j=0}^{\infty} c_j p^j$$

$$\begin{aligned} \text{Proof 1)} \quad \left| \sum_{j=m}^N c_j p^j \right|_p &\leq \max \left\{ |c_m p^m|_p, \left| \sum_{j=m+1}^N c_j p^j \right|_p \right\} \\ &\leq \max \{ p^{-m}, \dots, p^{-N} \} \\ &= p^{-m} \rightarrow 0 \\ &\quad \text{as } m \rightarrow \infty \end{aligned}$$

2) Claim $A = \left\{ \sum_{j=0}^{\infty} c_j p^j \right\}$ forms a field.
(+, ·, ÷)

“Subtraction” works if $c_m \neq 0$.

$$\begin{aligned} - \sum_{j=0}^{\infty} c_j p^j &= \left(\sum_{j=0}^{\infty} p^{j+1} - \sum_{j=0}^{\infty} p^j \right) - \sum_{j=0}^{\infty} c_j p^j \\ &= p \cdot p^0 + \sum_{j=1}^{\infty} p^j - \sum_{j=0}^{\infty} p^j - \left(\sum_{j=0}^{\infty} c_j p^j + c_m p^m \right) \\ &= (p - c_m) p^m + \sum_{j=m+1}^{\infty} (p - 1 - c_j) p^j \in A. \end{aligned}$$

p-adic number

Claim: $\mathbb{Q}_p = (A, |\cdot|_p)$. A is completed.

$\{x_n = \sum_{j=1}^{\infty} c_{jn} p^j\}$ is Cauchy $(|\cdot|_p)$

$$\forall k, \exists N \quad |x_{n_1} - x_{n_2}|_p = \left| \sum_{j=1}^{\infty} (c_{jn_1} - c_{jn_2}) p^j \right|_p < p^{-k}$$

$$\forall n_1, n_2 \geq N$$

$$\Rightarrow c_{jn_1} = c_{jn_2} \quad \text{if } j \leq k$$

$\forall j \quad \{c_{jn}\}_{n=1}^{\infty}$ Cauchy

$$c_j = \lim_n c_{jn}$$

$$x = \sum c_j p^j$$

we have $x = \lim_n x_n \quad (|\cdot|_p)$ □

Topology $\mathbb{Q}_p$.

$$\overline{B}(r, x) = \{y \in \mathbb{Q}_p : |x - y|_p \leq r\}$$

$$\forall x \in \mathbb{Q}_p. |x|_p \in \{p^{-m} : m \in \mathbb{Z}\}$$

$\forall r, \exists \varepsilon$ s.t.

$$\bar{B}(r, x) = B(\varepsilon, x) \quad \text{closed open set}$$

$$z \in \bar{B}(r, x) \Rightarrow \bar{B}(r, x) = \bar{B}(r, z)$$

$$y \in \bar{B}(r, x)$$

$$|x - z|_p = |x - y + y - z|_p \leq \max\{|x - y|_p, |y - z|_p\} \leq r$$

$$Z_p = \bar{B}(1, 0)$$

Define φ_1 by $x \in \mathbb{Q}_p$

$$\langle x, \varphi_1 \rangle = e^{2\pi i x} = \exp(2\pi i \sum_{j=0}^{\infty} c_j p^j) \exp(2\pi i \sum_{j=0}^{\infty} c_j p^j) = \exp(2\pi i \sum_{j=0}^{\infty} c_j p^j)$$

$$\text{homo. } \langle x+y, \varphi_1 \rangle = \langle x, \varphi_1 \rangle \langle y, \varphi_1 \rangle$$

$$\langle x, \varphi_1 \rangle = 1 \quad \forall x \in \mathbb{Z}_p \quad \text{cont.}$$

$$\Rightarrow \varphi_1 \in \hat{\mathbb{Q}}$$

$$\forall y \in \mathbb{Q} \quad \varphi_y$$

$$\langle x, \varphi_y \rangle = \langle xy, \varphi_1 \rangle$$

$$\hat{\mathbb{Q}} = \hat{\mathbb{Q}}$$

p-adic number

$$\hat{\mathbb{Q}}_p = \mathbb{Q}_p$$

4.9. If $\xi \in \hat{\mathbb{Q}}_p$, $\exists$ integers k s.t.
 $\xi(x) = 1, x \in B(p^{-k}, 0)$

Proof: ξ is continuous.

$$\{z \in T, |z-1| < 1\} \supset \xi(B(p^{-k}, 0))$$

Also, $B(p^{-k}, 0)$ subgroup.

