

Peter-Weyl Theorem

Recall: π : irreducible ^(unitary) rep. of Abelian G .

then $\pi: G \rightarrow \mathcal{L}(\mathbb{C}) \cong \mathbb{C}$

i.e. π is a character of G .

- if G is not Abelian
& G is compact,

$$\pi: G \rightarrow \mathcal{L}(\mathcal{H}_\pi), \quad \dim \mathcal{H}_\pi < \infty$$

so $\pi(x)$ can be viewed as a matrix.

($x \in G$) (choose basis of $\mathcal{H}_\pi$, $\pi(x) = (\pi_{ij}(x))_{i,j=1}^{d_\pi}$)

"coefficient functions" $\pi_{ij}(x)$ behave "sort of" like characters.

Today

Defn: L

Today: G is a compact group.

Defn: Let π be a unitary repⁿ of G ,
and $u, v \in \mathcal{H}_\pi$.

The function $\phi_{u,v} = \phi_{u,v}^\pi: G \rightarrow \mathbb{C}$,
 $\phi_{u,v}(x) = \langle \pi(x)u, v \rangle$

is called a matrix element of π .

if $\{e_i\}_i$ is an o/n basis of $\mathcal{H}_\pi$, define

$$\pi_{ij} = \phi_{e_j, e_i}(s) = \langle \pi(s)e_j, e_i \rangle.$$

Define $\mathcal{E}_\pi := \text{span} \{ \phi_{u,v} : u, v \in \mathcal{H}_\pi \} \subset C(G)$

$\mathcal{E} := \text{span} \bigcup_{\pi \in \hat{G}} \mathcal{E}_\pi$ "trigonometric polynomials on G ".

Peter-Weyl Theorem

a. E is dense in $C(G)$ and $L^p(G)$ $p < \infty$

b. $L^2(G) = \bigoplus_{[\pi] \in \hat{G}} E_{\pi}$ (orthogonal direct sum)
(for each $[\pi]$, fix an arbitrary representative π)

c. The set

$$\left\{ \sqrt{d_{\pi}} \pi_{i,j} : [\pi] \in \hat{G}, i, j = 1, \dots, d_{\pi} \right\}$$

is an o/n basis for $L^2(G)$.

d. Each $[\pi] \in \hat{G}$ occurs in the right regular repⁿ of G , with multiplicity d_{π} .

More precisely, for $[\pi] \in \hat{G}$,

$\forall i = 1, \dots, d_{\pi}$, the subspace of E_{π} spanned by the i th row of $(\pi_{i,j})$

is invariant under R , and $\pi \sim R|_{\text{this subspace}}$

Prop 5.6. Suppose π is a unitary repⁿ of G .

- $\mathcal{E}_\pi$ depends only on $[\pi]$.
- $\mathcal{E}_\pi$ is invariant under left & right translations.
& is a two-sided ideal in $L^1(G)$.
- $\dim \mathcal{E}_\pi \leq d_\pi^2$.

Proof. Suppose $T: \mathcal{H}_\pi \rightarrow \mathcal{H}_{\pi'}$ is a unitary equivalence $\pi \sim \pi'$, so that $\pi(x) = T^{-1} \pi'(x) T$.

Then $\forall u, v \in \mathcal{H}_\pi, \forall x \in G$,

$$\begin{aligned} \phi_{u,v}^\pi(x) &= \langle \pi(x)u, v \rangle = \langle T^{-1} \pi'(x) T u, v \rangle \\ &= \langle \pi'(x) T u, T v \rangle = \phi_{T u, T v}^{\pi'}(x). \end{aligned}$$

So $\mathcal{M}_\pi = \{\text{matrix elem. of } \pi\} \subseteq \{\text{matrix elem. of } \pi'\}$

$$\Rightarrow \mathcal{E}_\pi \subseteq \mathcal{E}_{\pi'} \quad \text{by symmetry, } \mathcal{E}_\pi = \mathcal{E}_{\pi'}.$$

if $x, y \in G, u, v \in \mathcal{H}_\pi$,

$$\begin{aligned} \phi_{u,v}^\pi(y^{-1}x) &= \langle \pi(y^{-1}x)u, v \rangle \\ &= \langle \pi(x)u, \pi(y)v \rangle \\ &= \phi_{u, \pi(y)v}^\pi(x) \end{aligned}$$

So $\mathcal{M}_\pi$ is invariant under left translation

$\Rightarrow \mathcal{E}_\pi$ " " (same for right transl.)

