

Spectral Theory in Hilbert space

$$H, T \in L(H) \rightarrow T = \sum \lambda P_\lambda$$

$$\rightarrow T = \int f_T dP$$

$\mathcal{A}$ - Banach $\ast$ -alg. H - Hilbert space.

$\pi: \mathcal{A} \rightarrow L(H)$ is $\ast$ -hom.

$\exists (X, \Omega) \quad P: \Omega \rightarrow L(H)$ s.t.

$$\pi(a) = \int f_a dP$$

Def 1. Let (X, Ω) be a measurable space, a set map $P: \Omega \rightarrow L(H)$ is called a spectral measure if

$$(1) P(\emptyset) = 0 \quad P(X) = id.$$

$$(2) P(E \cap F) = P(E)P(F) \quad \forall E, F \in \Omega$$

(3) $\{E_i\}$ - seq. pairwise disjoint subsets in Ω .

$$\rightarrow P(\bigcup_{i=1}^{\infty} E_i) = \text{SOT-} \lim_{n \rightarrow \infty} \sum_{i=1}^n P(E_i)$$

Pro: $\forall u, v \in H. \quad P_{u,v}(E) = \langle P(E)u, v \rangle \rightarrow P_{u,v}$ is measure.

Measure. $(u, v) \rightarrow P_{u,v}$ is a sesquilinear map.
($\lambda x + \mu y \rightarrow \lambda P_{u,v} + \mu P_{u,y}$)

Sesquilinear
f. d. m. $\rightarrow$
f. m. $\rightarrow$ w.
f. m. $\rightarrow$ b. y.

The R ()
if f is
f(u, v)

Supp f

$$P: \mathbb{R} \rightarrow \mathbb{R}$$

sequilinear
 $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$

$$f(x, y+z) = f(x, y) + f(x, z)$$

$$f(ax, by) = ab f(x, y) \quad \forall a, b \in \mathbb{C}$$

The R (Theorem 12.8 Rudin Functional Analysis)

if f is bounded sequilinear map, then $\exists T_f \in \mathcal{L}(H)$ s.t.

$$f(u, v) = \langle T_f u, v \rangle \quad \forall u, v \in H$$

$$\sup \{ |f(u, v)| : \|u\| = \|v\| = 1 \} < \infty$$

Integral

$\mathcal{B}(X)$ - all bounded measurable function on X

$$f \in \mathcal{B}(X) \longrightarrow \int f dP_{x, y}$$

$$(x, y) \longrightarrow \int f dP_{x, y} \quad ? \quad \text{bound sequilinear}$$

$$T_f \in \mathcal{L}(H) \rightarrow \langle T_f u, v \rangle = \int f dP_{u, v}$$

$$\int f dP = T_f$$

Prop $(x, y) \rightarrow \int f dP_{x, y}$ is bounded sequilinear

pf: 1. $\|u\| = 1$ $P_{u, u}$ - positive measure

$$\begin{aligned} \left| \int f dP_{u, u} \right| &\leq \int |f| dP_{u, u} \leq \|f\|_{\infty} P_{u, u}(X) \\ &= \|f\|_{\infty} \langle P(x), u, u \rangle \\ &= \|f\|_{\infty} \|u\|^2 = \|f\|_{\infty} < \infty \end{aligned}$$

2. $\|u\| = \|v\| = 1$

$$\begin{aligned} \left| \int f dP_{u, v} \right| &= \left| \frac{1}{2} \left(\int f dP_{u+u, v} + \int f dP_{u-u, v} \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\int f dP_{u, v+v} + \int f dP_{u, v-v} \right) \right| \\ &< \infty \end{aligned}$$

