

Recall

G loc. cpt Abelian group

$$\hat{G} = \{\text{characters of } G\}$$

$$= \{\text{continuous hom } G \rightarrow \mathbb{T}\}$$

$$= \{\text{irreducible representations of } G\}$$

$\hat{G}$ loc. cpt Abelian group

For $\mu \in M(\hat{G})$ we see

$$\varphi_\mu(x) = \int \langle x, \xi \rangle d\mu(\xi), \quad x \in G$$

$\mu \mapsto \varphi_\mu$ is injective

$$\varphi_\mu \in C_b(G)$$

$$B(G) := \{\varphi_\mu : \mu \in M(\hat{G})\} \quad B^p(G) := B(G) \cap L^p(G)$$

$$= \text{span } P(G) \quad (\text{Bochner's theorem})$$

Fourier transform $f \in L^1(G)$

$$\hat{f}(\xi) = \int \langle x, \xi \rangle f(x) dx$$

Haar measures on G and $\hat{G}$ are normalized s.e.

$$f(x) = \int \langle x, \xi \rangle \hat{f}(\xi) d\xi \quad \text{for } f \in B^1(G) \quad (\text{FIT 1})$$

§4.3 The Pontryagin duality theorem

$\hat{G}$ loc cpt, Abelian $\Rightarrow$ construct $\hat{\hat{G}}$

view $x \in G$ as a character on $\hat{G}$

$$\Phi(x): \hat{G} \rightarrow \mathbb{T} \quad \xi \mapsto \langle x, \xi \rangle$$

$\Phi: G \rightarrow \hat{\hat{G}}$ is a group homomorphism

Lemma 4.30 If $\phi, \psi \in C_c(\hat{G})$, then $\phi * \psi = \hat{h}$ for some $h \in B(G)$

In particular $F(B(G) \cap L^p(\hat{G}))$ is dense in $L^p(\hat{G})$ for $p \in [1, \infty)$

proof

$$f(x) = \int \langle x, \xi \rangle \phi(\xi) d\xi$$

$$g(x) = \int \langle x, \xi \rangle \psi(\xi) d\xi$$

$$h(x) = \int \langle x, \xi \rangle \phi * \psi(\xi) d\xi$$

Note $\phi, \psi, \phi * \psi \in L^1(\hat{G})$, thus $f, g, h \in B(G)$

For $k \in L^1(G) \cap L^2(G)$ we have

$$\begin{aligned} \left| \int f \bar{k} \right| &= \left| \iint \langle x, \xi \rangle \phi(\xi) \bar{k}(x) d\xi dx \right| \\ &= \left| \int \phi \bar{\hat{k}} \right| \leq \|\phi\|_2 \|\hat{k}\|_2 \stackrel{4.26}{=} \|\phi\|_2 \|k\|_2 \end{aligned}$$

So $f \in L^2(G)$ and similarly $g \in L^2(G)$

$$\begin{aligned} h(x) &= \iint \langle x, \xi \rangle \phi(\xi \eta^{-1}) \psi(\eta) d\eta d\xi \\ &= \iint \langle x, \xi \rangle \psi(\eta) \phi(\xi) d\eta d\xi = f(x) g(x) \end{aligned}$$

So $h \in L^1(G)$, hence $h \in B'(G)$

By FIT1 we have

$$\int \langle x, \xi \rangle (\varphi * \psi)(\xi) d\xi = h(x) = \int \langle x, \xi \rangle \hat{h}(\xi) d\xi$$

So since $\mu \mapsto \varphi_\mu$ is injective, $\hat{h} = \varphi * \psi$ $\square$

Lemma 4.31 Let G be a loc cpt group and H a subgroup. If H is loc cpt in the relative topology, then H is closed.

proof Exercise / read book

Lemma A Let $x \in G$ and $\{x_\alpha\}_{\alpha \in A}$ a net in G

TFAE

- 1) $x_\alpha \rightarrow x$ in G
- 2) $f(x_\alpha) \rightarrow f(x)$ for every $f \in B'(G)$
- 3) $\int \langle x_\alpha, \xi \rangle \hat{f}(\xi) d\xi \rightarrow \int \langle x, \xi \rangle \hat{f}(\xi) d\xi$ for every $f \in B'(G)$
- 4) $\Phi(x_\alpha) \rightarrow \Phi(x)$ in $\hat{G}$

In particular, G and $\Phi(G)$ are homeomorphic

Proof

Since $B'(G) \subseteq C_b(G)$, $1 \Rightarrow 2$ is follows by continuity. Conversely

If $x_\alpha \rightarrow x$, then there is a nbhd U of x and a cofinal $B \subseteq A$ s.t. $x_\beta \notin U$ for $\beta \in B$. By proposition 3.33 $(C_c(G) \cap P(G))$ dense in $C'(G)$, there is an $f \in B'(G)$ with $\text{supp } f \subseteq U$ and $f(x) \neq 0$.

