

$(e_n)_{n=1}^\infty$ monotone Schauder Basis
 notation $\tilde{e}_n = (\tilde{e}_i(n))_{i=1}^\infty$

Lemma 2.11. $(y_n)_{n=1}^\infty$ normalized block
 m.s.s. sequence in c_0 or ℓ_p
 $(y_n)_{n=1}^\infty$ isometrically equiv to
 $(e_n)_{n=1}^\infty$ & $[y_n]$ is range of
 contr. proj.

Remark 2.12 $0 < a \leq \|y_n\| \leq b < \infty$
 $\forall k \in \mathbb{N}$ (seminormalized)
 $\Rightarrow (y_n)_{n=1}^\infty$ equiv to $(e_n)_{n=1}^\infty$ & $[y_n]$ compl
 to a contr. proj.

Prop 2.13: $(x_n)_{n=1}^\infty$ norm seq in ℓ_p (resp c_0)
 and $\sum_{n=1}^\infty \|x_n\|_p < \infty$ (resp $\sum_{n=1}^\infty \|x_n\|_\infty < \infty$)
 Now $\exists (x_{n_k})_{k=1}^\infty$ that is basic seq, equiv to
 $(e_n)_{n=1}^\infty$ and s.t. $[x_{n_k}]_{k=1}^\infty$ complemented in ℓ_p
 (resp c_0)

Proof: proof of 1.3.10 yields subseq $(x_{n_k})_{k=1}^\infty$
 & block basic seq $(y_k)_{k=1}^\infty$ of $(e_n)_{n=1}^\infty$ s.t.
 $(x_{n_k})_{k=1}^\infty$ is basic, equiv to $(y_k)_{k=1}^\infty$ and $[x_{n_k}]_{k=1}^\infty$
 is complemented whenever $[y_k]_{k=1}^\infty$ is
 with corol 1.3.3 $(y_k)_{k=1}^\infty$ is seminormalized,
 so Remark 2.12 gives $\square$.

Theorem 2.14 (Pitt): $1 \leq p < \infty$,
 $X \subseteq \ell_p$ closed subspace,
 $T: X \rightarrow \ell_p$ bounded operator
 Then T is compact

Corollary 2.16: the spaces of
 the set $\{\ell_p\} \cup \{\ell_\infty\}$ are
 mutually non isomorphic if X is of
 dim subspace of any of those spaces,
 then X not isom to a subspace of any
 other

Definition: X is

Definition X, Y inf dim Banach spaces. X, Y are totally incomparable if they have no inf dim subspaces in common (up to isom)

Definition: $T: X \rightarrow Y$ (X, Y Banach), T is strictly singular if $\nexists$ an $E \subseteq X$ inf. dim st $T|_E$ is an isom onto its range

Thm 2.19. If $p \neq r$, then every bounded operator $T: \ell_p \rightarrow \ell_r$ is strictly singular

Prop 2.21: $Y \subseteq \ell_p$ (resp c_0) inf dim & closed then Y contains a closed subspace Z st Z is isom to ℓ_p (resp c_0) and compl in ℓ_p (resp c_0)

Proof: Y inf dim so $\forall n \in \mathbb{N} \exists x_n \in Y$ with $\|x_n\| = 1$ st $e_k(x_n) = 0 \forall k \leq n$
Now by Prop 2.13: $(x_n)_n$ has a subseq $(x_{n_k})_{k=1}^\infty$ that is basic, equiv to e_n , and s.t. $Z = [x_{n_k}]$ is compl.
 $\Rightarrow$ Thm 13.2: Z isom to ℓ_p $\square$

Prop 2.22 Y inf dim closed subsp of c_0 or ℓ_1 , Y is not reflexive

$1 \leq p \leq \infty$, $\ell_p(X) = (X \oplus X \oplus \dots)_p$
consists of all X -valued seq $x = (x(n))_n$ s.t. $(\|x(n)\|_X)_n \in \ell_p$ with norm:
* $1 \leq p < \infty$: $\|x\| = \left(\sum_{n=1}^\infty \|x(n)\|_X^p \right)^{1/p}$
* $p = \infty$: $\|x\| = \sup_n \|x(n)\|_X$

$c_0(X)$ consists of all X -valued seq $x = (x(n))_n$ s.t. $\lim_n \|x(n)\|_X = 0$. norm given by $\|x\| = \max_{1 \leq n \leq \infty} \|x(n)\|_X$

Thm 223. X, Y be Banach.
 X isom to a compl subspace of Y ,
 Y isom to a compl subspace of X
 also one of the following:

