

$(X, \|\cdot\|)$ Banach space

X^* dual Banach space.

$(x_n)_{n=1}^{\infty}$ sequence in X ,

$$\langle x_n \rangle = \text{span}\{x_n : n \in \mathbb{N}\}$$

$$[x_n] = \overline{\text{span}\{x_n : n \in \mathbb{N}\}}.$$

Chap 1 Bases.

Let $(X, \|\cdot\|)$ be a Banach space,
over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Def A seq $(e_n)_{n=1}^{\infty}$ in X is called a
basis for X if $\forall x \in X \exists \text{ unique } (a_n)_{n=1}^{\infty}$
of scalars s.t.

$$x = \sum_{n=1}^{\infty} a_n e_n$$

$$\text{i.e. } \left\| \sum_{n=1}^N a_n e_n - x \right\| \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Def A seq $(e_n^*)_{n=1}^{\infty}$ in X^* is called a
Schauder basis for X if $\exists$ seq $(e_n^*)_{n=1}^{\infty}$
in X^* s.t. $\|e_k^*(e_j)\| = \begin{cases} 1 & k=j \\ 0 & k \neq j \end{cases}$
iii) $\forall x \in X, x = \sum_{k=1}^{\infty} e_k^*(x) e_k$

e_n^* are called
associated w

Note $(e_n)_{n=1}^{\infty}$

$X \neq \text{span}$

Note If X is

$$X = \left\{ \sum_{k=1}^N a_k e_k \right\}$$

e_n^* are called biorthogonal functionals associated with $(e_n)_{n=1}^\infty$.

Note: $(e_n)_n$ not a Hamel basis.

$$X \neq \text{span}\{e_n : n \in \mathbb{N}\}$$

Note: If X has a basis, then X is separable:

$$X = \left\{ \sum_{j=1}^N a_j e_j : (a_j)_{j=1}^\infty \text{ in } \mathbb{Q}, N \in \mathbb{N} \right\} \text{ or } \mathbb{Q} \cup \{0\}.$$

Theorem. A seq $(e_n)_{n=1}^\infty$ in X is a Schauder basis for X if and only if it is a basis for X .

Pf $\Rightarrow$). Let $(e_n)_n$ basis. Define

$$e_k^* \left(\sum_{n=1}^\infty a_n e_n \right) = a_k, \quad k \in \mathbb{N}, \quad x = \sum_{n=1}^\infty a_n e_n \in X.$$

$$S_n(x) = \sum_{k=1}^n e_k^*(x) e_k = \sum_{k=1}^n a_k e_k.$$

Are well defined and linear, and

$$e_k^*(e_j) = \begin{cases} 1 & k=j \\ 0 & k \neq j \end{cases}.$$

By def $\forall x \in X \quad S_n x \rightarrow x$ (norm).

$$\text{Def. } \|x\| = \sup_{n \geq 1} \|S_n x\| < \infty \\ \geq \|x\|$$

Claim $(X, \|\cdot\|)$ also complete.

Pf. Let $(x_n)_n$ Cauchy in $(X, \|\cdot\|)$

Then also $\|\cdot\|$ -Cauchy, so $\exists x \in X$
st $\|x_n - x\| \rightarrow 0$.

For $k \in \mathbb{N}$: $(S_k x_n)_{n=1}^\infty$ is $\|\cdot\|$ -Cauchy
so $\exists y_k \in [e_1, \dots, e_k]$ st.
 $\|S_k x_n - y_k\| \rightarrow 0$ as $n \rightarrow \infty$.

$y_k, S_k x_n \in [e_1, \dots, e_k]$ and k -dim, so

$e_j^\#$ is ct on $[e_1, \dots, e_k]$ hence

$$e_j^\#(S_k x_n) \rightarrow e_j^\#(y_k).$$

For $j \in \{1, \dots, k\}$ $e_j^\#(S_k x_n) = e_j^\#(x_n)$.

so $e_j^\#(S_k x_n)$ indep of k , so

$e_j^\#(y_k)$ indep of k

!!
a.j.

Claim. Then $y_k = \sum_{j=1}^k a_j e_j$
 $\sum_{j=1}^\infty a_j e_j = x$ in $(X, \|\cdot\|)$, i.e.
 $\|x_n - x\| \rightarrow 0$.