Thus $\overset{=0 \text{ on } B}{f(x_\alpha)} \not\rightarrow \overset{\neq 0}{f(x)}$, so $2) \Rightarrow 1$

2) $\Leftrightarrow$ 3) follows from FIT 1
Reformulate 3) and 4

$$3) \int \Phi(x) \hat{f} \rightarrow \int \Phi(x) \hat{f} \text{ for all } f \in B'(G)$$

$$4) \int \Phi(x) g \rightarrow \int \Phi(x) g \text{ for all } g \in L'(G)$$

Therefore 3) $\Leftrightarrow$ 4) follows since $\mathcal{F}(B'(G))$ is dense in $L'(G)$
by Lemma 4.30 and $\|\Phi(x)\|_\infty = 1$ $\square$

Theorem 4.32 The map $\Phi: G \rightarrow \hat{G}$ is an isomorphism of
topological group

- Lemma A: Φ is a homeomorphism onto $\Phi(G) \subseteq \hat{G}$

- If $\Phi(x_1) = \Phi(x_2)$, then $\langle x_1, \xi \rangle = \langle x_2, \xi \rangle \forall \xi \in \hat{G}$
So by Gelfand-Raikov $x_1 = x_2$, so Φ is injective

- Surjectivity: $\Phi(G)$ is closed in $\hat{G}$ by Lemma 4.31

Suppose $x \in \hat{G} \setminus \Phi(G)$, pick a sym nbh V of 1 s.t.

$xV \cap \Phi(G) = \emptyset$. Pick nonnegative, nonzero $\phi, \psi \in C_c(G)$

with $\text{supp } \phi, \text{supp } \psi \subseteq xV$, $\phi * \psi \neq 0$, $\text{supp } \phi * \psi \subseteq xV$

$\Rightarrow \text{supp } \phi * \psi \cap \Phi(G) = \emptyset$ By Lemma 4.30 $h \in B'(G)$ s.t.
 $\phi * \psi = \hat{h}$. For $\gamma \in G$

$$0 = \hat{h}(\Phi(\gamma)) = \int \langle \xi, \Phi(\gamma) \rangle h(\xi) d\xi$$

$$= \int \langle \gamma, \xi \rangle h(\xi) d\xi$$

So by injectivity of $\mu \mapsto \Phi_\mu$ we have $h=0$
 i.e. $\Phi_\gamma = \hat{h} = 0$, a contradiction, so $\Phi(\gamma) = \hat{c}$ $\square$

Theore
 Topolog

Theorem 4.33. FIT2

If $f \in L^1(G)$ and $\hat{f} \in L^1(G)$, then

$$f(x) = \int \langle x, \xi \rangle \hat{f}(\xi) d\xi \quad \text{for a.e. } x \in G$$

If f is continuous, this holds for all $x \in G$.

proof Since

$$\hat{f}(\xi) = \int \langle x, \xi \rangle f(x) dx = \int \langle x, \xi \rangle \frac{f(x')}{\mu \in M(G)} dx$$

we have $\hat{f} \in B^1(G)$, and thus by FIT1

$$\hat{f}(\xi) = \int \langle \xi, x \rangle \hat{\hat{f}}(x) dx$$

so $\hat{f}(x') = \hat{\hat{f}}(x)$ for a.e. $x \in G$, i.e.

$$\underline{f}(x) = \int \langle x', \xi \rangle \hat{f}(\xi) d\xi = \int \langle x, \xi \rangle \hat{f}(\xi) d\xi \quad \text{for a.e. } x \in G$$

continuous □

Corollary 4.34 (Uniqueness)

If $\mu, \nu \in M(G)$, and $\hat{\mu} = \hat{\nu}$, then $\mu = \nu$.