Peter-Weyl Theorem

- $\mathcal{E}$ is dense in $C(G)$ and $L^p(G)$ $1 \leq p < \infty$
- $L^2(G) = \bigoplus_{[\pi] \in \hat{G}} \mathcal{E}_\pi$ (orthogonal direct sum)
(for each $[\pi]$, fix an arbitrary representative π)
- The set

$$\left\{ \sqrt{d_\pi} \pi_{i,j} : [\pi] \in \hat{G}, i, j = 1, \dots, d_\pi \right\}$$
 is an orthon basis for $L^2(G)$.

If $u, v \in \mathcal{H}_\pi$, $f \in L^1(G)$, then $\forall x \in G$,
 $(f * \phi_{u,v})(x)$

$$\begin{aligned} &= \int_G f(y) \phi_{u,v}(y^{-1}x) dy \\ &= \int f(y) \langle \pi(y^{-1}x)u, v \rangle dy \\ &= \int f(y) \langle \pi(x)u, \pi(y)v \rangle dy \\ &= \langle \pi(x)u, \int \overline{f(y)} \pi(y)v dy \rangle \\ &= \langle \pi(x)u, \pi(\bar{f})v \rangle \\ &= \phi_{u, \pi(\bar{f})v}(x). \end{aligned}$$

So $\mathcal{E}_\pi$ is a left ideal in $L^1(G)$.

If $\{e_i\}_{i=1}^{d_\pi}$ is an orthonormal basis of $\mathcal{H}_\pi$,
 $u = \sum_i u_i e_i$, $v = \sum_j v_j e_j$.

$$\begin{aligned}\phi_{u,v}(\pi) &= \langle \pi(x)u, v \rangle \\ &= \sum_{i,j} u_i \overline{v_j} \langle \pi(x)e_i, e_j \rangle \\ &= \sum_{i,j} u_i \overline{v_j} \pi_{j,i}(x).\end{aligned}$$

So $\dim \mathcal{E}_\pi \leq d_\pi^2$. □

Prop 5.7. If $\pi = \pi_1 \oplus \dots \oplus \pi_n$, then $\mathcal{E}_\pi = \sum_{j=1}^n \mathcal{E}_{\pi_j}$ (not necessarily direct)

Proof. We have $\mathcal{H}_\pi = \bigoplus_{j=1}^n \mathcal{H}_{\pi_j}$.

Let $\iota_j: \mathcal{H}_{\pi_j} \rightarrow \mathcal{H}_\pi$ be the inclusion map.

$[\sum_j \mathcal{E}_{\pi_j} \subseteq \mathcal{E}_\pi]$: if $u, v \in \mathcal{H}_{\pi_j}$, then $\forall x \in G$,

$$\begin{aligned}\phi_{u,v}^{\pi_j}(x) &= \langle \pi_j(x)u, v \rangle \\ &= \langle \pi(x) \iota_j(u), \iota_j(v) \rangle \\ &= \phi_{\iota_j(u), \iota_j(v)}^\pi(x).\end{aligned}$$

So $\mathcal{E}_{\pi_j} \subseteq \mathcal{E}_\pi$, and then $\sum_j \mathcal{E}_{\pi_j} \subseteq \mathcal{E}_\pi$.

Peter-Weyl Theorem

a. $\mathcal{E}$ is dense in $C(G)$ and $L^p(G)$ $p < \infty$

b. $L^2(G) = \bigoplus_{[\pi] \in \hat{G}} \mathcal{E}_{\pi}$ (orthogonal direct sum)

(for each $[\pi]$, fix an arbitrary representative π)

c. The set

$$\left\{ \sqrt{d_{\pi}} \pi_{i,j} : [\pi] \in \hat{G}, i, j = 1, \dots, d_{\pi} \right\}$$

is an o/n basis for $L^2(G)$.

