

Def 3.1.1. A basis $(u_n)_{n \geq 1}$ of a Banach space X is called unconditional if $\forall x \in X$

the sum $\sum_{n \geq 1} u_n^*(x) u_n$ converges unconditionally

Lemma 2.4.2 TFAE

- 1) $\sum x_n$ converges unconditionally
- 2) $\sum \varepsilon_n x_n$ converges for any $(\varepsilon_n)_{n \geq 1} \in \{-1, 1\}$
- 3) $\sum x_{n_k}$ converges $\forall (n_k)_{k \geq 1}$ increasing

Example 3.1.2 ℓ^p , $1 \leq p < \infty$, c_0

$(e_n)_{n \geq 1}$ - standard basis is unconditional (simple)

Consider c_0 with a basis

$$f_n = \sum_{k=1}^n e_k$$

$$c_0 \ni x = \sum a_n f_n \iff \sum_{n=1}^{\infty} a_n < \infty$$

choose $a_n = (-1)^n \cdot \frac{1}{n}$
 $\sum \varepsilon_n a_n f_n$ does not converge for $\varepsilon_n = (-1)^n$

∞, c_0
unconditional

Prop. 3.13. A bas. $\{e_n\}_{n=1}^\infty$
is unconditional iff $\exists k > 0$
s.t. $\forall N > 0, a_1, \dots, a_N, b_1, \dots, b_N$ s.t.
 $|a_i| \leq |b_i|$

$$\left\| \sum_{n=1}^N a_n e_n \right\| \leq k \left\| \sum_{n=1}^N b_n e_n \right\| (x)$$

Proof. Assume that $\{e_n\}_{n=1}^\infty$ uncond
 $t_1, \dots, t_N \in [-1, 1], T_{(t_n)} x \rightarrow x$
 $\sum a_n e_n \mapsto \sum t_n a_n e_n$

$T_{(t_n)}$ is bdd on finite sequences
as $(t_n)_{n=1}^N$ is convex combination of $\{1, -1\}$

$$\text{Fix } x \mapsto x = \sum a_n e_n$$

$$\|T_{(t_n)} x\| < \infty \text{ by Banach-Steinhaus}$$

$$\sup_{(t_n)} \|T_{(t_n)} x\| < \infty$$

$$\exists (t_n)_{n=1}^\infty \exists p_1 < p_2 < \dots < p_{N_k}$$

$$\text{s.t. } \left\| \sum_{k=p_{N_{k-1}}}^{p_{N_k}} t_k a_k e_k \right\| > 1$$

$$\text{Take } t_k = t_{p_k} \text{ for } p_{N_{k-1}} + 1 \leq k \leq p_{N_k}$$

Then $\sum t_k a_k e_k$ does not
converge

$$\|T_{(t_n)} x\| = \infty$$

So we have that

$$\sup_{(t_n)} \|T_{(t_n)} x\| < \infty \forall x \in X$$

$$\exists k > 0 \text{ s.t. } \|T_{(t_n)}\| < k$$

$$\text{Set } t_k = \frac{a_k}{b_k} \text{ (} b_k \neq 0 \text{)}$$

and (x) follows



Def Let X Banach space with
a basis $(e_n)_{n \geq 1}$. Then $(e_n)_{n \geq 1}$ is called
shrinking if $\forall x^* \in X^*$

$$\|x^*\|_N \rightarrow 0, N \rightarrow \infty$$

where

$$\|x^*\|_N = \sup \{ |x^*(x)|, x \in [e_n]_{n \geq N}, \|x\| \leq 1 \}$$

Example $X = \ell^1$, $(e_n)_{n \geq 1}$ standard basis
 $\Rightarrow x^* = (1, 1, 1, \dots)$

$$\|x^*\|_N = 1 \quad \forall N \geq 1 \Rightarrow (e_n)_{n \geq 1} \text{ is not shrinking}$$

Def 32.12 $(a_n)_{n \geq 1}$ is boundedly complete
if $\lambda (a_n)_{n \geq 1} \in \ell^1$ s.t.

$$\sup_N \left\| \sum_{n=1}^N a_n e_n \right\| < \infty \text{ we have that}$$

$$\sum_{n=1}^{\infty} a_n e_n \text{ converges.}$$

$\Rightarrow (e_n)_{n=1}$ is not shrinking

is boundedly complete

s.t.