In particular if $f, g \in L^1(G)$ and $\hat{f} = \hat{g}$, then $f = g$.

proof

This follows from injectivity of $\mu \mapsto \hat{\mu}$

$$\phi_\mu(\xi) = \hat{\mu}(\xi^{-1})$$

Corollary 4.36

G is discrete $\Leftrightarrow \hat{G}$ is compact

G is compact $\Leftrightarrow \hat{G}$ is discrete

Proposition 4.5, converse implications by duality

Proposition 4.37 If $f, g \in L^2(G)$; then $(fg)^\wedge = \hat{f} * \hat{g}$

Proof Suppose $f, g \in L^2(G) \cap \mathcal{F}(B'(G))$, so $f(x) = \hat{\varphi}(x^{-1})$ and $g(x) = \hat{\psi}(x^{-1})$ with $\varphi, \psi \in L^2(\hat{G}) \cap B'(\hat{G})$. Then

$$(\varphi * \psi)^\wedge(x^{-1}) = \iint \langle x, \xi \rangle \varphi(\xi \eta^{-1}) \psi(\eta) d\eta d\xi$$

$$= f(x) g(x)$$

By FITI applied to φ and ψ we have $\varphi = \hat{f}$ and $\psi = \hat{g}$

Moreover $\varphi * \psi \in L^1(\hat{G})$ and $fg \in L^1(G)$, so

$$\hat{f} * \hat{g}(\xi) = \varphi * \psi(\xi) \stackrel{\text{FITZ}}{=} (\varphi * \psi)^\wedge(\xi^{-1}) = (fg)^\wedge(\xi)$$

Now $f, g \in L^2(G)$, by Lemma 4.30 there are sequences f_n, g_n in $L^2(G) \cap \mathcal{F}(B'(G))$ that converge to f, g in $L^2(G)$

Then $f_n g_n \rightarrow fg$ in L^1 so $(f_n g_n)^\wedge \rightarrow (fg)^\wedge$ uniformly

Moreover by Planchet $\hat{f}_n \rightarrow \hat{f}$ $\hat{g}_n \rightarrow \hat{g}$ in $L^2(\hat{G})$, so

$$\hat{f}_n * \hat{g}_n \rightarrow \hat{f} * \hat{g} \text{ uniformly, so } (fg)^\wedge = \hat{f} * \hat{g} \quad \square$$

H be a closed subgroup of G and set
 $H^\perp = \{ \xi \in \hat{G} : \langle x, \xi \rangle = 1 \text{ for all } x \in H \}$
 which is a closed subgroup of $\hat{G}$.
Proposition 4.39 $(H^\perp)^\perp = H$ (uses $\hat{\hat{G}} \cong G$)

Theorem 4.40

Let H be a closed subgroup of G . Then

$$(G/H)^\wedge \cong H^\perp \text{ and } \hat{G}/H^\perp \cong \hat{H}$$

Corollary

Every character on H extends to a character on G .

Proof

If $\xi \in \hat{H}$, then $\xi \in \hat{G}/H^\perp$, so $\xi = \eta|_H$ where η is
 canonical projection $\hat{G} \rightarrow \hat{G}/H^\perp$

Proof of theorem 4.40

Let $\eta: G \rightarrow G/H$ be the canonical projection and define

$$\Phi: (G/H)^\wedge \rightarrow H^\perp \text{ by } \Phi(\eta) = \eta \circ \eta$$

$$\eta: G/H \rightarrow \Pi$$

$$G \rightarrow \Pi$$

which well-defined as for $x \in H$ we have

$$\eta \circ \eta(x) = \eta(1) = 1$$

Moreover Φ is a group isomorphism
 Homeomorphism: see book, bijectivity clear. So $(G/H)^\wedge \cong H^\perp$

Now by the duality theorem, we get

$$(\hat{G}/H^\perp)^\wedge \cong (\hat{H})^\wedge \stackrel{4.39}{=} \hat{H}$$

$$\text{so } \hat{G}/H^\perp \cong \hat{H}$$

Theorem 4.41
 Suppose χ is a character on H .

fine

isomorphism

$$\hat{\cdot} \cong H^\perp$$

Theorem 4.43

Suppose H is a closed subgroup of G . If $f \in L^1(G)$ and $\hat{f}|_{H^\perp} \in L^1(H^\perp)$, then

$$\int_H f(xy) dy = \int_{H^\perp} \hat{f}(\xi) \langle x, \xi \rangle d\xi$$

for almost every $x \in G$

Example:

If $G = \mathbb{R}$, $H = \mathbb{Z}$, then $H^\perp = \mathbb{Z}$. If we identify $\hat{\mathbb{R}}$ with $\mathbb{R}$ via $\langle x, \xi \rangle = e^{2\pi i x \xi}$

$$\sum_{n \in \mathbb{Z}} f(x+n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$$

proof

For $xH \in G/H$ define $F(xH) = \int_H f(xy) dy$ for $a \in xH$, Then $F \in L^1(G/H)$