- (i) $X \cong X^2 = X \oplus X$ & $Y \cong Y^2$
- (ii) $X \cong C_0(X)$ or $X \cong C_b(X)$, i.e. C_0
 then X isom to Y

Thm 224. Y compl inf dim subspace of
 ℓ_p (i.e. $p < \infty$) resp c_0 . Then Y isom
 to ℓ_p (resp c_0)

Proof. prop 221 gives inf dim subspace Z
 of Y s.t. Z is isom to ℓ_p (or c_0) & Z is
 comp in ℓ_p (resp c_0). Then Z also compl in Y
 and thus ℓ_p (resp c_0) is isomorphic to a compl
 subspace in Y . Since $\ell_p(\ell_p) = \ell_p$ (resp $c_0(c_0) = c_0$)
 (ii) of thm 223 holds and thus $\square$

Thm 2.3.1: X separable Banach space, then $\exists$ a cont operator $Q: \ell_1 \rightarrow X$ from ℓ_1 onto X .

Corollary 2.3.2: X sep Banach, then X is isometrically isom to a quotient of ℓ_1 .

Proof: Q as in thm 2.3.1, then it follows $\ell_1 / \ker(Q)$ is isomet. isom to X $\square$

Corollary 2.3.3: ℓ_1 has an uncount closed

Subspace
Proof: Let X be an infinite dimensional separable Banach space that is not isom to ℓ_1 . Thm 2.3.1 gives a Q that is not isom to ℓ_1 and $\ker(Q)$ is closed subsp of ℓ_1 . If $\ker(Q)$ is complemented in ℓ_1 , then

$\ell_1 = \ker(Q) \oplus M$, $M \subseteq \ell_1$ closed, but then $X = \ell_1 / \ker(Q) \cong M$. Now by thm 2.2.4 we have that X is isom to ℓ_1 .

Def: X Banach X has the Schur property if a seq (x_n) in X converges to 0 weakly iff (x_n) converges to 0 in norm.

thm 2.3.6: ℓ_1 has the Schur property.

Proof: (x_n) weakly null seq in ℓ_1 that does not converge to 0 in norm. prop 2.13: gives subseq of (x_n) (x_{n_k}) basic & equiv to $(e_n) \rightarrow \downarrow$

thm 2.3.7

thm 2.3.7 X Banach with Schur prop
Then subset W of X is weakly compact iff
 W is norm compact

Coroll 2.3.8 X reflexive Banach with Schur
Prop, then $\dim(X) < \infty$.
Proof. unit ball of X norm compact by
2.3.7, and thus $\dim(X) < \infty$

def X Banach, seq (x_n) is called
Weakly Cauchy if $\lim_{n \rightarrow \infty} \tilde{x}(x_n)$ exists
 $\forall \tilde{x} \in X^*$

def Banach space X is weakly
sequentially complete (WSC) if every
weakly Cauchy sequence in X converges
weakly

Prop 2.3.12 Every Banach space with
Schur prop is WSC (in part 1.)

Def. $(x_n)_{n=1}^{\infty}$ in Banach space X , a series
 $\sum_{n=1}^{\infty} x_n$ in X is unconditionally
convergent if $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges
for every permutation π

Lemma 2.12 Given $\sum x_n$ in X
 f.a.c.

- (a) $\sum x_n$ uncond conv
- (b) $\sum x_{n_k}$ converges $\forall$ increasing subseq $(n_k)_{k=1}^\infty$
- (c) $\sum \epsilon_n x_n$ converges for every choice of signs
- (d) $\forall \epsilon > 0 \exists N \in \mathbb{N}, \leq 1$

$F \subseteq \{n+1, n+2, \dots\}$ finite,
 where $\|\sum_{k \in F} x_k\| < \epsilon$

def. $\sum x_n$ in X is weakly unconditionally Cauchy (WUC) if $\sum_{n=1}^\infty |x^*(x_n)| < \infty$
 $\forall x^* \in X^*$

Prop 2.14 Suppose $\sum x_n$ converges unconditionally to some x , then:

- (i) $\sum x_{\pi(n)} = x \quad \forall \pi$ permutation
- (ii) $\sum x_n$ converges unconditionally $\forall A \subseteq \mathbb{N}$ infinite subset
- (iii) $\sum x_n$ WUC

Ex 2.15 $\sum e_n$ in c_0 , this WUC, however fails to converge weakly

Prop 2.17 $\sum x_n$ in Banach X , then $\sum x_n$ is WUC iff $\exists T: c_0 \rightarrow X$ bounded s.t. $T e_n = x_n$

Prop 2.18 $\sum x_n$ WUC in X , then $\sum x_n$ unconditionally convergent in X iff T of 2.17 is compact

c_0 ,
series fails to
converge

x_n in Banach X .