We have that

Example $X = c_0$, $(e_n)_{n=1}$ standard

$$a_n = 1 \quad \forall n \geq 1$$

$$\sup_N \left\| \sum_{n=1}^N a_n e_n \right\| = \left\| (1, 1, 1, \dots, 1, 0, 0, 0) \right\|$$
$$= 1$$

(i) $\Rightarrow$ (ii) by
(ii) $\Rightarrow$ (i)

(iii) $\Rightarrow$ (iv) (v)

(iv) $\Rightarrow$ (v)

$\exists M \in \mathbb{R}$ s.t. $\forall x \in X$

$$P_A + P_B = I$$

(iv) $\Rightarrow$ (iii)

$$\|P_A x\| = \lim_{n \rightarrow \infty} \|P_n x\|$$

Example $X = c_0, (e_n)_{n \geq 1}$ standard

$$a_n = 1 \quad n \geq 1$$

$$\sup_N \left\| \sum_{n=1}^N a_n e_n \right\| = \|(1, 1, 1, \dots, 0, 0, \dots)\| = 1$$

$$Z = [(e_n^*)_{n \geq 1}] \subset X^*$$

$$\forall x \in X^{**} \quad \exists y \in Y \quad \forall x \in X^* \quad \exists y \in Y \quad \forall x \in X^* \quad \exists y \in Y$$

Theorem 3.2.15. Let $(e_n)_{n \geq 1}$ be a

basis of X , $(e_n^*)_{n \geq 1}$ biorthogonal

basis. TFAE

i) $(e_n)_{n \geq 1}$ is totally complete for X

ii) $(e_n^*)_{n \geq 1}$ is shrinking in $Z = [e_n^*]$

and the sequence $(j(e_n) \mid_Z)_{n \geq 1}$

is a basis of Z^* biorth to $(e_n^*)_{n \geq 1}$

iii) $X \rightarrow j(X) \mid_Z$ is an isomorphism from X

onto Z^*

for the proof:

Prop 3.2.1 $(e_n^*)_{n \geq 1}$ basis of Z

$$e_n^{**} = j(e_n) \mid_Z, n \geq 1$$

Prop 3.2.3 $x \mapsto |x|_Z$ is

isomorphism from X into Z .

Prop 3.2.8 $(e_n^*)_{n \geq 1}$ is a basis

of X^* iff $(e_n)_{n \geq 1}$ is shrinking.

Prop 3.2.14 $(e_n)_{n \geq 1}$ is bounded, $\forall x^{**} \in X^{**}, \exists \sum x^{**}(e_n^*) e_n \in X$



Proof (i) $\Rightarrow$ (iii) we need to show
that $x \mapsto j(x)|_Z$ is an isomorphism
onto Z^* . Fix $h^* \in Z^*$. By Hahn-Banach

$$\exists x^{**} \in X^{**} \text{ s.t. } x^{**}|_Z = h^*$$

by Prop 3.2.14 we set

$$X \ni x = \sum x^{**}(e_n^*) e_n$$
$$x(e_n^*) = e_n^*(x) = x^{**}(e_n^*) = h^*(e_n^*) \Rightarrow j(x)|_Z = h^*$$

by constr. of x^{**}

(iii) $\Rightarrow$ (i) $x \mapsto j(x)|_Z$ is isomorphism from
 X onto Z^* . Since $(e_n)_{n \geq 1}$ is a basis of X ,
 $j(e_n)|_Z$ is a basis of Z^* . It is biorthogonal
basis is $(e_n^*)_{n \geq 1}$ (basis of Z).

