

$\pi(x)(z) = \xi(x)z \quad \forall z \in \mathbb{C}$
 ξ ct hom of $G \rightarrow \mathbb{T}$
 ξ called characters

$\hat{G} = \{\text{characters}\}$
dual group

$$\hat{G} \subseteq \mathcal{P}_1$$

$$\langle x, \xi \rangle = \xi(x)$$

$$\xi(f) := \int \langle x, \xi \rangle f(x) dx$$

$$\hat{G} \cong \sigma(L'(G))$$

$$\textcircled{1} \langle x, \xi^{-1} \rangle = \langle x^{-1}, \xi \rangle = \overline{\langle x, \xi \rangle}$$

$$f \mapsto \xi^{-1}(f) \stackrel{\textcircled{1}}{=} \int \overline{\langle x, \xi \rangle} f(x) dx$$

A comm BA, Gelfand
transform is map

$$\Gamma: A \rightarrow C[\sigma(A)],$$

$$x \mapsto \hat{x}, \quad \hat{x}(f) = f(x) \quad \forall f \in \sigma(A)$$

$$F: L'(G) \rightarrow C[\hat{G}]$$

$$x \mapsto \hat{f}, \quad \hat{f}(\xi) = \xi^{-1}(f)$$

Fourier transform

Prop 4.1

Prop 4.14 F is norm-decr $\ast$ hom
 from $L^1(G)$ to $C_0(\hat{G})$ and its
 range is dense subsp of $C_0(\hat{G})$
Proof non-unit thm 1.30 $\rightarrow F(L^1(G)) \subseteq C_0(\hat{G})$
 unital thm 1.13 a
 dense similar to 1.11 c by Stone-Weier
 since $F \ast \text{hom} \rightarrow L^1(G)$ symm (ie $\hat{x} = \bar{x}$)

$$\begin{aligned} \text{4.15 } (L_y f)^\wedge(\xi) &= \int \overline{\langle x, \xi \rangle} f(y^{-1}x) dx \\ &= \int \overline{\langle yx, \xi \rangle} f(x) dx = \overline{\langle y, \xi \rangle} \hat{f}(\xi) \\ \text{4.16 } \text{for } \eta \in \hat{G} \quad (\eta f)^\wedge(\xi) &= \int \overline{\langle x, \xi \rangle} \langle x, \eta \rangle f(x) dx \\ &= \int \overline{\langle x, \xi \rangle} \overline{\langle x, \eta^{-1} \rangle} f(x) dx \\ &= \hat{f}(\eta^{-1}\xi) = L_\eta \hat{f}(\xi) \end{aligned}$$

$M(G)$ space of all complex Radon measure
 on G , $\mu \ast \nu$ given by:

$$\begin{aligned} \int \phi d(\mu \ast \nu) &= \iint \phi(xy) d\mu(x) d\nu(y) \\ \forall \phi \in C_c(G) \quad \mu \in M(G) \text{ define} \\ \text{its Fourier trans by the bounded fcn} \\ \text{function } \hat{\mu} \text{ on } \hat{G} \text{ st} \\ \hat{\mu}(\xi) &= \int \overline{\langle x, \xi \rangle} d\mu(x). \end{aligned}$$

$$\begin{aligned}
 (\mu * \nu)^{\wedge}(\xi) &= \hat{\mu}(\xi) \hat{\nu}(\xi) \\
 &= \int \overline{\langle x, \xi \rangle} d(\mu * \nu)(x) \\
 &= \iint \overline{\langle xy, \xi \rangle} d\mu(x) d\nu(y) \\
 &= \iint \overline{\langle x, \xi \rangle} \overline{\langle y, \xi \rangle} d\mu(x) d\nu(y)
 \end{aligned}$$

For $\mu \in M(\hat{G})$ define the bounded cl function ϕ_{μ} on G by $\phi_{\mu}(x) = \int \langle x, \xi \rangle d\mu(\xi)$

Prop 4.18: the map $\mu \mapsto \phi_{\mu}$ is an isom. norm-decr. lin inj from $M(\hat{G})$ to the space of bounded cl functions on G with unif norm $(C_b(G))$