$$\{\mathcal{E}_{\pi} \subset \sum_j \mathcal{E}_{\pi_j}\}$$

$$\text{if } u, v \in \mathcal{H}_{\pi}, \quad u = \sum_j u_j, \quad v = \sum_j v_j \\ u_j, v_j \in \mathcal{H}_{\pi_j}$$

then

$$\begin{aligned} \phi_{u,v}(x) &= \sum_{j,k} \langle \pi_j(x) u_j, v_k \rangle \\ &= \sum_j \langle \pi_j(x) u_j, v_j \rangle + \underbrace{\sum_{j \neq k} \langle \pi_j(x) u_j, v_k \rangle}_{=0} \\ &= \sum_j \phi_{u_j, v_j}^{\pi_j} \\ &\in \sum_j \mathcal{E}_{\pi_j} \end{aligned}$$

$$\therefore \mathcal{E}_{\pi} \subseteq \sum_j \mathcal{E}_{\pi_j}$$

□

Theorem 5

Theorem 5.8 (Schur Orthogonality Relations)

Let π, π' be irreducible unitary rep^s of G .

a. $\pi \neq \pi'$, then $E_\pi \perp E_{\pi'}$ (in $L^2(G)$).

b. if $\{e_j\}_{j=1}^{d_\pi}$ is an o/n basis of $\mathcal{H}_\pi$,

then $\{\sqrt{d_\pi} \pi_{i,j} : i, j = 1, \dots, d_\pi\}$

is an o/n basis of E_π .

Prop 5.7: If $\pi = \pi_1 \oplus \dots \oplus \pi_n$

Proof: We have $\mathcal{H}_\pi = \bigoplus_{j=1}^n \mathcal{H}_{\pi_j}$

Let $\zeta_j: \mathcal{H}_{\pi_j} \rightarrow \mathcal{H}_\pi$

$[\zeta_j, E_{\pi_j}] \subseteq E_\pi$: if $u, v \in \mathcal{H}_{\pi_j}$

$$\phi_{u,v}^{\pi_j}(x) = \langle \pi_j(x)u, v \rangle$$

$$= \langle \pi(x) \zeta_j(u), \zeta_j(v) \rangle$$

$$= \phi_{\zeta_j(u), \zeta_j(v)}^\pi(x)$$

So $E_{\pi_j} \subseteq E_\pi$, and then $\sum$

Theorem 5.8 (Schur Orthogonality Relations)

Let π, π' be irreducible unitary rep^s of G .

a. $\pi \neq \pi'$, then $E_\pi \perp E_{\pi'}$ (in $L^2(G)$).

b. if $\{e_j\}_{j=1}^{d_\pi}$ is an o/n basis of $\mathcal{H}_\pi$,

then $\{\sqrt{d_\pi} \pi_{ij} : i, j = 1, \dots, d_\pi\}$

is an o/n basis of E_π .

Proof. a. Suppose $u, v \in \mathcal{H}_\pi$, $u', v' \in \mathcal{H}_{\pi'}$.

$$\langle \phi_{u,v}^\pi, \phi_{u',v'}^{\pi'} \rangle = \int_G \langle \pi(x)u, v \rangle \overline{\langle \pi'(x)u', v' \rangle} dx$$

$$= \int_G \langle \pi(x)u, v \rangle \langle v', \pi'(x)u' \rangle dx$$

$$= \int_G \langle \langle \pi(x)u, v \rangle v', \pi'(x)u' \rangle dx$$

$$= \left\langle \int_G \pi'(x^{-1}) \langle \pi(x)u, v \rangle v' dx, u' \right\rangle$$

$$= \langle \tilde{T}u, u' \rangle \quad (\text{WTS } = 0; \text{ suffices to show } \tilde{T} = 0, \\ \text{ie. } \tilde{T} \in C(\pi, \pi'))$$

Peter-Weyl Theorem

a. $\mathcal{E}$ is dense in $C(G)$ and $L^p(G)$ $p < \infty$

b. $L^2(G) = \bigoplus_{[\pi] \in \hat{G}} \mathcal{E}_{\pi}$ (orthogonal direct sum)
(for each $[\pi]$, fix an arbitrary representative π)

c. The set

$$\{\sqrt{d_{\pi}} \pi_{ij} : [\pi] \in \hat{G}, i, j = 1, \dots, d_{\pi}\}$$

is an o/n basis for $L^2(G)$.

More generally, if $T: \mathcal{H}_{\pi} \rightarrow \mathcal{H}_{\pi'}$,

define $\tilde{T} := \int_G \pi'(x^{-1}) T \pi(x) dx : \mathcal{H}_{\pi} \rightarrow \mathcal{H}_{\pi'}$

$$\text{then } \tilde{T}_{\pi}(y) = \int_G \pi'(x^{-1}) T \pi(xy) dx$$

$$= \int_G \pi'(yx^{-1}) T \pi(x) dx$$

$$= \pi'(y) \tilde{T} \quad \text{so } \tilde{T} \in C(\pi, \pi')$$

Since $\pi \neq \pi'$, Schur's lemma $\Rightarrow \tilde{T} = 0$.