$\Rightarrow (e_n^*)_{n \geq 1}$ is shrinking basis of Z (Prop 3.2.8)

(ii) $\Rightarrow$ (i) Fix $(a_n)_{n \geq 1} \in \mathbb{R}$ s.t.

$$\sup_n \left\| \sum_{n=1}^N a_n e_n \right\| < \infty$$

$$\sup_n \left\| \sum_{n=1}^N a_n e_n^* \right\| < \infty$$

is isomorphism from
 ... a basis of X ,
 ... its biorthogonal
 ... of Z .
 basis of Z (Prop 3.2.8)
 ...

$$\exists N_1 < N_2 < \dots < N_k < \dots, \exists x^{**} \in X^{**}$$

$$\sum_{n=1}^{N_k} a_n (e_n)(x^{**}) \xrightarrow{k \rightarrow \infty} x^{**}(x^{**}) \quad \forall x^{**} \in X^{**}$$

Let $h^* = x^{**}|_Z$

$$h^*(e_p) = x^{**}(e_p^*) = \lim_{k \rightarrow \infty} \sum_{n=1}^{N_k} a_n (e_n)(e_p^*)$$

$$= \lim_{k \rightarrow \infty} \sum_{n=1}^{N_k} a_n \delta_{np} = a_p, p \geq 1$$

$$h^* = \sum_{n=1}^{\infty} e_n^{**}(h^*) e_n^{**} = \sum_{n=1}^{\infty} h^*(e_n^*) e_n^{**}$$

$$= \sum_{n=1}^{\infty} a_n (e_n)|_Z$$

By Prop 3.2.3 $\forall \epsilon \exists M \neq k > l > M$
 $\epsilon > \left\| \sum_{n=l}^k a_n (e_n)|_Z \right\| \sim \left\| \sum_{n=l}^k a_n e_n \right\|$

$\Rightarrow \sum a_n e_n$ converges in X
 so $(e_n)_{n \geq 1}$ is bddly complete in X $\square$

na

i) $(e_n)_{n \geq 1}$ is bddly complete for X

ii) $(e_n^*)_{n \geq 1}$ is shrinking in $Z = [e_n^*]$
 and the sequence $(j(e_n)|_Z)_{n \geq 1}$
 is a basis of Z^* biorth to $(e_n^*)_{n \geq 1}$

iii) $X \rightarrow j(X)|_Z$ is an isomorphism from X
 onto Z^*

Example 3.2.16
 $X = c_0$ it does not have
 a bddly complete basis
 as c_0 is not a dual of any space

Thm 3.2.17 Let $(e_n)_{n \geq 1}$ basis of X ,
 (e_n^*) biorth. basis. TFAE

- (i) $(e_n)_{n \geq 1}$ is shrinking
- (ii) $(e_n^*)_{n \geq 1}$ is bddly complete in Z .
- (iii) $Z = X^*$

Proof Thm 3.2.15 for $(e_n)_{n \geq 1} = (e_n^*)_{n \geq 1}$ $\square$

Theorem 3.2.19 (James) Let X Banach,

- (i) $(e_n)_{n \geq 1}$ basis TFAE
- (ii) X reflexive
- (iii) $(e_n)_{n \geq 1}$ is both shrinking and bddly complete.

Theorem (Zippin) TFAE

- (i) X reflexive
- (ii) every basis in X shrinking
- (iii) every basis in X bddly complete

$\exists N, \epsilon > 0$
 $\sum_{n=1}^N a_n$
 let h^*
 $h^*(e_n)$

$h^* =$
 $=$

By prop
 $\epsilon > 0$

Bounded,

notly complete

Theorem 3.31 Let X Banach
with uncond basis $(u_n)_{n \in \mathbb{N}}$. TFAE

- (i) $(u_n)_{n \in \mathbb{N}}$ is not shrinking
- (ii) X contains a complemented subspace isom to ℓ^1
- (iii) $\exists$ complemented block basic sequence $(y_n)_{n \in \mathbb{N}}$, which is equivalent to the canonical basis of ℓ^1
- (iv) X contains a subspace isom to ℓ^1

Theorem 3.32

(i) $\dots$ bandedly complete

(ii) $\subset$

(iii) $\subset$

(iv) $\subset$

Corollary 3.33 Let X Banach

space. Then

- (i) Either every uncond basis is shrinking or none
- (ii) Either every uncond basis is bandedly complete or none

Theorem 3.34 Let X Banach with uncond basis. If X not reflexive then either ℓ^1 is complemented in X and hence X is nonseparable

$$\text{If } X = \ell^1 \oplus X_0 \quad \subset \oplus X_0 \\ X^* = \ell^\infty \oplus X_0^* \quad \subset \oplus X_0^*$$