Pf

given ϵ
choose n
large

Pf. $\|y_k - x\| \leq \|y_k - S_k x_m\| + \|S_k x_m - S_k x_n\|$
 $+ \|S_k x_n - x_n\| + \|x_n - x\|$
 given $\varepsilon > 0$
 choose n, m
 large
 $\leq \varepsilon + \|x_m - x_n\|$
 $+ \|S_k x_n - x_n\| + \varepsilon$
 so l.l. $\|y_k - x\| = 0, \square$

So $\sum_{j=1}^{\infty} a_j e_j = x$, so $S_k x = y_k$.

Now, let $\varepsilon > 0$. Take n_2 st. $\forall n, m \geq n_2$
 $\|x_n - x_m\| < \varepsilon$

Then

$$\|x_n - x\| = \sup_{k \geq 1} \|S_k(x_n - x)\|$$

and $\forall k \geq 1, n \geq n_2$:

$$\|S_k(x_n - x)\| \leq \|S_k x_n - S_k x_m\| + \|S_k x_m - S_k x\|$$

$$\leq \|x_n - x_m\| + \|S_k x_m - \underbrace{S_k x}_{y_k}\|$$

$$\leq \|x_n - x_m\| + \varepsilon + \|S_k x_m - y_k\|$$

$\|x_n - x\| < \varepsilon$
 so $x_n \rightarrow x$ in $(X, ||| |||)$.

So $(X, ||| |||)$ is also complete.

Now identity $i: (X, ||| |||) \rightarrow (X, || ||)$ is bi-ct since $||| ||| \geq || ||$.
 both Banach. so i^{-1} is also ct. Hence $||| |||$ and $|| ||$ equivalent.

$$\text{so } \exists K: \|S_n x\| \leq K \|x\|$$

$$\text{i.e. } \|S_n x\| \leq K \|x\| \text{ all } x$$

so every S_n is bdd.

$$e_k^* = S_k - S_{k-1} \text{ is also bdd.}$$

□

P. 1.1.4. Let $(e_n)_{n=1}^\infty$ Schauder basis for X .

$$S_n(x) = \sum_{k=1}^n e_k^*(x) e_k, \quad x \in X, n \in \mathbb{N}$$

partial sum projections

$$S_n^2 x = S_n x \quad (\text{all } x \in X)$$

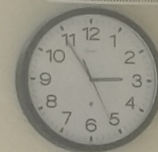
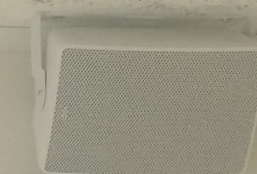
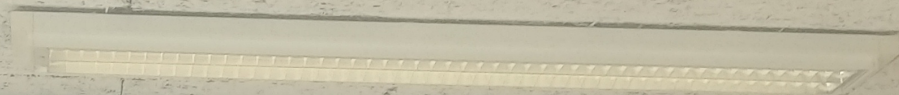
Then $\sup_k \|S_k\| < \infty$.

Pf. Each S_n is bdd lin and $\forall x \in X$

$$S_n x \rightarrow x \text{ so } \sup_{n \geq 1} \|S_n x\| < \infty$$

By Uniform Boundedness Principle

$$\sup_{n \geq 1} \|S_n\| < \infty$$



Def $K = \sup_{n \geq 1} \|S_n\|$ is called the basis constant
of $(e_n)_{n=1}^{\infty}$.

If $K=1$: $(e_n)_{n=1}^{\infty}$ is called monotone.

Prop $\|x\| = \sup_{n \geq 1} \|S_n x\|$.

then $\forall x \in X$, $\|x\| \leq \|x\| \leq K \|x\|$
so norms are equivalent

and $(e_n)_{n=1}^{\infty}$ is monotone for $(X, \|\cdot\|)$.

P1.1.7 If $S_n: X \rightarrow X$ bdd lin projections st

(i) $\dim S_n(X) = n$ (all n)

(ii) $\forall m, n$: $S_n S_m = S_m S_n = S_{\min\{n, m\}}$.

