

K = cpt. Haus. S.
 $C(K) = \{f = f \text{ real-valued continuous on } K\}$
 $C(K)$ is B. S.
 $\|f\|_{C(K)} = \|f\|_{\infty} = \max_{s \in K} |f(s)|$

$$\begin{aligned}
 A \subset K \\
 \mu(A) &= \inf \{ \mu(U) : A \subset U \} \\
 &= \sup \{ \mu(K) : K \subset A \}
 \end{aligned}$$

Th 4.1.1 (Riesz, R.T.) K cpt Haus. S.

$$C(K)^* \cong M(K) \text{ (isometrically isom.)}$$

$M(K) = \{ \mu : \mu \text{ finite signed regular Borel measure on } K \}$
 $\|\mu\| = |\mu|(K) \text{ (total variation)}$

$$\langle f, \mu \rangle = \int_K f d\mu$$

If K is metrizable every Borel measure is regular.

Th 4.1.2 (The Stone-Weierstrass Theorem)

(Real case) K - cpt Haus. S.
 A is a subalgebra of $C(K)$ contains constants. If A separates points of K .
 (for $s_1, s_2 \in K$ with $s_1 \neq s_2$, $\exists f \in A$ s.t. $f(s_1) \neq f(s_2)$)
 $\overline{A} = C(K)$.

Th 4.1.3

Separable

Proof:

K in
 for u
 $C(K)$

Th. 4.13. If K cpt Haus then (K) is separable $\Leftrightarrow K$ is metrizable

Proof: " $\Rightarrow$ " $S \rightarrow \delta_S$ (point mass of S)

K into $M(K)$ is a homeom for w^* -top of $M(K)$

$C(K)$ is separable $\xRightarrow{\text{Lemma 4.10}}$

$B_M(K)$ is metrizable (w^*)

$\Rightarrow K$ is metrizable

$\Leftarrow K$ is separable " $\Downarrow$ " a metric inducing top on K
and $(S_n)_{n=1}^\infty \subset K$ dense.

$d_n: K \rightarrow \mathbb{R}$ by $d_n(S) = d(S, S_n)$

$A \subset C(K)$ algebra generated by

$D = \{1_K, d_1, d_2, \dots\}$

$\overline{A} = C(K)$

$A_i \subset A$ with A_i is countable rational coefficients

$C(K)$ is separable.

X is totally disconnected if each point in X has a base of neighborhoods consisting of clopen sets (closed and open).

Cantor sets. $\Delta = \{0, 1\}^\mathbb{N}$ is totally disconnected cpt metric space.

Prop 4.14 K is totally disconnected cpt then

$A = \{f = \sum_{i=1}^n \alpha_i \chi_{U_i} \mid U_1, U_2, \dots, U_n \text{ disjoint and clopen}\}$

$\overline{A} = C(K)$

Th 4.1.5 (Borchers) Suppose,

K, L two Cpt Havn. s.t.

$C(K) \cong C(L)$ (isometrically isomorphic)

Then we have K and L are homeomorphic.

Banach algebra $(A, \cdot)$

$$\|xy\| \leq \|x\| \|y\|$$

• commutative if $xy = yx, x, y \in A$

• with unit $x = ex = xe, e \in A$

$C(K) \quad e = 1_K$

$$\|2fg\| \leq \|f^2 + g^2\|$$

f, g $\in C(K)$

$$\|2f(s)g(s)\| \leq f^2(s) + g^2(s)$$

Th 4.2.1 A commutative B.a with e

$\|e\| = 1$. Then $A \cong C(K)$ (isometric isomorphism)

$$\Leftrightarrow \|2ab\| \leq \|a^2 + b^2\| \quad (4.1)$$

Th 4.2.2

$$\Leftrightarrow \|a^2\| \leq \|a^2 + b^2\| \quad (4.2)$$

$$\Leftrightarrow \|a^2 - b^2\| \leq \|a^2 + b^2\| \quad \forall a, b \in A \quad (*)$$