Take $Tu = \langle u, v \rangle v$, then $0 = \langle \tilde{T}u, u \rangle = \langle \phi_{u,v}^{\pi}, \phi_{v,u}^{\pi'} \rangle$.

b. Set $u=e_i, u'=e_{i'}, v=e_j, v'=e_{j'}$.

Since $\pi=\pi', \tilde{T} \in C(\pi), \tilde{T}=cI$.

$$\begin{aligned} \text{So } \langle \pi_{i,j}, \pi_{i',j'} \rangle &= \langle \phi_{e_j, e_i}, \phi_{e_{j'}, e_{i'}} \rangle \\ &= \langle \tilde{T} e_i, e_{i'} \rangle \\ &= c \delta_{i,i'}. \end{aligned}$$

Note. $\text{tr } \tilde{T} = \text{tr}(cI) = c d_\pi$

$$\int_G \text{tr}(\pi(x)^* T \pi(x)) dx$$

$$\int_G \text{tr}(T) dx = \text{tr}(T)$$

Since $Tu = \langle u, e_j \rangle e_{j'}$,

$$\text{tr } T = \delta_{j,j'}, \text{ so}$$

$$c = \frac{\text{tr } T}{d_\pi} = \frac{\delta_{j,j'}}{d_\pi}.$$

$$\langle \pi_{i,j}, \pi_{i',j'} \rangle = \frac{1}{d_\pi} \delta_{i,i'} \delta_{j,j'}.$$

So $\{\sqrt{d_\pi} \pi_{i,j}\}_{i,j=1}^{d_\pi}$ are orthonormal.

Since $\dim \mathcal{E}_\pi \leq d_\pi^2$, the set is an orthonormal basis of $\mathcal{E}_\pi$. $\square$

Peter-Weyl Theorem

- $\mathcal{E}$ is dense in $C(G)$ and $L^p(G)$ $1 \leq p < \infty$.
- $L^2(G) = \bigoplus_{[\pi] \in \hat{G}} \mathcal{E}_\pi$ (orthogonal direct sum)
(for each $[\pi]$, fix an arbitrary representative π)
- The set

$$\{ \sqrt{d_\pi} \pi_{i,j} : [\pi] \in \hat{G}, i,j=1, \dots, d_\pi \}$$
 is an o/n basis for $L^2(G)$.

Recall:

$$\mathcal{E} := \text{span} \bigcup_{\substack{\pi \\ \text{f.d. rep's} \\ \text{of } G}} \mathcal{E}_\pi = \text{span} \bigcup_{\substack{\pi \\ \text{irred.} \\ \text{rep. of } G}} \mathcal{E}_\pi = \text{span} \bigcup_{[\pi] \in \hat{G}} \mathcal{E}_\pi$$

Prop 5.10. $\mathcal{E}$ is an algebra that separates points, is closed under conjugation, and contains constants.

Corollary. $\mathcal{E}$ is dense in $C(G)$.
(Stone-Weierstrass).

Proof of Prop.: "separates pt's" Gelfand-Raikov.

• "contains const." trivial rep $G \rightarrow \mathbb{C}$.

• "conjugation": if π is a rep of G ,
define the contragredient $\bar{\pi} : G \rightarrow \mathcal{H}_{\pi}^*$
 $\bar{\pi}(x) = \pi(x^{-1})^t$.

Then

$$\begin{aligned}\overline{\pi_{ij}(x)} &= \overline{\langle \pi(x)e_j, e_i \rangle} \\ &= \langle e_i, \pi(x)e_j \rangle \\ &= \langle \pi(x^{-1})e_i, e_j \rangle \\ &= \langle \pi(x^{-1})^t e_j, e_i \rangle \\ &= \bar{\pi}_{ij}(x).\end{aligned}$$

[E is an algebra]:

Need to show: if $(\pi, \pi') \in \hat{G}$ and we choose bases for $\mathcal{H}_{\pi}, \mathcal{H}_{\pi'}$,
then π_{ij}, π'_{kl} is a matrix element.