If $\xi \in H^\perp$ we have $\forall y \in H$

$$\langle xy, \xi \rangle = \langle x, \xi \rangle \langle y, \xi \rangle = \langle x, \xi \rangle$$

So using $H^\perp \cong (G/H)^\perp$

$$\begin{aligned} \hat{F}(\xi) &= \int_{G/H} \int_H f(xy) \overline{\langle xy, \xi \rangle} dy dx \\ &= \int_G f(x) \overline{\langle x, \xi \rangle} dx = \hat{f}(\xi) \end{aligned}$$

so if $\hat{f}(\xi)|_{H^\perp} \in L^1(H^\perp)$, we can use F.T. 2.6 to obtain

$$\int_H f(xy) dy = F(xH) = \int_{H^\perp} \langle xH, \xi \rangle \hat{F}(\xi) d\xi = \int_{H^\perp} \langle x, \xi \rangle \hat{f}(\xi) d\xi$$

for almost every $x \in G$

Representations of compact groups

π_1, π unitary representations of G

Intertwining operator is bdd lin $T: H_{\pi_1} \rightarrow H_{\pi_2}$

st $T\pi_1(x) = \pi_2(x)T \quad \forall x \in G$

$(\pi_1, \pi_2) =$ set of all intertwining op's

$\pi_1 \sim \pi_2$ if $\exists U \in (\pi_1, \pi_2)$ unitary.

Standing Assumption: G compact and $|G|=1$
 $\Rightarrow G$ unimodular

Lemma 5.1

Let π be a representation. Fix a unit vector $u \in H_\pi$ and define $T: H_\pi \rightarrow H_\pi$

$$Tv = \int \langle v, \pi(x)u \rangle \pi(x)u \, dx$$

Then T is positive, nonzero, compact and $T \in \mathcal{C}(\pi)$

proof for $v \in H_\pi$ we have

$$\langle Tv, v \rangle = \int \langle v, \pi(x)u \rangle \langle \pi(x)u, v \rangle \, dx$$

$$= \int |\langle v, \pi(x)u \rangle|^2 \, dx \geq 0$$

So T is positive. Moreover if $v=u$, then $|\langle u, \pi(x)u \rangle| > 0$ on a nbhd of 1, so $T \neq 0$

Let $\varepsilon > 0$. Since G is compact and $x \mapsto \pi(x)u$ is continuous

$\exists$ partition $E_1, \dots, E_n$ of G and point $x_j \in E_j$ s.t.

$$\|\pi(x)u - \pi(x_j)u\| < \frac{1}{2}\varepsilon \quad \text{for } x \in E_j$$

Then $x \in E_j$

$$\|\langle v, \pi(x)u \rangle \pi(x)u - \langle v, \pi(x_j)u \rangle \pi(x_j)u\|$$

$$\leq \|\langle v, [\pi(x) - \pi(x_j)]u \rangle \pi(x)u\|$$

$$+ \|\langle v, \pi(x_j)u \rangle [\pi(x) - \pi(x_j)]u\| < \varepsilon \|v\|$$

So $T_\varepsilon: H_\pi \rightarrow H_\pi$

$$T_\varepsilon = \sum_{j=1}^n |E_j| \langle v, \pi(x_j)u \rangle \pi(x_j)u$$

We have $\|Tv - T_\varepsilon v\| < \varepsilon \|v\|$

Finally $y \in G$

$$\pi(y)Tv = \pi(y) \int \langle v, \pi(x)u \rangle \pi(x)u \, dx$$

So $T_\varepsilon: H \rightarrow H$

$$T_\varepsilon = \sum_{j=1}^n |E_j| \langle v, \pi(x_j)u \rangle \pi(x_j)u$$

We have $\|Tv - T_\varepsilon v\| < \varepsilon \|v\|$. Since T_ε has finite rank, T is compact

Finally $y \in G$

$$\pi(y)Tv = \int \langle v, \pi(x)u \rangle \pi(yx)u dx$$

$$= \int \langle v, \pi(y^{-1})\pi(x)u \rangle \pi(x)u dx$$

$$= \int \langle \pi(y)v, \pi(x)u \rangle \pi(x)u = T\pi(y)v$$

So $T \in C(\pi)$ $\square$

Theorem 5.2

- 1) Every irreducible representation is finite-dimensional
- 2) Every representation is the direct sum of irreducible representations

Proof. 1) Suppose π is irreducible and take T as in 5.1

By Schur's Lemma $C(\pi)$ only contains cI for $c \in \mathbb{C}$, so $T = cI$ for some $c \neq 0$. Thus I is compact on H_π , so $\dim H_\pi < \infty$.

