



$T.C \rightarrow X$ weakly compact
 $X \rightarrow \ell_1$

$\{ \rightarrow T \text{ compact} \quad \begin{matrix} 2.4 & 10 \\ 2.3 & 7 \end{matrix}$

Thm 1.6's Eberlein Smoliar
 $A \subseteq X$, weak compact iff A weak sequentially compact

Let $(\Omega, \mathcal{E}, \mu)$ be a prob space

Vol. 1,
Lemma 5

Let Σ be a locally compact
 Hausdorff space with a positive Borel measure μ .

Let $\mathcal{L}_1(\mu)$ be the Lebesgue space
 of integrable functions. If $\{f_n\}_{n=1}^\infty \subset \mathcal{L}_1(\mu)$
 is a sequence of norm 1 and disjointly
 supported, then its norm equals the
 cardinality of the support.

Def 5.2. $\mathcal{S} \subset \mathcal{L}_1(\mu)$ is bounded.
 $\mathcal{S}$ is equi-integrable
 or uniformly integrable if
 $\forall \epsilon > 0 \exists \delta > 0 \forall E \in \Sigma$ if $\mu(E) < \delta$
 then $\sup_{f \in \mathcal{S}} \int_E |f| d\mu < \epsilon$, $\lim_{\mu(E) \rightarrow 0} \sup_{f \in \mathcal{S}} \int_E |f| d\mu = 0$

Ex: Let $f \in \mathcal{L}_1(\mu)$, $f \geq 0$
 $\mathcal{S} = \{f_n \in \mathcal{L}_1(\mu) : 0 \leq f_n \leq f\}$ is uniformly
 integrable. $\mathcal{S} = \{f_n \in \mathcal{L}_1(\mu) : 0 \leq f_n \leq f\}$



Lemma 5.25 $\mathcal{F} \subseteq L_1(\mu)$ bounded TFAE

i) $\mathcal{F}$ is unif int

ii) $\lim_{m \rightarrow \infty} \sup_{f \in \mathcal{F}} \int |f| d\mu = 0$
 $\int |f| > m$

Proof $\Rightarrow$ i) $f \in \mathcal{F}, \epsilon \in \Sigma, m > 0$ finite

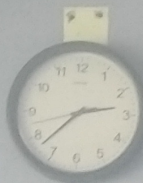
$$\int |f| d\mu = \int_{\epsilon \cap \{|f| \leq m\}} |f| d\mu + \int_{\epsilon \cap \{|f| > m\}} |f| d\mu$$

$$\leq m\mu(\epsilon) + \sup_{f \in \mathcal{F}} \int_{\{|f| > m\}} |f| d\mu$$

$\epsilon \in \Sigma$, pick $m > 0$ st $\sup_{f \in \mathcal{F}} \int |f| d\mu < \epsilon/2$

For $\epsilon \in \Sigma$ st $\mu(\epsilon) < \epsilon/(2m)$ use above

$$\sup_{f \in \mathcal{F}} \int |f| d\mu < \epsilon$$



25 $\mathcal{F} = L_1(\mu)$ barrel TFAE

in unit int

sup $\int |f| d\mu = 0$
so $f = 0$ a.e.

Suppose II fails

25 $\mathcal{F} \not\subset L_1^\infty$, as $\mathcal{F} \not\subset L_1^\infty$

If $\int |f_n| d\mu \geq \delta > 0$, for all $n \in \mathbb{N}$
 $\{ |f_n| > n\delta \}$

So, we may suppose from 5.2.7 that

$$f_n = \int_n \chi_{A_n} + \int_n \chi_{B_n} \text{ where } B_n = A_n$$

$\{A_n\}_{n=1}^\infty \subseteq \Sigma$ disjoint and $\{B_n\}_{n=1}^\infty$ unit int.

Note.

$$\mu(|f_n| > n\delta) < \frac{\int |f_n| d\mu}{n\delta} \rightarrow 0$$

$$\text{Thus, } \lim_{n \rightarrow \infty} \int |f_n| d\mu = 0$$

For all but finitely many n .

$$\int_{A_n} |f_n| d\mu \geq \frac{1}{2} \delta > 0 \text{ contradicts II}$$

Proof of S.2.9:

If $f_n \in L_1(\mu)$ weakly Cauchy

If f_n is not equiv to the canonical Ban

of l_1 (is not weakly Cauchy)

Here S.2.8: f_n is weakly equiv

Theorem 5.2.9

$L_1(\mu)$ is weakly sequentially compact

Even weakly Cauchy seq converges

(x_n) st $\forall x^* \in X^*$
 $\lim_{n \rightarrow \infty} x^*(x_n)$ exists

Theorem 5.2.8

I) f is relatively

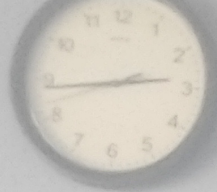
II) f is weakly

III) f does not
equiv to l_1

IV)

