Chapter 6: L_p for $1 \le p < \infty$

★ Banach structures of L_p

ex $L_p(\mu)$ Hilbert $\iff p=2$

ex $L_p(\mu)$ reflexive $\iff p>1$.

Thus $\boxed{L_1 \mid L_{1<p<2} \mid L_2 \mid L_{2<p<\infty} \mid \cancel{L_\infty}}$

Q: different? A: Yes!

Remark every infinite-dim. separable $L_p(\mu)$ space is isomorphic to ℓ_p or $\boxed{L_p[0,1]}$ §6.1

$L_p(\lambda)$ with λ Lebesgue measure on $[0,1]$.

Recall ℓ_1 and $L_1[0,1]$ is not isomorphic

Q: $\ell_p \simeq L_p[0,1]$? A: Yes iff $p=2$

§6.1 The Haar basis in L_p

Remark: we denote L_p for $L_p[0,1]$.

Now recall (CH3): A basis $(u_n)_{n=1}^{\infty}$ of X Banach is unconditional if $\forall x \in X$ the $\sum u_n^*(x) u_n$ converges to x uncond.

§6. The Haar basis in L_p

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<u>Now recall</u> (CH3): A basis $(u_n)_{n=1}^{\infty}$ of X Banach is <u>unconditional</u> if $\forall x \in X$ the $\sum u_n^*(x) u_n$ converges to x uncond.

<u>Some useful background</u>

First, $f \in L_p$ we define Fourier coefficients as

$$\hat{f}(n) \overset{\text{def}}{=} \langle f, e^{2\pi i t} \rangle = \int_0^1 f(t) \boxed{e^{-2\pi i n t}}\, dt$$

Now define $(u_n)_{n=1}^{\infty}$ with this ordering: $u_1 = 1$; $u_2 = e^{-2\pi i t}$, $u_3 = e^{2\pi i t}$, $u_4 = e^{-4\pi i t}$, ...

$u_n \overset{\text{def}}{=} e^{2\pi i t \omega_n t}$ $\forall n \in \mathbb{N}$ with $\omega_n = (-1)^{n+1} \lfloor \frac{n}{2} \rfloor$.

<u>Thrm</u> (Trigonometric sys in L_p)

(a) $p=2$: $(u_n)_n$ is o.n.

$$f(t) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i t}$$
$$= \sum_{n=1}^{\infty} \underbrace{u_n^*(f)}_{= \hat{f}(\omega_n)} .$$

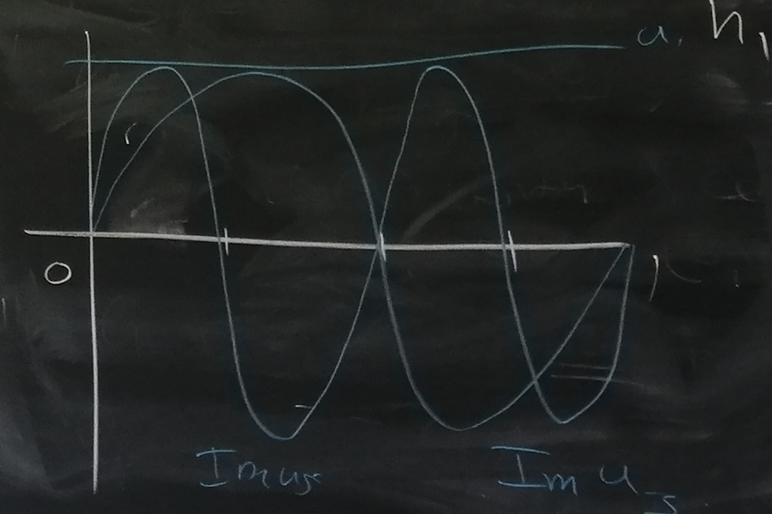
Note $(u_n)_{n=1}^{\infty}$ is unconditional in L_2.

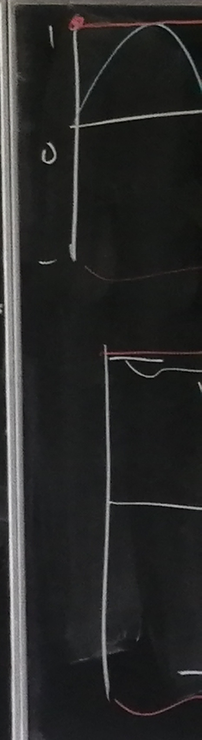
(b) $p \neq 1,2$ $\quad (u_n)_{n=1}^{\infty}$ is a conditional basis for L_p with

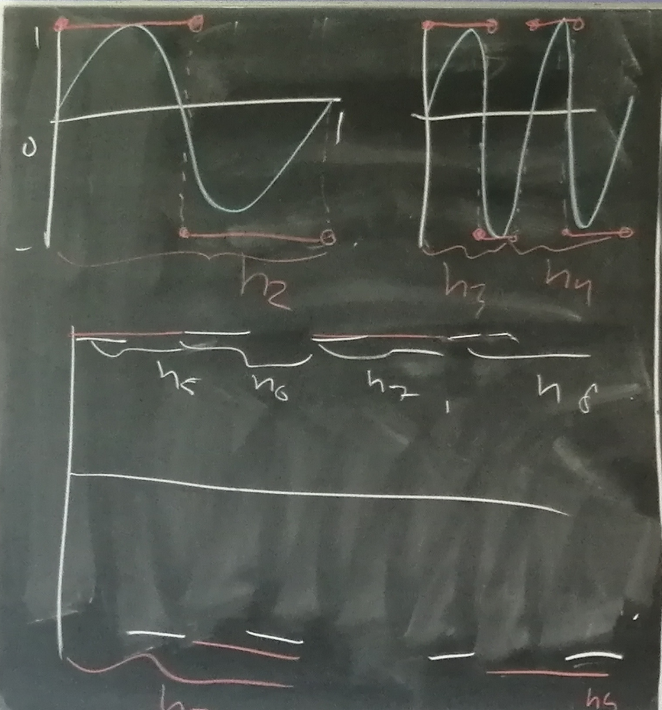
$$f = \sum_{n=1}^{\infty} u_n^*(f)\, u_n \quad \forall f \in L_p.$$

(c) $p = 1$ $\quad (u_n)_{n=1}^{\infty}$ is not a basis at all.