Th 4: The Integral w.r.t. P is a λ -hom
from $B(X) \rightarrow L(\mathcal{H})$

Pf: i. linear: $\forall u, v \in \mathcal{H}, f, g \in B(X), a, b \in \mathbb{C}$

$$\langle \int af + bg \, dP, u, v \rangle = \int af + bg \, dP_{u,v}$$

$$= a \int f \, dP_{u,v} + b \int g \, dP_{u,v}$$

$$= \langle a \int f \, dP, u, v \rangle + \langle b \int g \, dP, u, v \rangle$$

$$\rightarrow \int af + bg \, dP = a \int f \, dP + b \int g \, dP$$

ii. bounded:

$$\sup \langle \int f \, dP, u, v \rangle = \| \int f \, dP \| = \| f \|_{\infty} \leq 4 \| f \|_{\infty} \in P_{u,v}$$

iii. alg-hom: $\int fg \, dP = \int f \, dP \int g \, dP$

$$f = X_A, g = X_B, A, B \in \mathcal{S}$$

$$\int fg \, dP = P(A \cap B) = P(A)P(B)$$

$$f, g \text{ - step functions } f = \sum_i \alpha_i X_{A_i}, g = \sum_j \beta_j X_{B_j}$$

$f \in B(X)$ g - step function

$\exists$ f_n λ -step fun s.t. $f_n \rightarrow f$ uniformly



$\mathcal{A}$ -commutative C^* -alg

$$\sigma(\mathcal{A}) = \{h \in \mathcal{A}^* \mid h(ab) = h(a)h(b) \forall a, b \in \mathcal{A}, h \neq 0\}$$

$\sigma(\mathcal{A})$ - locally compact T_0 ($\mathcal{W}^*$ -top.)

$$\text{if } \mathcal{A} \text{ is unital} \rightarrow \begin{cases} h \in \sigma(\mathcal{A}), h(1) = 1 \\ \sigma(\mathcal{A}) \text{ compact.} \end{cases}$$

$\mathcal{B}(\sigma(\mathcal{A})) = \{ \text{Borel measurable subsets in } \sigma(\mathcal{A}) \}$

Def 5. $\{e_\lambda\}$ is called app. idemp. if

$$\begin{cases} e_\lambda \geq 0 & (e_\lambda^* = e_\lambda, \sigma(e_\lambda) \in \mathbb{R}_+) \\ \lambda \geq \mu \Rightarrow e_\lambda \geq e_\mu \\ \forall a \in \mathcal{A} \quad \lim_{\lambda} \|e_\lambda a - a\| = \lim_{\lambda} \|a e_\lambda - a\| = 0 \end{cases}$$

Th 6: Every C^* -alg has approximate identity.

$\Leftarrow$ An innovation to C^* -alg. $\Rightarrow$ William Arveson Th. 1.8.2

Def 7. $\varphi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ is called a non-degenerated $*$ -rep. of $\mathcal{A}$ on $\mathcal{H}$

if (1) φ is $*$ -hom

$$(2) \quad \varphi(a)V = 0, \forall a \in \mathcal{A} \Rightarrow V = 0$$

$$(2)' \quad \forall V \in \mathcal{H}, V \neq 0, \exists a \in \mathcal{A} \text{ s.t. } \varphi(a)V \neq 0$$

$$(2)'' \quad \varphi(\mathcal{A}) \cdot \mathcal{H} := \{ \varphi(a)V = a \cdot V, V \in \mathcal{H} \} \text{ is norm dense in } \mathcal{H}$$

$(2) \Rightarrow (2)''$

$$\forall u \in (\varphi(\mathcal{A}) \cdot \mathcal{H})^\perp \Rightarrow \langle u, \varphi(a^*) \varphi(a) u \rangle = \langle \varphi(a) u, \varphi(a) u \rangle$$

$$\Rightarrow \varphi(a)u = 0 \stackrel{(2)}{\Rightarrow} u = 0$$

$(2)'' \Rightarrow (2)$

Suppose u

$\forall V \in \mathcal{H}$

$\langle \varphi(a)u, V \rangle$

$\Rightarrow V \in$

1.8.2

and x -rep. of $\mathcal{A}$ on $\mathcal{H}$

(2)' $\Rightarrow$ (2)

Suppose $u \in \mathcal{H}$ s.t. $\varphi(a)u = 0 \quad \forall a \in \mathcal{A}$.