V) In all

$\lim_{n \rightarrow \infty} \sum_{k=1}^n f_k(x)$



$L_1(\mu)$ bounded TFAE

if $\|d\mu\| = 0$
imp

II fails

$\infty \in L_1, \text{ and } \infty$

Cor S 2.11:

C_0 is not isomorphic to a subspace
of $L_1(\mu)$

Proof: 2.4.15, 2.4.11

□

Proof of S 2.9:

that $f_n \in L_1(\mu)$ w

that f_n is not equiv

of f_1 (is not weak

Here S.2.8: f



§5.3 Weak Convergence in $M(\mathbb{R})$.

Ex. 1 Dunford-Pettis Property

Let X, Y be Banach spaces and

bounded, linear operator $T: X \rightarrow Y$ is completely ct

or Dunford-Pettis Operator if

if $W \subset X$ weakly compact, $T(W) \subset Y$ is norm compact.

Prop 4.2

T is Dunford-Pettis operator iff

T is weak to norm sequentially ct

(if $\{x_n\}_{n=1}^{\infty} \subset X$ converges weakly to $x \in X$ then $\{Tx_n\}_{n=1}^{\infty} \subset Y$ converges in norm to $Tx \in Y$.)

Proof:

$\Rightarrow$ Suppose $\{x_n\}_{n=1}^\infty \subseteq X$

that is weakly null but

$$\|Tx_n\| \geq \delta > 0.$$

Define $W = \{x_n \mid n \geq 1\} \cup \{0\} \subseteq X$

The W is weakly compact. Then $T(W) \subseteq Y$ is norm compact. Then W contains a converging sequence $\{x_{n_k}\}_{k=1}^\infty$ converging to $y \in Y$. $\sum_{k=1}^\infty y_{n_k} = 0$

$\Leftarrow$ Let $W \subseteq X$ be weakly compact.

Let $\{y_n\}_{n=1}^\infty \subseteq T(W)$ then pick $\{x_n\}_{n=1}^\infty$

s.t. $T(x_n) = y_n$. Then $\exists \{x_{n_k}\}_{k=1}^\infty$ subsequence of $\{x_n\}$ converging weakly to some $x \in X$

$\sum_{k=1}^\infty y_{n_k} \neq 0$ converges in norm to Tx

$T(W)$ is norm compact $\square$

Def 5.13 A Banach space X has Rumford Pettis Property (RPP)

if, for all $T: X \rightarrow Y$ that are weakly compact then T is Rumford Pettis Operator.

Ex: C_0 has RPP

- ℓ_1 " "
- X infinite dimensional reflexive does not have RPP.



Theorem 5.4.5: X Banach. Then X has RPP

$\iff \forall \{x_n\}_{n=1}^{\infty} \subseteq X$ converges weakly to 0

$\forall \{x_n^*\}_{n=1}^{\infty} \subseteq X^*$ " " "

then $\{x_n^*(x_n)\}_{n=1}^{\infty}$ converges to 0

Prop 5.4.2

T is Dunford-Pettis operator $\iff$

T weak to norm sequentially c.t.

$\left(\exists \{x_n\}_{n=1}^{\infty} \subseteq X \text{ converges weakly to } x \in X \text{ then } \{Tx_n\}_{n=1}^{\infty} \text{ converges in norm to } Tx \in X \right)$

Proof:

Suppose

Then $T^*(K_{Y^*}) \subseteq X^*$ is relatively weakly compact

We may assume that

$\{T^*y_n\}_{n=1}^\infty$ is weakly convergent to some x^* in

$\sum \{T^*y_n - x^*\}_{n=1}^\infty$ is weakly null

$$\text{Thus } (T^*y_n - x^*)(x_n) \rightarrow 0$$

Because $x^*(x_n) \rightarrow 0$, I

$$T^*y_n(x_n) = \|Tx_n\| \rightarrow 0 \quad \text{by } \Delta$$

← Suppose $\{x_n\}_{n=1}^\infty \subseteq X$ weakly null
 $\{x_n^*\}_{n=1}^\infty \subseteq X^*$

Define: $T: X \rightarrow C_0$

$$x \mapsto (x_n^*(x))_{n=1}^\infty$$

Let $\{e_i\}_{i=1}^\infty$ be the canonical basis of ℓ_1

$$T^*(e_i) = x_i^*$$

NS: $T^*(\mathbb{R}_1)$ is contained in
the convex hull of $\{x_n^*\}_{n=1}^\infty$

T^* is weakly compact

$T \rightarrow T$ is DP

$$|x_n^*(x_n)| \leq \max_k |x_k^*(x_n)| = \|Tx_n\|_\infty$$

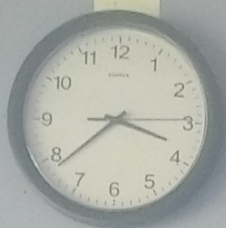
Then $T^*(B_{Y^*}) \subseteq \bar{X}$

We may assume $\{T^*y_n^*\}_{n=1}^\infty$ is weakly

$$\sum \{T^*y_n^* - \dots\}$$

Thus $(T^*y_n^*)$

Because $x^*(T^*y_n^*(x_n))$



X has DPP
by 4.0

Thm 5.4.6: (Dunford Pettis Theorem)

(i) if μ is σ -finite then $L_1(\mu)$ has DPP.