(iii) $\forall x \in X$, $S_n x \rightarrow x$ as $n \rightarrow \infty$

then any sequence $(e_k)_{k=1}^{\infty}$ in X st. $e_k \neq 0$
 $e_1 \in S_1(X)$, $e_k \in S_k(X)$, $S_{k-1} e_k = 0$ all $k \geq 2$,
is a basis for X with partial sum projections S_n .

Pf $E_n = S_n(X)$ n -dim subspace of X
 $E_1 \subseteq E_2 \subseteq \dots$ since

$$E_n = S_n(X) = S_{n+1} S_n(X) \\ \subseteq S_{n+1}(X) = E_{n+1}.$$

S_n projection, so $\ker(S_n) = (I - S_n)(X)$.

$$E_{n+1} \cap \ker(S_n) = \text{span}\{e_{n+1}\}.$$

So $\exists e_n^*: X \rightarrow \mathbb{R}$ st. $\forall x \in X$ in E_n ,
 $p_n^*(x) e_{n+1} = S_{n+1}(I - S_n)x$.

Then $e_n^* \in X^*$, $e_n^*(e_m) = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$

$$\text{and } \forall x \in X: S_n(x) = \sum_{k=1}^n \frac{(S_k(x) - S_{k-1}(x))}{S_k - S_{k-1}} \\ = \sum_{k=1}^n e_k^*(x) \rightarrow x$$

Hence $(e_n)_n$ is a Schauder basis with biorth. fund. (e_n^*) .

Def $K = \sup_{n \geq 1}$

of $(e_n)_{n=1}^\infty$

If $K=1$: (c)

Prop. $\|x\| =$

then $\forall x \in X$

and $(e_n)_n$

Does every Banach space have a basis?

1973: Per Enflo: no!

Def. $(e_n)_n$ is called a basic sequence if $(e_n)_n$ is a basis for $[e_n]$.

P. 1.15 (Grunblum's criterion).
 $(e_k)_{k=1}^\infty$ in X with $e_k \neq 0$ (all k)

is a basic sequence if and only if
 $\exists K \geq 0 \forall (a_k)$ seq. of scalars and
 $m, n \in \mathbb{N}, m \leq n$.

$$\left\| \sum_{k=1}^m a_k e_k \right\| \leq K \left\| \sum_{k=1}^n a_k e_k \right\|$$

$$\text{Pf. } \Rightarrow \left\| \sum_{k=1}^m a_k e_k \right\| = \left\| S_m \left(\sum_{k=1}^n a_k e_k \right) \right\| \\ \leq \underbrace{\sup_{k \geq 1} \|S_k\|}_{K \text{ on } [e_n]} \left\| \sum_{k=1}^n a_k e_k \right\|$$

$$\Leftrightarrow E = \text{span} \{ e_k \} \\ \text{by } S_n \left(\sum_{j=1}^n a_j e_j \right)$$

$$\Leftarrow) E = \text{span}\{e_k : k \in \mathbb{N}\}, \quad S_m: E \rightarrow [e_k]_{k=1}^m$$

$$S_m\left(\sum_{j=1}^n a_j e_j\right) = \sum_{k=1}^{\min\{n, m\}} a_k e_k.$$

By assumption S_m is bdd lin with $\|S_m\| \leq K$

Extends to $S_m: [e_n] \rightarrow [e_k]_{k=1}^n$, $\|S_m\| \leq K$.

For $x \in E$ we have

$$S_n S_n x = S_n S_n(x) = S_{\min\{n, \infty\}}(x).$$

So also also on $[e_n]$.

Also $S_n x \rightarrow x$ for all $x \in [x_n]$.

Indeed,

$$D := \{x \in [e_n] : S_n(x) \rightarrow x\} \supseteq E$$

D is closed (by a seq argument)

$E \parallel$ dense $[x_n]$

So $D = [e_n]$.

By P1.1.7, $(e_n)_n$ is a basis for $[x_n]$ with (S_n) as partial sum projections. $\square$

§12 Example

$T = \{e^{i\theta} : \theta \in [0, 2\pi]\}$ unit circle in $\mathbb{C}$.

$C(T)$ complex valued cont'n functions with $\|\cdot\|_\infty$.

$$e_n(\theta) = e^{in\theta}, n \in \mathbb{Z}$$

Is $e_0, e_1, e_{-1}, e_2, e_{-2}, \dots$ a basis for $C(T)$?