$\forall v \in \mathcal{H}$,

$$\langle \varphi(a)v, u \rangle = \langle v, \varphi(a^*)u \rangle = \langle v, 0 \rangle = 0$$

$$\Rightarrow v \in (\varphi(\mathcal{A})\mathcal{H})^\perp \stackrel{(2)'}{\Rightarrow} v = 0$$

pro $\varphi(e_\lambda) \rightarrow \text{id}$ (sot)

pf: $\forall u \in \mathcal{H}$, (to show $\|\varphi(e_\lambda)u - u\| \rightarrow 0$)

$\exists \{a_i\}_{i=1}^\infty \subset \mathcal{A}$ s.t. $\|a_i\| \leq 1$ and $\|a_i u\| \rightarrow 0$.

$$\begin{aligned} \|u - \varphi(e_\lambda)u\| &\leq \|u - \varphi(a_i)u\| + \|\varphi(a_i)u - \varphi(a_i e_\lambda)u\| + \|\varphi(a_i e_\lambda)u - \varphi(e_\lambda)u\| \\ &\leq \|u - \varphi(a_i)u\| + \|\varphi\| \|a_i - a_i e_\lambda\| \|u\| + \|\varphi(e_\lambda)\| \|\varphi(a_i)u - u\| \rightarrow 0 \end{aligned}$$

Homework:

is a "C*-"

T.F.A.E

(1) $\exists p$

(2) $f_n \rightarrow$

uniformly

$$\| \int f g d\mu - \int f d\mu \int g d\mu \|$$

$$\leq \| \int f g d\mu - \int f_n g d\mu \| + \| \int f_n g d\mu - \int f d\mu \int g d\mu \|$$

$$\leq 4 \| f - f_n \|_\infty + \| \int f_n d\mu \cdot \int g d\mu - \int f d\mu \int g d\mu \|$$

$$\leq 4 \| f - f_n \|_\infty \| g \|_\infty + 4 \| f - f_n \|_\infty (4 \| g \|_\infty)$$

$$\rightarrow 0$$

$$\int f d\mu = (\int f d\mu)^*$$

$$f = \sum \alpha_i \chi_{A_i} = \bar{f}$$

$$\int f d\mu = \sum \alpha_i \mu(A_i)$$

$$\bar{f} = \sum \bar{\alpha}_i \chi_{A_i}$$

$$\int \bar{f} d\mu = \sum \bar{\alpha}_i \mu(A_i) = (\sum \alpha_i \mu(A_i))^* = (\int f d\mu)^*$$

$$f \in L^1(\mu) \quad f_n \rightarrow f \text{ a.e.} \quad (f_n \rightarrow f \checkmark)$$

$$\| \int \bar{f} d\mu - (\int f d\mu)^* \|$$

$$\leq \| \int \bar{f} d\mu - \int \bar{f}_n d\mu \| + \| \int \bar{f}_n d\mu - (\int f_n d\mu)^* \|$$

$$\leq 4 \| \bar{f} - \bar{f}_n \|_\infty + 4 \| f_n - f \|_\infty \rightarrow 0$$

Remark: Suppose $I = (B(X) \rightarrow L(H))$

is a *** unital $*$ -hom.

T.F.A.E

(1) $\exists p: \Omega \rightarrow L(H)$ spectral measure s.t.

$$I(f) = \int f d p \quad \forall f \in B(X)$$

(2) $f_n \rightarrow f$ a.e. $\Rightarrow I(f_n) \rightarrow I(f)$ (SOT)

The spectral theorem

Thy Let $\mathcal{A}$ be a C^* -alg on Hilbert space

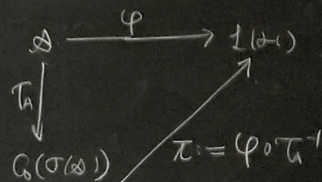
$\varphi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ is a $*$ -rep of $\mathcal{A}$ on $\mathcal{H}$
if $\mathcal{A}$ is unital or φ is non-degenerated.