Identify $\mathcal{H}_{\pi} \cong \mathbb{C}^n, \mathcal{H}_{\pi'} \cong \mathbb{C}^{n'}$,

consider $\mathbb{C}^{nn'}$ as the space of $(n \times n')$ -matrices over $\mathbb{C}$.

Define $\pi \otimes \pi' : G \rightarrow \mathbb{C}^{nn'}$ by

$$(\pi \otimes \pi')(x) = \pi(x) \bar{\pi}'(x^{-1}) \in \mathbb{C}^{nn'} \quad (T \in \mathbb{C}^{nn'})$$

Peter-Weyl Theorem:

- $\mathcal{E}$ is dense in $C(G)$ and $L^p(G)$ $p < \infty$
- $L^2(G) = \bigoplus_{[\pi] \in \hat{G}} \Sigma_{\pi}$ (orthogonal direct sum)
(for each $[\pi]$, fix an arbitrary representative π)
- The set

$$\left\{ \sqrt{d_{\pi}} \pi_{i,j} : [\pi] \in \hat{G}, i, j = 1, \dots, d_{\pi} \right\}$$
 is an o/n basis for $L^2(G)$.

The matrix units $e_{j,l} \in \mathbb{C}^{n \times n}$ are an o/n basis

$$e_{j,l} = \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 1 & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix}$$

Compute $\phi_{\pi \otimes \pi'}(x)$

$$\begin{aligned} & \langle (\pi \otimes \pi')(x) e_{j,l}, e_{i,k} \rangle \\ &= \sum_{m,m'} [(\pi \otimes \pi')(x) e_{j,l}]_{m,m'} \overline{[e_{i,k}]_{m,m'}} \\ &= [(\pi \otimes \pi')(x) e_{j,l}]_{i,k} = [\pi(x) e_{j,l} \overline{\pi'(x^{-1})}]_{i,k} \end{aligned}$$

force $m=i, m'=k$

$$\begin{aligned}
&= \sum_m [\pi(x)]_{i,m} [e_{j,l} \overline{\pi'(x^{-1})}]_{m,k} \\
&= \sum_m \pi_{i,m}(x) \sum_{m'} \underbrace{[e_{j,l}]_{m,m'}}_{\substack{\text{forces } m=j \\ m'=l}} [\overline{\pi'(x^{-1})}]_{m',k} \\
&= \pi_{i,j}(x) [\overline{\pi'(x^{-1})}]_{l,k} \\
&= \pi_{i,j}(x) \overline{\pi'_{k,l}(x)}
\end{aligned}$$

So, $\pi_{i,j} \overline{\pi'_{k,l}}$ is a matrix element (of $\pi \otimes \pi'$). $\square$

Proof of Peter-Weyl theorem trivial (done already). $\square$

§5.3 Fourier Analysis on Compact Groups

Peter-Weyl tells us: if $f \in L^2(G)$,

$$f = \sum_{\pi \in \hat{G}} \sum_{i,j=1}^{d_\pi} c_{i,j}^\pi \pi_{i,j} \quad (\text{converge in } L^2(G))$$

$$\text{where } c_{i,j}^\pi = d_\pi \int f(x) \overline{\pi_{i,j}(x)} dx.$$

This depends on choices of basis for each $\pi \in \hat{G}$!

Now what?

Defn. If $f \in L^1(G)$ and π irreducible ctr,
define $\hat{f}(\pi): \mathcal{H}_\pi \rightarrow \mathcal{H}_\pi$ by

$$\begin{aligned}\hat{f}(\pi) &= \int_G f(x) \pi(x^{-1}) dx \\ &= \int_G f(x) \pi(x)^* dx.\end{aligned}$$

If we fix a basis $\{e_i\}$ for $\mathcal{H}_\pi$,

$$\begin{aligned}[\hat{f}(\pi)]_{ij} &= \langle \hat{f}(\pi) e_j, e_i \rangle \\ &= \int f(x) \langle \pi(x^{-1}) e_j, e_i \rangle dx \\ &= \int f(x) \langle e_j, \pi(x) e_i \rangle dx \\ &= \int f(x) \overline{\pi_{ji}(x)} dx = \frac{1}{d_\pi} c_{ji}^\pi.\end{aligned}$$

Then

$$\begin{aligned}\sum_{i,j=1}^{d_\pi} c_{ij}^\pi \pi_{ij}(x) &= d_\pi \sum_{i,j} [\hat{f}(\pi)]_{ji} \pi_{ij}(x) \\ &= d_\pi \operatorname{tr}(\hat{f}(\pi) \pi(x))\end{aligned}$$

independent of choice of basis.