No.

Fourier coefficients of $f \in C(T)$:

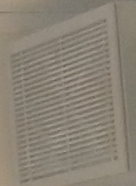
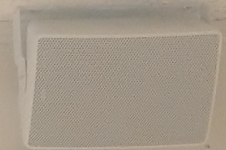
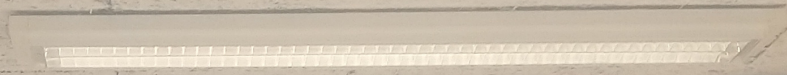
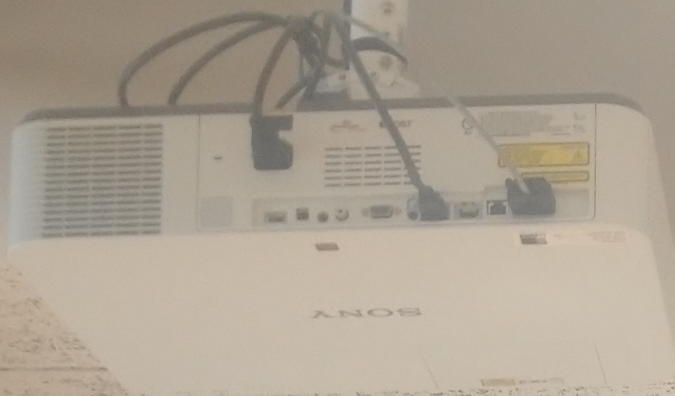
$$\hat{f}(n) = \int_{-\pi}^{\pi} f(t) e^{-int} \frac{dt}{2\pi}, n \in \mathbb{Z}$$

$$e_n^*(f) = \hat{f}(n) \text{ biorthog to } (e_n)_n.$$

$$T_n(f) = \sum_{k=-n}^n \hat{f}(k) e_k$$

Cesàro mean

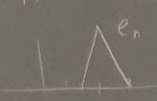
$$A_n = \frac{1}{n} (-T_1 + \dots + T_n),$$



$f \in C[0,1]$
 $f(t) = \frac{1}{2t}$ $n \in \mathbb{Z}$
 (n) basis for $(e_n)_n$.
 $f_k(t) = e_k$
 $T_k = T_n$

Thm
 • $\forall f \in C([t]) \quad \|A_n f - f\|_\infty \rightarrow 0, n \rightarrow \infty$
 so in part $\text{span} \{e_n : n \in \mathbb{N}\} = X$.
 • $\liminf_{n \rightarrow \infty} \|T_n\| = \infty$ so $(T_n)_n$ are
 not unif bdd, so partial sum
 proj of $e_0, e_1, \dots$ are not unif bdd
 Hence $e_0, e_1, e_2, \dots$ not a basis.

Thm. $C[0,1]$ and $C([t])$ do have monotone bases.
 "Pf": take $q_1 = 0, q_2 = 1, (q_n)_n$ distinct dense in $[0,1]$
 Define $e_1(t) = 1, t \in [0,1]$

$$e_n(q_k) = \begin{cases} 1 & n=k \\ 0 & k=0, \dots, n-1 \end{cases}$$
 affine in between

 then $(e_n)_n$ is a monotone basis for $C[0,1]$
 Megginson, Introduction to Banach Space Theory

§1.3 Equivalence of bases and basic sequences

Def X, Y Banach spaces

$(x_n)_n$ basic seq in X

$(y_n)_n$ basic seq in Y .

(x_n) and (y_n) are called equivalent

if $\forall$ seq of scalars $(a_j)_{j=1}^\infty$ behave

$$\sum_{j=1}^{\infty} a_j x_j \text{ converges} \iff \sum_{j=1}^{\infty} a_j y_j \text{ converges}$$

Thm 1.3.2

(x_n) and (y_n) are equivalent

$\iff \exists$ isomorphism $T: [x_n] \rightarrow [y_n]$

s.t. $T x_n = y_n$.

T is lin bij., T bdd

and T^{-1} bdd.