Lemma 4.2.3 A commutative real B.a, with $e, \|e\| = 1$

Then $(4.1) \Leftrightarrow (4.2) \Leftrightarrow (*)$

Proof: $(4.1) \Rightarrow (*)$

$$\|2 \frac{a+b}{a} \frac{a-b}{b}\| \leq \| \frac{(a+b)^2}{a^2} + \frac{(a-b)^2}{b^2} \|$$

$$\|a^2 - b^2\| \leq \|a^2 + b^2\| \quad (*)$$

$(4.1) \Rightarrow (4.2)$

Put $b = e$ in (4.1) . we get

$$2\|a\| \leq \|a^2 + e\| \leq \|a^2\| + 1$$

$a \in A, \|a\| = 1$ we have

$$2 \leq \|a^2\| + 1, \|a^2\| \geq 1$$

Therefore, $1 \leq \left\| \frac{a^2}{\|a\|^2} \right\| \quad a \neq 0$

$$\|a\|^2 \leq \|a^2\| \leq \|a\|^2$$

$$a, b \in A$$

$$\begin{aligned} \|a\|^2 &= \|a^2\| = \frac{1}{2} \|a^2 - b^2 + b^2 + a^2\| \\ &\leq \frac{1}{2} (\|a^2 - b^2\| + \|b^2 + a^2\|) \\ &\stackrel{(*)}{\leq} \|a^2 + b^2\| \end{aligned}$$

$\Rightarrow (4.2)$
(4.1) $\Rightarrow (*)$ $\nabla (*)$

$a, b \in A$ then

$$\begin{aligned} \|a^2 - b^2\|^2 &\leq \|\underbrace{a^2 - b^2}_a + \underbrace{b^2}_b\|^2 \\ &= \|(a^2 + b^2)^2\| \leq \|a^2 + b^2\|^2 \end{aligned}$$

$(*) \Rightarrow (4.1)$ $x, y \in A$. Pick $a, b \in A$.

$$\text{S.t. } \begin{cases} x = a - b \\ y = a + b \end{cases} \Leftrightarrow \begin{cases} a = \frac{x+y}{2} \\ b = \frac{y-x}{2} \end{cases}$$

$$\begin{aligned} \|2xy\| &= 2\|a^2 - b^2\| \\ &\stackrel{(*)}{\leq} 2\|a^2 + b^2\| \\ &= 2\left\|\left(\frac{x+y}{2}\right)^2 + \left(\frac{y-x}{2}\right)^2\right\| \\ &= \|x^2 + y^2\| \end{aligned}$$

Def 4.2.4. A c.b.a. $\|e\|=1$
The state space of

$$S = \{\varphi \in A^*, \|\varphi\| = \varphi(e) = 1\}$$

$S \subset A^*$. $x \in S$ state
 S is w^* -cpt.

$$A_+ := \overline{\{a^2 : a \in A\}}$$

Lemma 4.2.5. A c.b.a. we have

- i) If $x, y \in A_+$, then $xy \in A_+$
- ii) If $x \in A_+$, $\lambda \geq 0$, $\lambda x \in A_+$

Pro 4.2.6. A c.b.c. $\|e\|=1$.

then

i) If $x \in A$ with $\|x\| \leq 1$, then $e+x \in A_+$

ii) $A = A_+ - A_+$

$$\|x-y\| \leq \|x+y\| \quad x, y \in A_+ \quad (4.3)$$

$$\|x\| \leq \|x+y\| \quad x, y \in A_+ \quad (4.4)$$

$$a, b \in A$$

$$a^2 = x \quad b^2 = y$$

Lemma 4.2.7. Suppose A sat. (4.4)

Then $\varphi(x) \geq 0$ whenever $\varphi \in S$, $x \in A_+$

Proof: Take $x \in A_+$, $\|x\|=1$ By Pro 4.2.6

$e-x \in A_+$ and by (4.4)

$$\|e-x\| \leq \|e-x+x\| = 1$$

$\varphi \in S$ we have

$$\begin{aligned} 1 = \|\varphi\| &\geq \|\varphi\| \|e-x\| \\ &\geq \varphi(e-x) \\ &= \varphi(e) - \varphi(x) = 1 - \varphi(x) \end{aligned}$$

$$\varphi(x) \geq 0$$

□

$x \in S$ is nonempty convex subset of a vector space.