So we get a "Fourier Inverse Theorem" (FIT 3)

$$f(x) = \sum_{[\pi] \in \hat{G}} d_\pi \operatorname{tr}(\hat{f}(\pi) \pi(x)) \quad (\text{in } L^2(G))$$

Parseval:

$$\begin{aligned} \|f\|_2^2 &= \sum_{[\pi] \in \hat{G}} \sum_{i,j=1}^{d_\pi} \frac{1}{d_\pi} |L_{i,j}^\pi|^2 \\ &= \sum_{[\pi] \in \hat{G}} d_\pi \operatorname{tr} [\hat{f}(\pi)^* \hat{f}(\pi)] \\ &= \sum_{[\pi] \in \hat{G}} d_\pi \|\hat{f}(\pi)\|_{HS}^2. \end{aligned}$$

Lemma $\hat{f}(\pi)^* = \pi(\bar{f}) = \int_G \bar{f}(x) \pi(x) dx.$ $\square$

Basic properties:

$$\begin{aligned} (af + bg)^\wedge(\pi) &= a\hat{f}(\pi) + b\hat{g}(\pi). \\ (f * g)^\wedge(\pi) &= \hat{g}(\pi) \hat{f}(\pi) \\ (f^*)^\wedge(\pi) &= \hat{f}(\pi)^* \\ (L_x f)^\wedge(\pi) &= \hat{f}(\pi) \pi(x^{-1}) \\ (R_x f)^\wedge(\pi) &= \pi(x) \hat{f}(\pi). \quad \square \end{aligned}$$

Defn. π fin. dim. unitary rep. of G .

The character of π , $\chi_\pi: G \rightarrow \mathbb{C}$, is

$$\chi_\pi(x) = \operatorname{tr}(\pi(x)) \quad (\chi_\pi = \chi_{[\pi]})$$

$$\begin{aligned}
& \text{tr}(\hat{f}(\pi)\pi(x)) \quad (\text{appears in FIT3}) \\
&= \text{tr}\left(\int f(y)\pi(y^{-1})dy \pi(x)\right) \\
&= \text{tr}\left(\int f(y)\pi(y^{-1}x)dy\right) \\
&= \int f(y)\text{tr}(\pi(y^{-1}x))dy \\
&= \int f(y)\chi_\pi(y^{-1}x)dy \\
&= (f \times \chi_\pi)(x).
\end{aligned}$$

So (FIT3) can be written as

$$f = \sum_{[\pi] \in \hat{G}} d_\pi f \times \chi_\pi \quad (\text{FIT4})$$

& therefore $f \mapsto d_\pi f \times \chi_\pi$
is the orthogonal projection $L^2(G) \rightarrow E_\pi$.

It follows that

$$\chi_\pi \times \chi_{\pi'} = \begin{cases} 0 & \pi \nmid \pi' \\ d_{\pi'}^{-1} \chi_\pi & \pi \sim \pi' \end{cases}$$

Lemma 5.22,

if $f \in ZL^1(G)$ and $[\pi] \in \hat{G}$,

$$\text{then } P_{\pi}(f) = d_{\pi} f * \chi_{\pi} \\ = \langle f, \chi_{\pi} \rangle \chi_{\pi}.$$

Proof Note that central functions have scalar Fourier coefficients: $\hat{f}(\pi) = c_{\pi} I$.

(i.e. $\hat{f}(\pi) \in \mathbb{C}(\pi)$).

$$\begin{aligned} \hat{f}(\pi) \pi(x) &= \int f(y) \pi(y^{-1}x) dy \\ &= \int f(y y^{-1}) \pi(y) dy \\ &= \int f(y^{-1}x) \pi(y) dy \quad (f \text{ is central}) \\ &= \int f(y) \pi(\pi y^{-1}) dy = \pi(\pi) \hat{f}(\pi). \end{aligned}$$

$$\text{So } \hat{f}(\pi) = c_{\pi} I$$

$$\text{Now } c_{\pi} = \frac{1}{d_{\pi}} \text{tr}(\hat{f}(\pi))$$

$$= \frac{1}{d_{\pi}} \int f(y) \text{tr}(\pi(y)^*) dy$$

$$= \frac{1}{d_{\pi}} \int f(y) \overline{\chi_{\pi}(y)} dy$$

$$\begin{aligned} \text{So } d_{\pi} f * \chi_{\pi} &= d_{\pi} \text{tr}(\hat{f}(\pi) \pi) \\ &= d_{\pi} \text{tr}(c_{\pi} \pi) \\ &= d_{\pi} c_{\pi} \text{tr}(\pi) \\ &= \langle f, \chi_{\pi} \rangle \chi_{\pi}. \quad \square \end{aligned}$$