Cor 1.3.3

(x_n) and (y_n) are equivalent

$$\iff \exists C \forall \text{ scalar } (a_j)_{j=1}^{\infty} \quad \frac{1}{C} \left\| \sum_{j=1}^{\infty} a_j y_j \right\| \leq \left\| \sum_{j=1}^{\infty} a_j x_j \right\| \leq C \left\| \sum_{j=1}^{\infty} a_j y_j \right\|$$

Def (e_n) basis

$p_0 = 0$, (a_n)

s.t.

$u_n =$

and $u_n \neq$

is called a

Ex in ℓ_p is

$(1, 2, 7, 0, \dots)$

Def. (e_n) basis for X , (p_n) strictly increasing in $\mathbb{N}$
 $p_0 = 0$, $(a_n)_{n=1}^\infty$ scalars. A seq $(u_n)_{n=1}^\infty$ in X

$$u_n = \sum_{j=p_{n-1}+1}^{p_n} a_j e_j$$

and $u_n \neq 0$.

It is called a block basic sequence of

Ex. in ℓ_p : $(11000\dots)$ (e_n)
 $(001100\dots)$
 $(000011\dots)$
 $(12700\dots), (000\dots 2500\dots), \dots$

1.7.5. If (e_n) basis for X , with basic constant K .
 $(u_k)_k$ is a block basic seq. of $(e_n)_n$, then $(u_k)_k$
 is a basic sequence with basic constant $\leq K$.

Def. Notation as in def. For scalars $(b_k)_k$.

$$\sum_{k=1}^m b_k u_k = \sum_{k=1}^m \sum_{j=p_{k-1}+1}^{p_k} a_j e_j = \sum_{j=1}^{p_m} c_j e_j$$

$$\left\| \sum_{k=1}^m b_k u_k \right\| = \left\| \sum_{j=1}^{p_m} c_j e_j \right\| \leq K \left\| \sum_{j=1}^{p_m} c_j e_j \right\| \quad c_j = a_j b_k \text{ for } j=p_{k-1}+1, \dots, p_k$$

$$= K \left\| \sum_{k=1}^m b_k u_k \right\|$$

Grünblum criterion: $(u_k)_k$ basic sequence, const $\leq K$. $\square$

§ 1.5 Constructing basic sequences.
 X Banach space (not finite dim)

Lemma 1.5.1

$S \subseteq X^*$, $0 \in S^{wk}$, $0 \notin S^{||}$,
 E finite dim subspace of X^* . Then
 $\forall \varepsilon > 0 \exists x^* \in S$ st.

$$\forall x^* \in E, \lambda \in K: \|e^* + \lambda x^*\| \geq (1 - \varepsilon) \|e^*\|$$

• weak* topology on X^* is the weakest topology for which all maps

$$x^* \mapsto x^*(x), \quad X^* \rightarrow K \quad (x \in V)$$

are continuous

Thm 1.5.2 $S \subseteq X^*$ st $0 \in S^{wk}$, $0 \notin S^{||}$

$\forall \varepsilon > 0 \exists$ basic seq in S with norm $< 1 + \varepsilon$

P.P. Let $\varepsilon > 0$. Fix $\varepsilon_n > 0$ with

$$\sum_{n=1}^{\infty} \varepsilon_n < \infty \text{ and } \prod_{n=1}^{\infty} (1 - \varepsilon_n) > \frac{1}{1 + \varepsilon}$$

Take $x_1^* \in S \setminus \{0\}$, $E_1 = [x_1^*]$

[1 s.t.] $\exists x_2^* \in S$ st $\forall e^* \in E_1, \lambda \in K$.

$$\|e^* + \lambda x_2^*\| \geq (1 - \varepsilon_1) \|e^*\|$$

$E_2 = [x_1^*, x_2^*]$, $\exists x_3^* \in S$ st $\forall e^* \in E_2$

$$\|e^* + \lambda x_3^*\| \geq (1 - \varepsilon_2) \|e^*\| \text{ etc.}$$

Then $x_n^* \in S$ st

$$\left\| \sum_{h=1}^n \lambda_h x_h^* \right\| = \left\| \sum_{h=1}^n \lambda_h \right\|$$