Proof: norm-decr since $\xi(x) \in \mathbb{T} \forall x$
 $\|\mu\| = \|\mu\|(\hat{G})$ and $|\int \langle x, \xi \rangle d\mu(\xi)| \leq \int |\langle x, \xi \rangle| d\|\mu\|(\xi) \forall x$

$$\begin{aligned}
 \phi_{\mu} = 0 &\rightarrow 0 = \int \langle x, \xi \rangle d\mu(\xi) \\
 \forall f \in L^1(G): 0 &= \int f(x) \int \langle x, \xi \rangle d\mu(\xi) dx \\
 &= \iint \langle x, \xi \rangle f(x) dx d\mu(\xi) \\
 &= \iint \overline{\langle x, \xi^{-1} \rangle} f(x) dx d\mu(\xi) \\
 &= \int \hat{f}(\xi^{-1}) d\mu(\xi) = 0 \\
 \text{so } \forall f \in \mathcal{F}(L^1(G)): 0 &= 0 \\
 \Rightarrow \mu &= 0 \quad \square
 \end{aligned}$$

Lemma. $\mu \in \Pi(\hat{G})$ positive, then ϕ_μ is of pos type on G

Proof $f \in L^1(G)$

$$\begin{aligned} & \iint f(x) \bar{f}(y) \phi_\mu(y^{-1}x) dy dx \\ &= \iint \int f(x) \bar{f}(y) \langle y^{-1}x, \xi \rangle d\mu(\xi) dy dx \\ &= \iint f(x) \bar{f}(y) \langle y^{-1}, \xi \rangle \langle x, \xi \rangle d\mu(\xi) dy dx \\ &= \iint f(x) \langle x, \xi \rangle \overline{\int f(y) \langle y, \xi \rangle dy} dx d\mu(\xi) \\ &= \int \hat{f}(\xi) \overline{\int f(x) \langle x, \xi \rangle dx} d\mu(\xi) \end{aligned}$$

$$= \int \hat{f}(\xi) \overline{\hat{f}(\xi)} d\mu(\xi) = \int |\hat{f}(\xi)|^2 d\mu(\xi) \geq 0 \quad \square$$

thm 4.19 $\phi \in \mathcal{P}(G) \exists! \mu \in M(\hat{G})$

st. μ positive, $\phi = \phi_\mu$

Proof. $M_0 = \{ \mu \in M(\hat{G}) \mid \mu \geq 0 \text{ \& } \mu(\hat{G}) \leq 1 \}$

M_0 compact in weak* top of $M(\hat{G})$

$\{\mu_\alpha\}$ a net in M_0 converging to $\mu \in M_0$

$$f \in L^1(G) \int f(x) \phi_{\mu_\alpha}(x) dx$$

$$= \iint f(x) \langle x, \zeta \rangle d\mu_\alpha(\zeta) dx$$

$$= \int \hat{f}(\zeta) d\mu_\alpha(\zeta) \rightarrow \int \hat{f}(\zeta) d\mu(\zeta)$$

$$= \int f(x) \phi_\mu(x) dx$$

$\phi_\mu, \phi_{\mu_\alpha} \in \mathcal{P}_0$ since $\mu, \mu_\alpha \in M_0$ so

$$\|\mu\| = \mu(\hat{G}) \leq 1 \text{ by prop 4.18 } \|\phi_\mu\| \leq \|\mu\| \leq 1$$

$\phi_{\mu_\alpha} \rightarrow \phi_\mu$ in the weak* top of $\mathcal{P}_0 \subseteq L^\infty$

$\sum \mu \mapsto \phi_\mu$ is ct from M_0 to $\mathcal{P}_0$,

$S(M_0)$ compact, convex $\subseteq \mathcal{P}_0$

$$\hat{G} \subseteq S(M_0), 1 \in S(M_0)$$

thm 3.25 $\rightarrow \hat{G} = E(\mathcal{P}_0)$, lemma 3.20

$\hat{G} \cup \{0\} = E(\mathcal{P}_0)$. Krein Milman thm.