Dfn. A function $f: G \rightarrow \mathbb{C}$ is called central
 (or a class function)
 if it is constant on every
 conjugacy class of G :

$$f(x) = f(yxy^{-1}) \quad \forall x, y \in G$$

or equivalently if

$$f(xy) = f(yx) \text{ a.e. } x, y \in G.$$

Example: χ_π is central: $\chi_\pi(yxy^{-1}) = \text{tr}(\pi(y)\pi(x)\pi(y)^{-1})$
 $= \text{tr}(\pi(x)) = \chi_\pi(x).$

Let $ZL^p(G) = \{f \in L^p(G) : f \text{ central}\}$
 $ZC(G) = \{f \in C(G) : f \text{ central}\}$

Prop 5.21. $L^p(G)$ & $C(G)$ are Banach algebras under convolution,
 and $ZL^p(G)$, $ZC(G)$ are their centres.

Proof: $\|f * g\|_p \leq \|f\|_p \|g\|_p$ (Young's inequality)
 $\leq \|f\|_p \|g\|_p \quad \forall p \geq 1$, since $|G| = 1$.
 $f \in Z(L^p(G))$ iff $\forall g \in L^p(G)$, (p-notation)

$$\int f(xy) g(y^{-1}) dy = \int g(y) f(y^{-1}x) dy = \int f(yx) g(y^{-1}) dy \quad \text{a.e. } x \in G.$$

iff $f(xy) = f(yx) \text{ a.e. } x, y \in G$, $\omega f \in ZL^p(G)$. $\square$

Prop 5.23.

$\{x_\pi : [\pi] \in \hat{G}\}$ is an o/n basis for $\mathbb{Z}L^2(G)$.

Proof. We know orthonormality.

Furthermore if $f \in \mathbb{Z}L^2(G)$,

$$f = \sum_{[\pi] \in \hat{G}} P_{E_\pi} f = \sum_{[\pi] \in \hat{G}} \langle f, x_\pi \rangle x_\pi. \quad \square$$

(Peter-Weyl)

Prop 5.25. $\text{span}\{x_\pi : [\pi] \in \hat{G}\}$ is dense in $\mathbb{Z}C(G)$ & $\mathbb{Z}LP(G)$ $p < \infty$.
(uses "generalised summability" of Peter-Weyl expansion.)

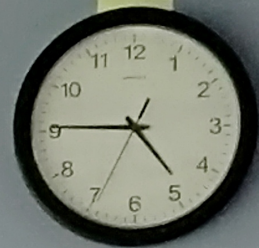
Thm 5.26: For $f \in \mathbb{Z}LP(G)$, $[\pi] \in \hat{G}$ ($1 \leq p < \infty$),

define $h_\pi(f) := d_\pi \langle f, x_\pi \rangle$.

Then $[\pi] \mapsto h_\pi$ is a homeomorphism

$(\hat{G}, \text{discrete}) \rightarrow o(\mathbb{Z}LP(G))$

(same holds for $\mathbb{Z}C(G)$.)



Prop 5.27: let G be finite.

a. $\#(\hat{G}) = \#\{\text{conjugacy classes of } G\}.$

b. $\#(G) = \sum_{[\pi] \in \hat{G}} d_{\pi}^2.$

Proof:

a. $\mathbb{C}L^2(G)$ has two bases:

1. $\{\chi_{\pi} : [\pi] \in \hat{G}\}$

2. $\{1_E : E \text{ is a conjugacy class of } G\}$ ✓

b. $\dim L^2(G) = \sum_{[\pi] \in \hat{G}} d_{\pi}^2,$
" $\#(G).$ □

So $\hat{f}(\pi)$

Now $c_{\pi} = \frac{1}{d_{\pi}}$

$= \frac{1}{d_{\pi}}$

$= \frac{1}{d_{\pi}}$

So d_{π}