then $\exists$ spectral measure $P: \mathcal{B}_{\sigma(\varphi)} \rightarrow \mathcal{L}(\mathcal{H})$ st

$$\varphi(x) = \int \lambda x \, dP \quad \forall x \in \mathcal{A}$$

T_A - Gelfand transform (isometrically $*$ -iso)

Pf:



π is continuous $*$ -hom

$\forall u, v \in \mathcal{H}$, define $\pi_{u,v}(f) = \langle \pi(f)u, v \rangle \quad \forall f \in C(\sigma(\varphi))$

R-rep $\Rightarrow \exists P_{u,v} = \mathcal{B}_{\sigma(\varphi)} \rightarrow \mathbb{C}$ st $\pi_{u,v}(f) = \int f \, dP_{u,v} \quad \forall f \in C(\sigma(\varphi))$

$$\{P_{u,v}\} \longrightarrow P$$

$(u, v) \rightarrow \int \cdot dP_{u,v}$ is sesquilinear ($\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{B}(\mathcal{O}(\mathcal{H}))$)

$$\circ \int \cdot dP_{u_1+u_2, v} = \int \cdot dP_{u_1, v} + \int \cdot dP_{u_2, v}$$

$$\Leftrightarrow P_{u_1+u_2, v} = P_{u_1, v} + P_{u_2, v}$$

$$\forall f \in \underline{G(\mathcal{O}(\mathcal{H}))} \quad \pi_{u_1+u_2, v}(f) = \pi_{u_1, v}(f) + \pi_{u_2, v}(f)$$

$$\begin{aligned} \int f dP_{u_1+u_2, v} &= \int f dP_{u_1, v} + \int f dP_{u_2, v} \\ &= \int f d(P_{u_1, v} + P_{u_2, v}) \end{aligned}$$

$$\circ P_{u_1+u_2, v} = P_{u_1, v} + P_{u_2, v}$$

$$P_{au, bv} = a\bar{b}P_{u, v}$$

Fix $g \in \mathbb{B}(\mathcal{O}(\mathcal{H}))$

(to show $(u, v) \rightarrow \int g dP_{u, v}$ is sesquilinear & bounded)

$$\circ \left| \int g dP_{u, v} \right| \leq \|g\|_0 \|P_{u, v}\| < \infty$$

$$\exists T_g \in \mathcal{L}(\mathcal{H}) \text{ s.t. } \langle T_g u, v \rangle = \int g dP_{u, v} \quad \forall u, v \in \mathcal{H}$$

$$\text{Next, } g \rightarrow T_g \quad (\mathbb{B}(\mathcal{X})) \rightarrow \mathcal{L}(\mathcal{H})$$

is $*$ -homomorphism

$$\text{linear } \forall a, b \in \mathbb{C} \quad f, g \in \mathbb{B}(\mathcal{X})$$

$$\begin{aligned} \langle (T_{af+bg})u, v \rangle &= \int (af+bg) dP_{u, v} = a \int f dP_{u, v} + b \int g dP_{u, v} \\ &= \langle aT_f u, v \rangle + \langle bT_g u, v \rangle \end{aligned}$$

$$T_{af+bg} = aT_f + bT_g$$



$$T_g = (T_g)^* \rightarrow$$

$\forall u, v \in \mathcal{H}$:

$$\begin{aligned} \langle T_g u, v \rangle &= \int g d\mu_{u,v} = \overline{\int g d\overline{\mu_{u,v}}} = \overline{\int g d\mu_{v,u}} \\ &= \langle T_g v, u \rangle = \langle u, T_g v \rangle = \langle T_g^* u, v \rangle \end{aligned}$$

$$\text{def: } (T_{gf} = T_g T_f)$$

$$f, g \in C(\sigma(A)) \quad T_{gf} = \pi(gf) = \pi(g)\pi(f) = T_g T_f$$

$$\forall u, v \in \mathcal{H} \quad \int gf d\mu_{u,v} = \langle T_g T_f u, v \rangle = \int g d\mu_{T_f u, v}$$