2) By Zorn's Lemma there is a maximal family $\{M_\alpha\}$ of mutually orthogonal, irreducible, invariant subspaces of H_π . Take $N = (\bigoplus M_\alpha)^\perp$.

Suppose $N \neq \{0\}$, then N is invariant by proposition 3.1

Let T be as in 5.1 associated to $\pi|_N$. Since T is c.p.a. nonzero, normal, by Theorem 1.5.2 it has a nonzero eigenvalue λ with eigenspace M . Then $\dim M < \infty$ as T is compact.

Since $T \in \mathcal{L}(\pi)$ we have for $v \in M$
 $\lambda \pi(x)v = \pi(x)Tv = T\pi(x)v$
 $\pi(x)v \in M$, so M is invariant under π .
 By induction (M finite dim, invariant) there is an
 irreducible invariant subspace $M' \subseteq M \subseteq N$.
 This is a contradiction with the maximality of
 $\{M_\alpha\}$, as we could add M . So $N = \{0\}$. $\square$
 $H_\pi = \bigoplus M_\alpha$

$[\pi]$ equivalence class of rep π
 $\hat{G} = \{[\pi] : \pi \text{ irreducible representations of } G\}$

Fix representation ρ
 For $[\pi] \in \hat{G}$ let $M_\pi = \overline{\text{span}} \{L \text{ irreducible subspaces of } H_\rho : \rho|_L \sim \pi\}$
 $\rho|_L \sim \pi$
 $\rho|_L \in [\pi]$

Proposition 5.3

- 1) If $[\pi] \neq [\pi']$, then $M_\pi \perp M_{\pi'}$
- 2) If N is an irreducible subspace of M_π , then $\rho|_N \sim \pi$

$$H_\rho = \bigoplus_{[\pi] \in \hat{G}} M_\pi \quad (\text{unique})$$

$$= \bigoplus_{[\pi] \in \hat{G}} \bigoplus_{\substack{\rho|_L \sim \pi \\ L \text{ irreducible}}} L_\pi \quad (\text{not unique})$$

$\rho|_L \sim \pi \quad \alpha \in A_\pi$

$\text{mult}(\pi, \rho) = \text{cardinality of } [\pi] \text{ in } \hat{G}$
 Proposition 5.4
 $\text{mult}(\pi, \rho) = \dim C(\pi, \rho)$
 proof
 Let $M_\pi = \bigoplus_{\alpha \in A_\pi} L_\alpha$ and

ations of G

irreducible subspaces of H_π
s.t. $p \sim \pi$
 $p \in [\pi]$

then $p \sim \pi$

$\text{mult}(\pi, p) = \text{cardinality of } A_\pi$

Proposition 5.4

$\text{mult}(\pi, p) = \dim C(\pi, p)$

Proof

Let $M_\pi = \bigoplus_{\alpha \in A_\pi} L_\alpha$ and $U_\alpha \in C(\pi, p)$ unitary. Fix $u \in H_\pi$

and $v_\alpha = U_\alpha u$. Then $\{v_\alpha\}$ is orthonormal in M_π . Set $V = \overline{\text{span}} \{v_\alpha\}$

Then $\dim V = \text{mult}(\pi, p)$. To prove $T \mapsto T_u$ is
a bijection $C(\pi, p) \rightarrow V \subseteq H_\pi$

Injectivity: If $T \in C(\pi, p)$ is nonzero, then for $v \in H_\pi$

$$Tv = 0 \implies T\pi(x)v = p(x)Tv = 0$$

So $N(T)$ is invariant under π . But π irreducible, so $N(T) = \{0\}$

Well-defined

$R(T)$ subspace of H_p and $p \sim \pi$, so $R(T) \in M_\pi$.
Write $T_u = \sum_{\alpha \in A_\pi} u_\alpha$, since $p \in C(\pi, p)$,

$p_\alpha T = c_\alpha U_\alpha$ for $c_\alpha \in \mathbb{C}$ by Schur's lemma. So

$$u_\alpha = p_\alpha T u = c_\alpha v_\alpha, T u \in V$$

Surjective

$V = \sum_{\alpha \in A_\pi} c_\alpha v_\alpha \in V$, then $T = \sum c_\alpha U_\alpha \in C(\pi, p)$

and $v = T u$