$\mathcal{P}_0 = \overline{\text{conv}(E(\mathcal{P}_0))} \subseteq S(M_0)$
and thus $\mathcal{P}_0 = S(M_0)$.
scalar multiplication $\rightarrow \forall \phi \in \mathcal{P}$ etc.

$$\mathcal{B}(G) = \{ \phi_\mu : \mu \in M(\hat{G}) \},$$

$$\mathcal{B}^p(G) = \mathcal{B} = \mathcal{B}(G) \cap L^p(G) \quad \forall p < \infty$$

$$\mathcal{B}(G) = \text{span}(\mathcal{P}(G))$$

$$\forall f, g \in C_c(G) : f + g \in \mathcal{B}(G),$$

& $\mathcal{B}^p$ is dense in $L^p(G)$

For $\phi \in \mathcal{B}(G)$

$$\underline{M_\phi}$$

Lemma

$$\exists p \in C$$

Proof. h.c.

$$g = h^*$$

For $\phi \in \mathcal{B}(G)$, its inverse is denoted by μ_ϕ ($\phi \mu_\phi = \phi$).

Lemma 4.20. If $K \subset \hat{G}$ compact, then $\exists f \in C_c(G) \cap \mathcal{P}$ st $\hat{f} \geq 0$ on $\hat{G}$, $\hat{f} > 0$ on K .

Proof. $h \in C_c(G)$ with $\hat{h}(1) = \int h = 1$, then $g = h^* * h$ then $\hat{g} = |\hat{h}|^2 \rightarrow \hat{g} \geq 0$ and $\hat{g}(1) = 1$.
 $\exists$ Neighborhood V of 1 in $\hat{G}$ st $\hat{g} > 0$ on V .

Say $K \subseteq \bigcup_{j=1}^n V_j$, $f = \left(\sum_{j=1}^n \chi_{V_j} \right) g$
 $\hat{f}(\xi) = \sum_{j=1}^n \hat{g}(\xi, \xi_j)$ by 4.16, $\rightarrow \hat{f} \geq 0$
 and $\hat{f} > 0$ on K corollary 3.16. $g \in \mathcal{P}$.
 $\Rightarrow f \in \mathcal{P}$ & $f \in C_c(G)$ $\square$

Lemma 4.21. $f, g \in \mathcal{B}^+$ then $\hat{f} d\mu_g = \hat{g} d\mu_f$

Proof. For $h \in C_c(G)$: $\int \hat{h}(\xi) d\mu_f(\xi)$
 $= \iint \langle x', \xi \rangle h(x) dx d\mu_f(\xi) = \int h(x) \phi_{\mu_f}(x'/k)$
 $= \int h(x) f(x') dx = h * f(1)$

Substituting $h+g$ or $h \cdot f$ for h & g for f

$$- \int \hat{h} \hat{g} d\mu_g = (h+g) * f(1)$$

$$- \int \hat{h} \hat{f} d\mu_g = (h * f) * g(1)$$

Since $\mathcal{F}(L^1(G))$ dense $C_c(\hat{G})$

$$\rightarrow \hat{g} d\mu_g = \hat{f} d\mu_g \quad \square$$

thm 4.22: $f \in B'$ then $\hat{f} \in L^1(\hat{G})$
 and if Haar measure $d\bar{s}$ on $\hat{G}$ is
 suitably normalized relative to
 the given Haar measure dx on
 G then $d\mu_f(\bar{s}) = \hat{f}(\bar{s}) d\bar{s}$,

(Fourier) Inversion Formula:

$$f(x) = \int \langle x, \bar{s} \rangle \hat{f}(\bar{s}) d\bar{s}$$

For $\psi \in C_c(\hat{G})$ Lemma 4.20 $\exists f \in L^1(G) \cap B'$
 st $\hat{f} \geq 0$ always & $\hat{f} > 0$ on $\text{supp } \psi$

$$I(\psi) = \int \frac{\psi}{\hat{f}} d\mu_f$$

if another such function then 4.21

$$\int \frac{\psi}{\hat{f}} d\mu_f = \int \frac{\psi}{\hat{f} \hat{g}} \cdot \hat{g} d\mu_f = \int \frac{\psi}{\hat{f} \hat{g}} \hat{f} d\mu_g = \int \frac{\psi}{\hat{g}} d\mu_g$$

linear & positive straightforward

$$\hat{g} \in B': I(\hat{g}\psi) = \int \frac{\psi}{\hat{f}} \hat{g} d\mu_f = \int \psi d\mu_g$$