$$\int \cdot f d\mu_{u,v} = \int \cdot d\mu_{T_f u, v} \quad \text{on } C(\sigma(A))$$

$$\rightarrow \forall h \in B(\sigma(A)), \int h f d\mu_{u,v} = \int h d\mu_{T_f u, v} \quad \forall f \in C(\sigma(A))$$

$$\begin{aligned} \int h f d\mu_{u,v} &= \int h d\mu_{T_f u, v} \stackrel{\text{def}}{=} \langle T_h (T_f u), v \rangle = \langle T_g u, T_h^* v \rangle \\ &= \int f d\mu_{u, T_h^* v} \quad \forall f \in C(\sigma(A)) \end{aligned}$$

$$\Rightarrow h d\mu_{u,v} = d\mu_{u, T_h^* v} \cdot \int h g d\mu_{u,v}$$

$$\Rightarrow \forall g \in B(\sigma(A)) \quad \int g h d\mu_{u,v} = \int g d\mu_{u, T_h^* v}$$

$$\langle T_{hg} u, v \rangle = \langle T_g u, T_h^* v \rangle$$

$$\begin{aligned} \Rightarrow T_{hg} &= T_h T_g \\ \forall h, g \in B(\sigma(A)) \end{aligned}$$

$$\langle u, v \rangle = \langle T_g u, T_h^* v \rangle$$

$$f \in \omega(\mathcal{B}_1)$$

$$= \int g d\mu_{u, T_h^* v}$$

$$= \langle T_g u, T_h^* v \rangle$$

$$= \langle T_h T_g u \rangle$$

Defn $P: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$, by

$$P(E) = T_{X_E} \quad \forall E \in \mathcal{B}(\mathcal{H})$$

$$P(\emptyset) = T_{X_\emptyset} = 0$$

$$P(E \cap F) = T_{X_{E \cap F}} = T_{X_E \cdot X_F} = T_{X_E} T_{X_F} = P(E) P(F)$$

$$\{E_i\} \text{ pairwise disjoint } E = \bigcup_{i=1}^{\infty} E_i$$

$$X_{\bigcup_{i=1}^n E_i} \rightarrow X_E, \quad \forall u, v \in \mathcal{H}$$

$$\lim_{n \rightarrow \infty} \left\langle \sum_{i=1}^n P(E_i) u, v \right\rangle = \lim_{n \rightarrow \infty} \left\langle \int \sum_{i=1}^n X_{E_i} d\mu, u, v \right\rangle$$

$$= \lim_{n \rightarrow \infty} \int \sum_{i=1}^n X_{E_i} d\mu_{u,v} \rightarrow \int X_E d\mu_{u,v} \Rightarrow P(E) = \text{SOT-} \lim_{n \rightarrow \infty} \sum_{i=1}^n P(E_i)$$

Homework: Suppose $I = (B(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}))$

is a ^{III} unital $\ast$ -hom.

T.T.A.E

(1) $\exists P: \mathcal{B} \rightarrow \mathcal{L}(\mathcal{H})$ spectral measure s.t

$$I(f) = \int f dP \quad \forall f \in B(\mathcal{H})$$

$$(2) f_n \rightarrow f \text{ a.e.} \Rightarrow I(f_n) \rightarrow I(f) \text{ (SOT)}$$

unif. by bounded convergence



$$\pi(f) = T_f = \int f d\mu \quad \forall f \in G(\sigma(\mathcal{X}))$$

$$\forall x \in \mathcal{X} \quad \pi(T_x) = \int T_x d\mu$$
$$\varphi(x)$$

$$P(\sigma(\mathcal{X}))$$

$$\text{if } \mathcal{X} \text{ is unital, } \xrightarrow{\text{min-de}} \varphi(e) = \text{id}$$