$$\xi(B(p^{-k}, 0)) = \{1\} \quad \square$$

4.9. $\xi \in \hat{\mathbb{Q}}_p^*$, $\langle p^0, \xi \rangle = \langle 1, \xi \rangle = 1$ and $\langle p^{-1}, \xi \rangle \neq 1$

Then

$$\langle p^{-k}, \xi \rangle = \exp(2\pi i \sum_{j=0}^k c_j p^{-j}), \quad k=1,2,\dots$$

$c_j \in \{0, 1, \dots, p-1\}, j \geq 0$ also $c_0 \neq 0$

Proof: $w_k = \langle p^{-k}, \xi \rangle$ then

$$w_{k+1}^p = \langle p^{-k-1}, \xi \rangle^p = \langle p \cdot p^{-k-1}, \xi \rangle = w_k(x)$$

As $w_0 = 1, w_1^k = w_0, w_1 = \exp(2\pi i c_0 p^{-1})$

$$c_0 \in \{1, \dots, p-1\}$$

If $w_k = \exp(2\pi i \sum_1^k c_j p^{-j})$

we need to show w_{k+1} also.

$$\begin{aligned} w_{k+1} &\stackrel{(*)}{=} \exp[2\pi i \sum_1^k c_j p^{-j}] \exp[2\pi i c_{k+1} p^{-k-1}] \\ &= \exp[2\pi i \sum_1^{k+1} c_j p^{-j}] \exp[2\pi i c_{k+1} p^{-k-1}] \\ &= \exp(2\pi i \sum_1^{k+1} c_j p^{-j}) \end{aligned}$$

□

4.11 $\xi \in \hat{Q}$ $\langle 1, \xi \rangle = 1$ and $\langle p^{-1}, \xi \rangle \neq 1$

$\exists y \in Q, |y|=1$ s.t. $\xi = \xi_y$

Proof: $y = \sum_0^\infty c_j p^j$ with $\{c_j\}$ as 4.10

For $k \geq 1$

$$\begin{aligned} \langle p^{-k}, \xi \rangle &\stackrel{4.10}{=} \exp(2\pi i \sum_1^k c_j p^{-j}) \\ &= \exp(2\pi i \sum_{j=k}^{-1} c_{j+k} p^j) \cdot \langle \sum_0^\infty c_{j+k} p^j, \xi_1 \rangle \\ &\stackrel{\text{addition}}{=} \langle \sum_{j=k}^\infty c_{j+k} p^j, \xi_1 \rangle \\ &= \langle p^k y, \xi_1 \rangle \stackrel{4.10}{=} \langle p^{-k}, \xi_y \rangle \\ &\quad \xi = \xi_y \end{aligned}$$

□

p-adic number

$$\hat{\mathbb{Q}}_p = \mathbb{Q}_p$$

4.12 $\mathbb{Q}_p = \hat{\mathbb{Q}}_p$ $y \mapsto \zeta_y$ is an isom
of topological.

Proof: $y \mapsto \zeta_y$ injective homo.

We claim ' $\rightarrow$ ' onto.

$\zeta \in \hat{\mathbb{Q}}$ If $\zeta = 1$, then $\zeta = \zeta_0$.

If $\zeta \neq 1$ Lemma 4.9

$\exists j$ s.t.

$$\langle p^j, \zeta \rangle = 1$$

Define η by $\langle x, \eta \rangle = \langle p^j x, \zeta \rangle$

$$\text{As } \langle p^0, \eta \rangle = 1 \quad \langle p^{-1}, \eta \rangle \neq 1.$$

Lemma 4.11 $\exists z \in \mathbb{Q} \cdot |z| = 1$ s.t.

$$\eta = \zeta_z$$

$$\text{So } \zeta = \zeta_y \quad y = p^{-j} z.$$

$y \rightarrow \xi_y$ is home

$$N(j, k) = \{ \xi \in \hat{Q}_p, | \langle x, \xi \rangle - 1 | < j^{-1} \text{ for } |x| \leq p^k \}$$

$\{ N(j, k) \}$ is neighborhood base at 1 for $\hat{Q}_p$

$$\xi_y (\{ x : |x| \leq p^k \}) \subset \{ z : |z - 1| \leq j^{-1} \}$$

$$1 = \langle x, \xi_y \rangle = \langle xy, \xi_1 \rangle \quad \forall x \text{ with } |x| \leq p^k$$

$$\Leftrightarrow |y| \leq p^{-k}$$

□

$$\forall \xi \in \hat{Q} \quad \langle 1, \xi \rangle = 1$$

$$\exists y \in Q, |y| = 1$$

$$\text{Proof: } y = \sum_{j=0}^{\infty} a_j p^j$$

For $k \geq 1$

$$\langle p^{-k}, \xi \rangle = 1$$