$I \neq 0$

$$-416 \cdot (\eta \bar{s})^* = \hat{f}(\eta \bar{s})$$

$$-d\mu_f(\eta \bar{s}) = d\mu_{\eta f}(\bar{s})$$

$$\begin{aligned} \int \langle x, \bar{s} \rangle d\mu_f(\eta \bar{s}) &= \int \langle x, \eta \bar{s} \rangle d\mu_f(\bar{s}) \\ &= \langle x, \eta \rangle f(x) = (\eta f)(x). \end{aligned}$$

f with $\hat{f} > 0$ on $\text{supp}(\psi) \cap \text{supp}(\eta \psi)$
 we have.

$$I(\eta \psi) = \int \frac{\psi(\eta \bar{s})}{\hat{f}(\bar{s})} d\mu_f(\bar{s})$$

$$\begin{aligned} &= \int \frac{\psi(\bar{s})}{\hat{f}(\eta \bar{s})} d\mu_f(\eta \bar{s}) = \int \frac{\psi(\bar{s})}{(\eta f)(\bar{s})} d\mu_{\eta f}(\bar{s}) \\ &= I(\psi) \rightarrow I \text{ transl-invar} \end{aligned}$$

$$\begin{aligned}
 & \chi(\eta\bar{z})^*(\bar{z}) = \hat{f}(\eta\bar{z}) \\
 & d\mu_{\eta\bar{z}}(\eta\bar{z}) = d\mu_{\eta\bar{z}}(\bar{z}) \\
 & \int \langle x, \bar{z} \rangle d\mu_{\eta\bar{z}}(\eta\bar{z}) = \int \langle x, \eta\bar{z} \rangle d\mu_{\eta\bar{z}}(\bar{z}) \\
 & = \langle x, \eta \rangle \hat{f}(x) = (\eta\hat{f})(x) \\
 & \text{with } \hat{f} > 0 \text{ on } \text{supp}(\chi) / \text{supp}(\chi\eta) \\
 & \text{have} \\
 & (L_{\eta}\psi) = \int \frac{\psi(\eta\bar{z})}{\hat{f}(\bar{z})} d\mu_{\eta\bar{z}}(\bar{z}) \\
 & \int \frac{\psi(\bar{z})}{\hat{f}(\eta\bar{z})} d\mu_{\eta\bar{z}}(\eta\bar{z}) = \int \frac{\psi(\bar{z})}{(\eta\hat{f})(\bar{z})} d\mu_{\eta\bar{z}}(\bar{z}) \\
 & = I(\psi) \rightarrow I \text{ trans. invar.}
 \end{aligned}$$

thus $I(\psi) = \int \psi(\bar{z}) d\bar{z}$ where $d\bar{z}$ is Haar measure on $\hat{G}$

for $f \in \mathcal{B}'$, $\psi \in C_c(\hat{G})$:

$$\int \psi(\bar{z}) \hat{f}(\bar{z}) d\bar{z} = I(\psi \hat{f}) = \int \psi d\mu_{\hat{f}}$$

$\rightarrow \hat{f}(\bar{z}) d\bar{z} = d\mu_{\hat{f}}(\bar{z})$ so $\hat{f} \in L^1(\hat{G})$
and also

$$f(x) = \int \langle x, \bar{z} \rangle \hat{f}(\bar{z}) d\bar{z} \quad \square$$

Coroll. 4.24: If $\rho \in L^1(G) \cap \mathcal{B}$ then $\hat{\rho} \geq 0$.

Def. the dual Haar measure of dx is the Haar meas. st thm 4.22 holds

$$f(x) = \int \langle x, \zeta \rangle \hat{f}(\zeta) d\zeta$$

Example 1. $\hat{\mathbb{R}} = \mathbb{R}, \langle x, \zeta \rangle = e^{2\pi i x \zeta}$
 $g(x) = e^{-\pi x^2}; \hat{g}(\zeta) = \int e^{-\pi i \zeta x - \pi x^2} dx$