x is extrem point of S if $x = \lambda x_1 + (1-\lambda)x_2$ with $0 < \lambda < 1$ then $x = x_1 = x_2$

$\mathcal{E}S$ - collection of all extrem points of S

Lemma 4.2.8. A sat. (4.4)

$$K := \{ \varphi \in S : \varphi(xy) = \varphi(x)\varphi(y) \text{ for } x, y \in A \}$$

Then K is a cpt. Her. w^* -top. of A^* contains the set $\mathcal{E}S$.

Proof of Th 4.2.1 (Th 4.2.2)

Let $J: A \rightarrow C(K)$ by
 $Jx(\varphi) = \varphi(x), x \in A, \varphi \in K$

J is an algebra hom.
 $J(e) = 1, \|J\| = 1.$

To prove J is isometry, that is,
 $\|x\| \leq 1$ whenever $\|Jx\|_{C(K)} \leq 1$

Claim: $x \in A$ s.t. $\|Jx\|_{C(K)} \leq 1$. Then
 $\exists \delta > 0$ there exists $t_2 > 0$ s.t.
 $\|e - t_2(1+\delta)e - t_2x\| < 1.$

Proof of claim.

Suppose it is not true. $x \in A$ with $\|Jx\| \leq 1$
 s.t. for some $\delta > 0$ have
 $\|e - t(1+\delta)e - tx\| \geq 1, \forall t > 0.$

By Hahn-Banach theorem, φ with $\|\varphi\| = 1$ and
 $\varphi(e - t(1+\delta)e - tx) \geq 1, \forall t > 0.$

In particular, $\varphi \in S$ and $\varphi((1+\delta)e + x) \leq 0.$

So $|\varphi(x)| \geq 1+\delta.$

By Krein-Milman theorem and lemma 4.2.8, $\varphi \in K$
 $|\varphi(x)| \geq 1+\delta. \|Jx\|_{C(K)} > 1. \quad \leftarrow$

Pro 4.2.6(i), $\|Jx\| \leq 1.$

$\Rightarrow (1+\delta)e + x \in A_+$ for $\delta > 0$

So $e + x \in A_+$

$-x$, we have $e - x \in A_+$ By (4.3)

$\|x\| = \frac{1}{2} \|e+x + e-x\| \leq \frac{1}{2} \|e+x+e-x\| = 1$

J is an isometry, $J(A) = C(K)$

Ref: $C(K)$ is order-complete. $\square$

$A, B \subset C(K)$ $f \leq g, f \in A, g \in B$
 then $h \in C(K)$ s.t.
 $f \leq h \leq g.$

map $V: F \rightarrow C(K)$ F subspace
of B.S. X is sublinear
H.i) $V(ax) = aV(x)$ $a \geq 0, x \in F$
ii) $V(x+y) \leq V(x) + V(y)$, $x, y \in F$.

A sublinear map V is minimal
if there is no $U: X \rightarrow C(K)$ s.t.
 $U(x) \leq V(x)$, $\forall x \in X$, $U \neq V$

Lemma 4.3.3. X B.S. F subspace X
Suppose $V: X \rightarrow C(K)$, $W: F \rightarrow C(K)$
are sublinear. s.t.
 $W(y) + V(-y) \geq 0$
 $y \in F$

If $C(K)$ is order-complete, then the map,
 $V \wedge W: X \rightarrow C(K)$ given by
 $V \wedge W(x) = \inf \{ V(x-y) + W(y) : y \in F \}$
is well defined and sublinear.