WUC $\Leftrightarrow \exists$
s.t. $T e_n = x_n$

WUC in X ,
absolutely convergent

$\Rightarrow T$ is compact

Prop 2.0.9: $\sum x_n$ in X is uncond
conv $\Leftrightarrow \sum t_n x_n$ converges uncond

$\forall (t_n) \in \ell_\infty$

Thm 2.4.10: $T: c_0 \rightarrow X$ bounded,

$\{e_n\}$ a.e.

(i) T is compact

(ii) T is weakly compact

(iii) T is strictly singular

def We say that Banach space X
contains a copy of Banach space Y $\Leftrightarrow$
 X contains a closed subspace E that
is isom to Y

Thm 2.4.11: In order that every WUC
series in X to be uncond conv,
it is necessary & sufficient that X contains
no copy of c_0

Proof: X has no copy of c_0 , $\sum x_n$ is a WUC series
 $\Rightarrow$ 2.4.7 gives $T: c_0 \rightarrow X$ s.t. $T(e_n) = x_n$.
 T must be strictly singular. Since every inf
dim subspace of c_0 contains a copy of c_0 ,
 $\Rightarrow T$ is compact by 2.4.10,
so 2.4.8 gives $\sum x_n$ converges
unconditionally. converse by
example 2.4.5 $\square$

Thm 2.11.11 (Orlicz-Pettis)
 $\sum x_n$ series in X st every
 subseries $\sum x_{n_k}$ converges
 weakly, then $\sum x_n$ converges
 unconditionally in norm

Corollary 2.11.15: X Banach,
 weakly sequentially complete,
 then every WUC series is
 uncondit conv.

Proof: If $\sum x_n$ is WUC, then $\sum x^*(x_n)$ is
 abs convergent $\forall x^* \in X^*$, which is equiv
 to saying that $\sum x^*(x_{n_k})$ converges for
 each subseries $\sum x_{n_k}$ $\forall x^* \in X^*$. Hence,

$\sum_{k=1}^{\infty} x_{n_k}$ is weakly Cauchy and thus
 weakly convergent by hypothesis. Thus

$\sum x_n$ converges uncond in norm by
 thm 2.11.11

Def Banach space Y is called
injective if $\forall X$ Banach, $E \subseteq X$
 closed subspace, $T: E \rightarrow Y$ bounded

$\exists \tilde{T}: X \rightarrow Y$ extension of T
 If additionally $\tilde{T}$ can be chosen
 ≥ 1 $\|\tilde{T}\| = \|T\|$ then Y is
 called isometrically injective

Prop 25.2. ℓ_∞ is isometrically injective
 also
 If X has subspace E that is isom to ℓ_∞ ,
 then E is complemented in X .

def. $A \subseteq \mathbb{N}$, define

$$\ell_\infty(A) = \left\{ \xi = (\xi(k))_{k=1}^\infty \in \ell_\infty : \xi(k) = 0 \ \forall k \notin A \right\}$$

Thm 25.4. $T: \ell_\infty \rightarrow \ell_\infty$ bounded
 and st. $T(\xi) = 0 \ \forall \xi \in c_0$, then
 $\exists A \subseteq \mathbb{N}$, infinite and st $T(\xi) = 0$
 $\forall \xi \in \ell_\infty(A)$.

Thm 25.5 There is no bounded
 Projection from ℓ_∞ onto c_0 .

Proof If P is such a projection, apply Thm 25.4 to
 $T = I - P$, then we have $A \subseteq \mathbb{N}$ infinite st
 $P(\xi) = \xi \ \forall \xi \in \ell_\infty(A)$
 because $\ell_\infty(A) \not\subset c_0$, Thus
 P is not a projection on c_0

Corollary 25.6. c_0 is not isomorphic
 to a dual space

Thm 257: Every separable Banach space embeds isometrically in ℓ_∞ .

Thm 258: X separable Banach space, $E \subseteq X$ closed subspace, $T: E \rightarrow C_0$ a bounded operator. Then $\exists$ an operator $\tilde{T}: X \rightarrow C_0$ s.t. $\tilde{T}|_E = T$ and $\|\tilde{T}\| \leq 2\|T\|$.

Corollary 25.9: $E \subseteq X$ closed subspace, X separable Banach. If E isom to C_0 , then $\exists$ projection P from X onto E .

Proof: Suppose $T: E \rightarrow C_0$ is an isom. and thm 258 gives $\tilde{T}: X \rightarrow C_0$. Then

$P = T^{-1} \circ \tilde{T}$, P is projection from X onto E .