we find $(\hat{g})'(\zeta) = -2\pi i \zeta \hat{g}(\zeta)$,

$$\hat{g}(0) = 1 \rightarrow \hat{g} = g \text{ so}$$

$$g(x) = \int e^{-2\pi i x \zeta - \pi \zeta^2} d\zeta$$

$$\rightarrow g(x) = g(-x) \quad \text{if } \int e^{2\pi i x \zeta - \pi \zeta^2} d\zeta$$

$$\hat{f}(\zeta) = \int f(x) e^{-2\pi i x \zeta} dx$$

$$f(x) = \int \hat{f}(\zeta) e^{2\pi i x \zeta} d\zeta$$

Proposition 4.25

① If G compact, Haar measure chosen s.t. $|G|=1$, then the dual measure on $\hat{G}$ is counting measure

② If G discrete, Haar measure chosen s.t. it is counting measure, then the dual measure on $\hat{G}$ is the one s.t. $|\hat{G}|=1$.

Proof: ① $g=1$, prop 4.11 $\rightarrow \hat{G}$ is orthonormal set in $L^2(\hat{G})$

$$\hat{g}(\zeta) = \int \overline{\langle x, \zeta \rangle} g(x) dx = \int \overline{\langle x, \zeta \rangle} 1 dx = \begin{cases} 1 & \text{if } \zeta=1 \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow \hat{g} = \chi_{\{1\}}$. then

$$\sum_{\zeta \in \hat{G}} \langle x, \zeta \rangle \hat{g}(\zeta) = \sum_{\zeta \in \hat{G}} \langle x, \zeta \rangle \chi_{\{1\}}(\zeta) = \langle x, 1 \rangle = 1 = g(x)$$

$\int \langle x, \zeta \rangle \hat{g}(\zeta) d\zeta$ for $\hat{G}$ counting measure

② $g = \chi_{\{1\}}, \hat{g}(\zeta) = \int \overline{\langle x, \zeta \rangle} g(x) dx = \int \overline{\langle x, \zeta \rangle} \chi_{\{1\}}(x) dx = \overline{\langle 1, \zeta \rangle} = \chi_{\{1\}}(\zeta)$

Prop 4.11 $\hat{G}$ orthon set in $L^2(\hat{G})$

for Haar measure on $\hat{G}$ s.t. $|\hat{G}|=1$

$\forall x \in G, \zeta \mapsto \langle x, \zeta \rangle$ is char. on $\hat{G}$

$$\begin{aligned} \int \langle x, z \rangle dz &= \int 1 \langle x, z \rangle dz = \begin{cases} 1 & \text{if } \langle x, z \rangle = 1 \\ 0 & \text{else} \end{cases} \\ &= \chi_{\{1\}}(x) = g(x) \quad \square \end{aligned}$$

Example 2 (Sinbad). $\hat{\mathbb{T}} \cong \mathbb{Z}$, now.

Lebesgue meas $\frac{d\theta}{2\pi}$ on $\mathbb{T}$ and counting meas on $\mathbb{Z}$ are dual. $\langle \alpha, n \rangle = \alpha^n$ for $\alpha \in \mathbb{T}$,

$$n \in \mathbb{Z} \rightarrow \alpha = e^{i\theta}, \theta \in [0, 2\pi) \text{ so}$$

$$\begin{aligned} \langle \theta, n \rangle &= e^{in\theta} \quad \hat{f}(n) = \int e^{-in\theta} f(\theta) \frac{d\theta}{2\pi} \\ f(\theta) &= \int e^{in\theta} \hat{f}(n) dn = \sum_{n=-\infty}^{\infty} e^{in\theta} \hat{f}(n) \end{aligned}$$

Exam 3(a) - $G = \mathbb{Z}_k, \hat{G} \cong \mathbb{Z}_k$

$$\begin{aligned} \langle m, n \rangle &= e^{2\pi i mn/k} \\ \hat{f}(n) &= \int \langle m, n \rangle \hat{f}(m) dm = \sum_{m=0}^{k-1} e^{2\pi i mn/k} \hat{f}(m) \frac{1}{k} \\ \hat{f}(m) &= \int \overline{\langle m, n \rangle} f(n) dn = \sum_{n=0}^{k-1} e^{-2\pi i mn/k} f(n) \end{aligned}$$



Thm 4.26. the Fourier transform on $L^1(G) \cap L^2(G)$ extends uniquely to a unitary isom from $L^2(G)$ to $L^2(\hat{G})$.