$$\varphi(e) = \int T_e d\mu = \int 1 d\mu = \text{id} \quad \sigma(\mathcal{X}) = \sigma(\mathcal{X}) \cup \{0\}$$

$$\text{if } \mathcal{X} \text{ is not unital, } \tilde{\mathcal{X}} = \mathcal{X} \oplus \mathbb{C}e \quad \tilde{\varphi}(x + \lambda e) = \varphi(x) + \lambda \text{id}$$

$$P = B(\sigma(\tilde{\mathcal{X}}) \rightarrow \mathbb{C}e)_{\text{id}} = P(\sigma(\tilde{\mathcal{X}})) = P(\sigma(\mathcal{X})) + P(\{0\})$$

$$\forall x \in \mathcal{X} \quad \text{we know } \frac{t}{T} < T$$

$$(T_x \chi)_0 = 0 \Rightarrow \int T_x \chi d\mu = 0 \Rightarrow \int T_x \chi \cdot \chi_{\{0\}} d\mu = 0$$

$$\Rightarrow \int T_x \chi d\mu \int \chi_{\{0\}} d\mu = 0$$

$$\Rightarrow \varphi(x) \cdot P(\{0\}) = 0$$

$$\Rightarrow \forall u \in \mathcal{H} \quad \varphi(x) P(\{0\}) u = 0 \quad \forall x \in \mathcal{X}$$

$$\Rightarrow \varphi \text{ is non-de.} \Rightarrow P(\{0\}) u = 0 \quad \forall u \in \mathcal{H}$$

$$\Rightarrow P(\{0\}) = 0$$

The spectral theorem.

Theorem. Let $\mathcal{A}$ be a C^* -alg on Hilbert space $\mathcal{H}$.

$\varphi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ is a $*$ -rep of $\mathcal{A}$ on $\mathcal{H}$ if φ is non-degenerated.

then $\exists$ spectral measure $P: \mathcal{B}_{\sigma(\mathcal{A})} \rightarrow \mathcal{L}(\mathcal{H})$ s.t.

$$\varphi(x) = \int T_\lambda x dP \quad \forall x \in \mathcal{A}$$

T_λ - Gelfand transform (isometrically $*$ -iso)

Coroll. (Spectral Theorem I Th 1.44)

$\mathcal{A}$ is commutative C^* -subalg of $\mathcal{L}(\mathcal{H})$, $\varphi \in \mathcal{A}$

$\Rightarrow \forall T \in \mathcal{A} \quad T = \int \lambda dP \quad P$ - spectral measure $= \sigma(\mathcal{A}) \rightarrow \mathcal{L}(\mathcal{H})$

$$\varphi = \varphi: \mathcal{A} \hookrightarrow \mathcal{L}(\mathcal{H})$$

Theorem (The. 1.54): $\mathcal{A}$ - commutative Banach $*$ -alg. φ is a non-degenerated $*$ -rep of $\mathcal{A}$ on $\mathcal{H} \Rightarrow \exists$ spectral measure $P: \sigma(\mathcal{A}) \rightarrow \mathcal{L}(\mathcal{H})$ s.t.

$$\varphi(x) = \int T_\lambda x dP \quad (\forall x \in \mathcal{A})$$

non-degenerated: $\{B = \overline{\varphi(A)}^{\|\cdot\|} \mid B \text{ is a } C^* \text{-subalg of } \mathcal{L}(\mathcal{H})\}$

$$P = \sigma(B) \quad P_\alpha = \sigma(\mathcal{A})$$

$$\mathcal{A} \xrightarrow{\varphi} B \xleftarrow{\psi} L(\mathcal{A})$$

def $\widehat{\varphi}: \sigma(B) \rightarrow \sigma(\mathcal{A})$ by

$$\widehat{\varphi}(h) = h \circ \varphi \quad \forall h \in \sigma(B)$$

$$\forall a \in \mathcal{A}, \quad \widehat{a}(\widehat{\varphi}(h)) = \widehat{\varphi(a)}(h) \quad (1)$$