For $n \geq n_1$:

$$\left\| \sum_{h=1}^n \lambda_h x_h^* \right\| \geq (1 - \varepsilon_1)$$

$$\geq \prod_{j=1}^{\infty} (1 - \varepsilon_j)$$

So $\forall n \geq n_1$.

$$\left\| \sum_{h=1}^n \lambda_h x_h^* \right\| \geq (1 - \varepsilon)$$

Then $(x_n^*) \in S$ s.t. $\forall (x_n)$ in K

$$\left\| \sum_{i=1}^{n+1} \lambda_i x_i^* \right\| = \left\| \sum_{i=1}^n \lambda_i x_i^* + \lambda_{n+1} x_{n+1}^* \right\|$$

$$\geq (1 - \varepsilon_n) \left\| \sum_{i=1}^n \lambda_i x_i^* \right\|$$

For $n \geq n_0$

$$\left\| \sum_{i=1}^n \lambda_i x_i^* \right\| \geq (1 - \varepsilon_{n-1}) \dots (1 - \varepsilon_n) \left\| \sum_{i=1}^n \lambda_i x_i^* \right\|$$

$$\geq \prod_{j=1}^{\infty} (1 - \varepsilon_j) \left\| \sum_{i=1}^n \lambda_i x_i^* \right\| \geq \frac{1}{1+\varepsilon} \left\| \sum_{i=1}^n \lambda_i x_i^* \right\|$$

So $\forall n \geq n_0$

$$\left\| \sum_{i=1}^n \lambda_i x_i^* \right\| \leq (1+\varepsilon) \left\| \sum_{i=1}^n \lambda_i x_i^* \right\|$$

Grinblum: $(x_n^*)_n$ is a basic seq for X^* and basis const $\leq 1+\varepsilon$.

Cor. Every inf dim Banach space has a basic sequence.

Even $\forall \varepsilon > 0$ a basic seq with basis const $\leq 1+\varepsilon$.

Pf. If X is inf dim, then X^{**} is inf. dim. so

Norm topo on X^{**} is strictly stronger than the weak* topo.

Consider $X \subseteq (X^*)^*$.

Norm topo of X is strictly stronger than weak* topo of $(X^*)^*$.

So $\exists$ net $(x_\alpha)_{\alpha \in A}$ in X s.t. $x_\alpha \rightarrow 0$ w.r.t. weak* topo of $(X^*)^*$.

Take $S = \{x_\alpha : \alpha \in A\} \subseteq X \subseteq (X^*)^*$ but not $\|x_\alpha\| \rightarrow 0$.
 s.t. $0 \in \overline{S}^{w*}$, of $\overline{S}^{w*}$

By T 1.5.2: $\exists$ seq $(x_n^*)_{n \in \mathbb{N}}$ in S
which is basic seq for $(X^*)^*$.

Hence $x_n^* \in S \subseteq X$.

and (x_n^*) basic seq in X

(as norms of $(X^*)^*$ and of X
coincide on X), $\square$

Thm 1.5.2. $S \subseteq X^*$ s.t. $0 \in \overline{S}^{w*}, 0 \notin S^{**}$

$\forall \varepsilon > 0 \exists$ basic seq in S with norm $< 1 + \varepsilon$.

Thm. 1.5.6 (Kadets and Pelczyński, 1965)

$S \subseteq X$ bdd. Equivalent:

(a) S does not contain a basic seq

(b) $\overline{S}^w$ is w -compact and $0 \notin \overline{S}^w$

Weak topo on X is the weakest topo on X s.t.
all maps $x \mapsto x^*(x)$ ($x^* \in X^*$) are
contin.

L1.61 If (x_n) basic seq and x a weak cluster point (limit pt or accumulation pt of $\{x_n : n \in \mathbb{N}\}$, then $x = 0$
 (i.e. cluster pt of A : $\forall \text{neigh}^U \text{ of } x$
 $\exists a \in A \cap U \setminus \{x\}$)

T1.63 (Eberlein-Smulian) $A \subseteq X$ Equiv:

- (a) A is (rel) weakly compact
- (b) A is (rel) weakly seq compact
- (c) A is (rel) weakly relatively compact

- (a): every open cover of A has a finite subcover.
- (b): every seq in A has a convergent subseq with limit in A .
- (c) every countable open cover of A has a finite subcover.