Proof. $f \in L^1(G) \cap L^2(G)$:
 $f + f^* \in L^1 \cap \mathcal{P}$ (cor 3.16),
 $\mathcal{F} + \text{con} \rightarrow (f + f^*)^\wedge = |f|^\wedge$,
 $\int |f(x)|^2 dx = f + f^*(1) = \int (f + f^*)(\xi) d\xi$
 $= \int |f|^\wedge(\xi) d\xi \rightarrow f \mapsto \hat{f}$ isometry in $\ell^2 \rightarrow \psi \hat{f} = 0$ a.e. $\forall f \in L^1 \cap L^2$, lemma 4.10

$L^1 \cap L^2$ dense in $L^2 \rightarrow$ extends uniquely to an isom from $L^2(G)$ to $L^2(\hat{G})$

let $\psi \in L^2(\hat{G})$, $\psi \perp \hat{f} \forall f \in L^1 \cap L^2$
 $0 = \int \psi \overline{(L_x f)^\wedge} = \int \psi(\xi) \langle x, \xi \rangle \bar{f}(\xi) d\xi$
 $\forall f \in L^1 \cap L^2 \forall x \in G$.

$\psi \hat{f} \in L^1(\hat{G})$ since $\psi, \hat{f} \in L^2(\hat{G})$ so:
 $\psi(\xi) \bar{\hat{f}}(\xi) d\xi \in M(\hat{G})$ so prop 4.18
 $\rightarrow \psi \hat{f} = 0$ a.e. $\forall f \in L^1 \cap L^2$, lemma 4.10

$\rightarrow \psi = 0$ a.e.
 $\text{range}(\mathcal{F}_{L^2})^\perp \subseteq \text{range}(\mathcal{F}_{L^1 \cap L^2})^\perp = \{0\}$
 is closed & dense in $L^2(\hat{G})$
 $\rightarrow$ surjective $\square$

Corollary 4.27. If G compact, $|G|=1$ then $\hat{G}$ is an orthonormal basis for $L^2(\hat{G})$.

Theorem
 Let $p_0, p_1 \in \mathcal{L}$
 let $q_0, q_1 \in \mathcal{L}$
 define f_0
 and $q_0 =$
 $(\mu_0, \mu_1), (\mu_1, \mu_0)$
 $T: L^2(\mu_0) \rightarrow L^2(\mu_1)$
 lin. st. T

Theorem Riesz-Thorin

Let $p_0, p_1 \in [1, \infty]$ with $p_0 \leq p_1$.

Let q_0, q_1 st $\frac{1}{p_0} + \frac{1}{q_0} = \frac{1}{p_1} + \frac{1}{q_1} = 1$,

define for $0 < \theta < 1$: p_θ st $\frac{1}{p_\theta} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$

and q_θ st $\frac{1}{q_\theta} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}$

$(\mu_0, \mu_1), (\mu_2, \mu_2)$ σ -finite measure spaces.

$T: L^{p_0}(\mu_1) + L^{p_1}(\mu_1) \rightarrow L^{q_0}(\mu_2) + L^{q_1}(\mu_2)$

ln st T maps $L^{p_0}(\mu_1)$ boundedly into $L^{q_0}(\mu_2)$

and $L^{p_1}(\mu_1)$ boundedly into $L^{q_1}(\mu_2)$.

now $\forall \theta \in (0, 1)$ that

T maps $L^{p_\theta}(\mu_1)$ boundedly into $L^{q_\theta}(\mu_2)$

and also

$$\|T\|_{L^{p_\theta} \rightarrow L^{q_\theta}} \leq \left(\|T\|_{L^{p_0} \rightarrow L^{q_0}} \right)^{1-\theta} \left(\|T\|_{L^{p_1} \rightarrow L^{q_1}} \right)^{\theta}$$

thm 4.20: let $1 \leq p \leq 2, q$ st $\frac{1}{p} + \frac{1}{q} = 1$

if $f \in L^p(G)$ then $\hat{f} \in L^q(\hat{G})$ and

$$\|\hat{f}\|_q \leq \|f\|_p$$