$\mathcal{A}$ is unital $\Rightarrow \varphi(e)$ is unit of $B \Rightarrow \widehat{\varphi}(e) = id$

$\widehat{\varphi}$ is homeomorphism from $\sigma(B)$ onto $\widehat{\varphi}(\sigma(B)) \subseteq \sigma(\mathcal{A})$

$$\exists P_0: B_{\sigma(B)} \rightarrow L(\mathcal{A}) \text{ s.t. } \forall T \in B, \quad T = \int_{\sigma(B)} T dP_0$$

$$\text{define } P: B_{\sigma(B)} \rightarrow L(\mathcal{A})$$

$$P(E) = P_0(\widehat{\varphi}^{-1}(E)) \quad \forall E \in B_{\sigma(\mathcal{A})} \quad (2)$$

$$\forall a \in \mathcal{A}$$

$$\varphi(a) = \int_{\sigma(B)} \widehat{\varphi(a)} dP_0 = \int_{\sigma(B)} \widehat{\varphi(a)}(h) dP_0(h)$$

$$\stackrel{(1)}{=} \int_{\sigma(B)} \widehat{a}(\widehat{\varphi}(h)) dP_0(h) \stackrel{(2)}{=} \int_{\sigma(\mathcal{A})} \widehat{a} dP$$

G -bimodular compact Lie group

$\widehat{G}$ -dual $\xi \in \widehat{G}$

$$\widehat{G} \cong \sigma(L^1(G)) \text{ by (4.1) } \xi(f) = \int_G x \cdot \xi \cdot f(x) dx$$

$$\pi: G \rightarrow L^1(G) \xleftrightarrow{1:1} \pi(f) = \int_G f(x) \pi(x) dx$$

$$\Rightarrow p: B_{\widehat{G}} \rightarrow L^1(G) \text{ s.t. } \pi(f) = \int_{\widehat{G}} \widehat{f}(\xi) dP_{\xi}, \forall f \in L^1(G)$$

$$\pi(x) = \text{SOT-}\lim_{u \rightarrow 1} \pi(L_x \psi_u) \text{ } (\psi_u \text{ is approximate identity of } L^1(G))$$

$$\pi(L_x \psi_u) = \int_{\widehat{G}} \widehat{L_x \psi_u}(\xi) dP_{\xi} \stackrel{(4.14)}{=} \int_{\widehat{G}} \widehat{\xi}(x) \widehat{\psi_u}(\xi) dP_{\xi} \quad (3)$$

$$\text{WOT-}\lim_{u \rightarrow 1} \pi(L_x \psi_u) = \int \widehat{\xi}(x) dP_{\xi}$$

$\widehat{\psi_u} \rightarrow 1$ uniformly on compact subset of $\widehat{G}$

as $u \rightarrow 1$
 $u, v \in \widehat{H} \Rightarrow P_{u,v}$ is a finite measure on $\widehat{G}$

$\exists K \subseteq \widehat{G}$ s.t. $|P_{u,v}(E)| / |G \setminus K| < \epsilon$

$\exists$ open set V s.t. $u \subseteq V, 1 \in u$

$$|\widehat{\psi_u}(\xi) - 1| < \frac{\epsilon}{P_{u,v}(K)} \quad \forall \xi \in K$$

$$\forall u \subseteq v, \pi(\widehat{\psi_u}) = \int \widehat{\xi}(x) dP_{\xi}$$

$$\leq \int_{\widehat{G}} |\widehat{\xi}(x)| |\widehat{\psi_u}(\xi) - 1| dP_{u,v}(\xi)$$

$$< \int_{\widehat{G} \setminus K} |\widehat{\xi}(x)| |\widehat{\psi_u}(\xi) - 1| dP_{u,v}(\xi)$$

$$+ \int_K |\widehat{\xi}(x)| |\widehat{\psi_u}(\xi) - 1| dP_{u,v}(\xi)$$

$$\leq 2P_{u,v}(G \setminus K) + \epsilon < 3\epsilon$$