(ii) if K compact and Hausdorff then $\mathcal{C}(K)$ has DPP.

Cor 5.4.6: K as in thm then $M(K)$ has DPP

Thm 5.4.7: if $T: L_1(\mu) \rightarrow L_1(\nu)$ then T^* is compact.
or $T: \mathcal{C}(K) \rightarrow \mathcal{C}(K)$

Norm

the c

$\subseteq$

T^*

$\subseteq T$

(X^*)

S.S.C. Weakly compact operator on $C(K)^D$ spaces.

2.13
Def. A operator $T: X \rightarrow Y$ is strictly singular

if there is no infinite $E \subseteq X$ subspace st.
 $T|_E$ isom onto its range

Thm 5.5.1 Let K be compact and Hausdorff
 $T: C(K) \rightarrow X$ weakly compact then T
is strictly singular.

Proof. Let $Y \subseteq C(K)$ st $T|_Y$ isom

Since T weakly compact $T(B_Y)$ is relatively weakly compact

So B_Y weakly compact, since $C(K)$ has DPP, B_Y is compact
Hence $T|_Y$ finite dim □

Remark. Same
for Steiner

Theorem (Pełczyński)
 $T \in C(K)^D$



Consider $\mathcal{C}(K)^n$

totally singular

X subspace of

Thm 5.5: Let K be compact and Hausdorff.
 $T: \mathcal{C}(K) \rightarrow X$ weakly compact then T
 is strictly singular.

Proof: Let $Y \subseteq \mathcal{C}(K)$ s.t. $T|_Y$ norm

Since T weakly compact $T(B_Y)$ is relatively weakly compact

So B_Y weakly compact, since $\mathcal{C}(K)$ has DPP, B_Y is compact

Hence $T|_Y$ is finite dim



Remark: From both lemmas for L_p
 the theorem

Theorem (Peteziński): If $T: \mathcal{C}(K) \rightarrow X$ is bounded, then
 if $T|_Y$ fails to be weakly compact then
 $\mathcal{E} \subseteq \mathcal{C}(K)$ closed subspace norm to C_0 s.t.
 $T|_{\mathcal{E}}$ is an isom.

Cor S
 is weakly
 from T

Thm
 Spe

if T

$\Rightarrow F$

Remark Same holds true for L_∞ (the theorem)

Theorem (Pettis) Let $T: T(K) \rightarrow X$ be a linear operator. If T is not weakly compact then $\exists \varepsilon \subseteq T(K)$ closed subspace norm to C_0 st. $T|_\varepsilon$ is not norm.

Cor 5.6.4 X Banach. If no subspace is isomorphic to C_0 . Then every operator from $T(K) \rightarrow X$ is weakly compact.

Thm 5.6.5 Suppose X is a Banach space, and $T: X \rightarrow Y$ is a bounded linear operator.

If T is not weakly compact, then

$\exists F \subseteq X$ closed st $F \cong \ell_\infty$ and $T|_F$ is not norm.

5.7

Prop 5.7.3 $X \subseteq L_1(\mu)$ inf dim complemented subspace then X contains a complemented subspace isomorphic to ℓ_1 .

Prop 5.7.1 $L_1(\mu)$ on $T(K)$ do not have inf dim complemented subspace.



Prop 5.74.

K compact, metric space. $X \subseteq C(K)$ a complex

Then X contains a complemented subspace isomorphic to c_0 .

Thm 5.75 ℓ_∞ is prime

(Every complemented
subspace of X
is isomorphic to X)

Cor 5.76 There are no infinite separable
injective Banach spaces.

Conjecture 5.77! Every infinite complemented
subspace of $L_1(\mu)$ is isomorphic to $L_1(\mu)$.

Conjecture 5.78: Every infinite complemented
subspace of $C[0,1]$ is isomorphic to a $C(K)$ space
for the right choice of K (Compact
metric)

Remark: Same for
the theorem

Theorem (Pełczyński)

$T: C(K) \rightarrow C(K)$

If T is finite rank

$\varepsilon \subseteq C(K)$ closed

$T|_\varepsilon$ is an