Proof If X is not $\mathcal{R}$, then by Thm 3.2.13
 $(x_n)_{n \geq 1}$ alternated shrinking or not
 totally complete, and no desired
 follows from Theorem 3.3.1, 3.3.2 $\square$

James space

$(\xi(n))_{n \geq 1} = \mathbb{R}$ -valued sequence

$\xi \in \mathcal{J}$ if

(i) $\xi(n) \rightarrow 0$ as $n \rightarrow \infty$

(ii) $\sup \sum_{j=1}^k (\xi(p_j) - \xi(q_{j+1}))^2$

where $1 \leq p_1 < p_2 < \dots < p_k$

$\mathcal{J}$ is a Banach space endowed with the
 norm

$$\|\xi\|_{\mathcal{J}} = \frac{1}{\sqrt{2}} \sup \left\{ \left((\xi(p_1) - \xi(q_1))^2 + \sum_{k=1}^N (\xi(p_k) - \xi(q_k))^2 \right)^{1/2} \right\}$$

$$\|\xi\|_0 = \sup \left\{ \left(\sum_{k=1}^N (\xi(p_k) - \xi(q_{k+1}))^2 \right)^{1/2} \right\}$$

where $1 \leq p_1 < p_2 < \dots < p_N$

Exercise $\|\cdot\|_{\mathcal{J}}$ and $\|\cdot\|_0$ are equivalent

bounded metric

$$\sum_{n=1}^{\infty} (s(p_n) - s(p_{n-1}))^{\frac{1}{2}}$$

$$\{s(p_n)\}^{\frac{1}{2}}$$

Prop 3.4.4 The basis (entire)

is shrinking.

Example 3.4.5 (entire, not totally complete)

$$\left\| \sum_{n=1}^N e_n \right\| = 1, \quad \left\| \sum_{n=1}^{\infty} e_n \right\| = 1$$

$\sum e_n$ diverges

By Thm 3.2.11 Since J has a shrinking basis, $J^{\perp}$ has the following form

$$(a_n)_{n \geq 1} \text{ s.t. } \sum_{n=1}^{\infty} a_n e_n = 0$$

$$\sup_N \left\| \sum_{n=1}^N a_n e_n \right\|$$

$$\sup_N \left\| (a_1, \dots, a_N, 0, 0, \dots) \right\|_2$$

$$= \lim_N \left\| (0, a_1, \dots, a_N, 0, 0, \dots) \right\|_2$$

$$= \left\| (0, a_1, \dots, a_n) \right\|_2$$

Theorem 3.4.6

A subspace of $J^{\perp}$ is isomorphic to J .

Theorem 3.3.4 Let X be a Banach space with a basis.

If X is not reflexive then either X or X^* is not complemented in X and hence X is not separable.

$$X = \ell^1 \oplus X_0 \quad \ell^1 \oplus X_0$$

$$X^* = \ell^\infty \oplus X_0^* \quad \ell^\infty \oplus X_0^*$$



Proof let $\xi \in J^{**}$ let

$$L(\xi) = \lim_{n \rightarrow \infty} \xi(n)$$

$$\xi = (L(\xi), L(\xi), \dots) + (\xi(1) - L(\xi), \xi(2) - L(\xi), \dots)$$

$$= L(\xi)(1, 1, \dots, 1, \dots) + (\xi(1) - L(\xi), \xi(2) - L(\xi), \dots)_{n \geq 1}$$

$$(1) \quad \xi(n) - L(\xi) \rightarrow 0, n \rightarrow \infty$$

$$(2) \quad \|\xi - L(\xi)\|_J = \|\xi\|_J < \infty$$

$$S: J^{**} \rightarrow J$$

$$S(\xi) = (-L(\xi), \xi(1) - L(\xi), \dots, \xi(n) - L(\xi), \dots)$$

$$\|S(\xi)\|_J = \|\xi\|_J$$

Cor. 34.7.

Thus, J is not reflexive,

$J^{**} \hookrightarrow$ separable $\Rightarrow$ by Thm 334

J has noncountable dual basis.