Proof: $\forall x \in X$.
 $V(x-y) + V(-x) \geq V(-y)$
 $\Rightarrow V(x-y) + W(y) \geq V(-y) - V(-x) + W(y)$
 $\geq -V(-x)$
all $y \in F$ $-V(-x) \in \{ V(x-y) + W(y) : y \in F \}$
 $C(K)$ $V \wedge W: X \rightarrow C(K)$ by
 $V \wedge W(x) = \inf \{ \dots \}$

Lemma 4.3.4. Let $V: X \rightarrow C(K)$ sublinear.
If $C(K)$ order comp. there exists a minimal
sublinear map $W: X \rightarrow C(K)$ with
 $W(x) \leq V(x)$ $x \in X$.

Lemma 4.3.5. $C(K)$ order comp. and
 $V: X \rightarrow C(K)$ is sublinear.
If V is minimal, then V is linear map.

Th 4.3.6. K cpt. Hwn. T hwn

$C(K)$ is isometrically injective (Def 2.5.1)

$\Leftrightarrow C(K)$ is order complete.

Proof: " $\Leftarrow$ " Let $E \subset \mathbb{R}(K)$
and let $S: E \rightarrow C(K)$ be linear map.

$\|S\| = 1$. For any $x \in E$

$$-\|x\| \leq S(x)(k) \leq \|x\| \quad k \in K$$

that is,

$$-\|x\| 1_K \leq S(x) \leq \|x\| 1_K \quad (4.8)$$

$V_0: \mathbb{R} \rightarrow C(K)$ by $V_0(x) = \|x\| 1_K$

$$(4.8) \quad S(x) \geq -V_0(-x) \quad \forall x \in E$$

$V = V_0 \wedge S$ as in Lemma 4.3.3

By Lemma 4.3.4 $T: \mathbb{R} \rightarrow C(K)$ minimal sublinear

$T \leq V$. Lemma 4.3.5, T is linear.

On E , $T(x) \leq S(x)$ and $T(-x) \leq S(-x)$

$$T|_E = S$$

Finally, $T(x) \leq \|x\| 1_K$ and $T(-x) \leq \|x\| 1_K$

$$\|T\| = 1$$

" $\Rightarrow$ " $C(K)$ is isometrically injective.

$$\text{Id}: C(K) \rightarrow C(K)$$

$P: \mathcal{L}(C(K)) \rightarrow C(K)$ $\|P\| = 1$

$A, B \subset C(K)$ s.t. $f \in A, g \in B \Rightarrow f \leq g$

For $s \in K$ put $a(s) = \sup_{f \in A} f(s) \leq \|g_0\|$

$a \in \mathcal{L}(C(K))$. Let $h = P(a) \in C(K)$

" $f \leq h \leq g$ $f \in A, g \in B$

$P(1_K) = 1_K, \|P\| = 1$ $b \in \mathcal{L}(C(K))$ $b \geq 0$

we have $\|P \circ \lambda b\| \leq 1$ for $0 \leq \lambda \leq \frac{2}{\|b\|}$

P is positive map. $Pb \geq 0$ $b \geq 0$ in $\mathcal{L}(C(K))$



$$f \in A, f \leq a$$

$$f = pf \leq pa = h$$

$$h \in A^u$$

$$h \in B^L$$

$C(K)$ is order complete $\square$

Th 4.3.7 Σ B.S

is isometrically injective iff
it is isometrically isomorphic to an
order-complete $C(K)$ -space.

Th 4.4.8 K ^{en} uncountable cpt
metric space $C(K) \cong C([0,1])$ (isomorphism)

Th 4.5.2 K a cpt metric space

TFAE

i) K is countable and has finite Cantor-Bendixson index

ii) $C(K) \cong C_0$ ^{Prop 3.5.4 ii)} $\uparrow$ $C([0,1])$ fail

iii) $C(K)$ embeds in a space with unibasis

iv) $C(K)$ has property (21)

Def 3.5.1

If K countable cpt
 $U =$ isolated points of K

$$K' = K \setminus U$$

$$K^{(n)} = (K^{(n-1)})'$$

K has finite Cantor-Bendixson index
if $K^{(n)}$ is finite for some n

$$K^{(n+1)} = \emptyset$$