Now visualise the u_n:







Def The <u>Haarsystem</u> is the $(h_n)_{n=1}^{\infty}$ on $[0,1]$ with $h_1 = 1$ and for $\boxed{n = 2^k + s}$, $k \in \mathbb{N}$ and $1 \le s \le 2^k$, then

$$h_n \overset{def}{=} \mathbb{1}_{\left[\frac{2s-2}{2^{k+1}}, \frac{2s-1}{2^{k+1}}\right)} - \mathbb{1}_{\left[\frac{2s-1}{2^{k+1}}, \frac{2s}{2^{k+1}}\right)}$$

ex $n = 7 = 2^2 + 3 \Rightarrow h_n =$

2^{nd} time we halved it and 3^{rd} version

Def The $\left[\frac{s-1}{2^k}, \frac{s}{2^k}\right)$ are called dyadic intervals

Probability measure space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $X \in L_1(\Omega, \mathcal{A}, \mathbb{P})$ be a <u>real-valued random var.</u>

$$\mathbb{E}(X) \overset{def}{=} \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega) = \int_{\mathbb{R}} x \, d\mathbb{P}_X$$

$$\mathbb{P}_X(B) \overset{\text{def}}{=} \mathbb{P}(X \in B) = \mathbb{P}(X^{-1}(B)),$$

$$\forall B \in \mathcal{B}(\mathbb{R}).$$

$$\mathbb{P}_X(B) = \int_B f_X \, d\lambda$$

$d\mathbb{P}_X = f_X \, d\lambda$.

then we have (Radon - Nikodym)

$$\mathbb{E}(X) = \int_{\mathbb{R}} x \, f_X(x) \, dx$$

(Ω, Σ', μ) to be prop measure.

And $\Sigma'' \subset \Sigma'$ sub-σ-algebra.

<u>Def</u> For any $f \in L^1(\Omega, \Sigma', \mu)$.

the <u>conditional expectation</u> of f on Σ'' is the unique function (Radon-Nikod) as $\mathbb{E}(f \mid \Sigma'') \in L^1(\Omega, \Sigma'', \mu)$

and $\int_E f \, d\mu = \int_E \mathbb{E}(f \mid \Sigma'') \, d\mu \quad \forall E \in \Sigma''$

<u>Lemma 6.12</u> The lin. map

$\mathbb{E}(\cdot, \Sigma'') : L_p(\Sigma') \longrightarrow L_p(\Sigma'')$

is a projection with

$\| \mathbb{E}(\cdot \mid \Sigma'') = 1 \quad \forall p$.

<u>Proof</u> proj. ✓

$\| \mathbb{E}(\cdot, \Sigma'') \| \geq 1$

$\| \mathbb{E}(f, \Sigma'') \| \leq \| f \|_p$.

$\| \mathbb{E} \| \leq 1 \Rightarrow$

$\square$

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Prop 6.13 The Haar sys. is a monotone basis in $L_p \ \forall p$

proof σ-algebra's $(\mathcal{B}_n)_{n=1}^{\infty}$ defined $\mathcal{B}_1 = \{\emptyset, [0,1]\}$ and

$\mathcal{B}_n = \sigma(\mathcal{F}_n)$ for $n = 2^k + s$

$$\mathcal{F}_n = \left\{ \begin{array}{ll} \left[\frac{j-1}{2^{k+1}}, \frac{j}{2^{k+1}} \right), & j = 1, \ldots, s \\ \left[\frac{j-1}{2^k}, \frac{j}{2^k} \right), & j = s+1, \ldots, 2^k \end{array} \right\}$$

Fix p and n define

$\mathbb{E}_n \overset{\text{def}}{=} \mathbb{E}(\cdot \mid \mathcal{B}_n)$

Lem. $\mathbb{E}_n : L_p \rightarrow \underline{L_p([0,1], \mathcal{B}_n, \lambda)}$

$= L_p(\mathcal{B}_n)$

is a norm-one linear proj.

Recall (prop 1.17) Suppose $S_n : X \to X$ bounded lin. proj s.t.

(i) $\dim S_n(X) = n$

(ii) $S_n \circ S_m = S_m \circ S_n = S_{\min\{m,n\}}$

(iii) $S_n(x) \xrightarrow[n]{} x \quad \forall x \in X$.

$(e_k)_{k=1}^{\infty} \subset X$ s.t. $e_1 \in S_1(X)$

and $e_k \in S_k(X) \cap S_{k-1}^{-1}(0)$

will form a basis for X.

We consider proj. $(\mathbb{E}_n)_{n=1}^{\infty}$

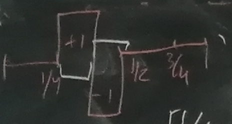
(i) $\mathbb{E}_n$ is cdv. onto ✓

(ii) Simply observe $|\mathcal{B}_m| \leq |\mathcal{B}_n|$ whenever $m \leq n$, ✓

(iii) Banach-Steinhaus theorem

$\mathbb{E}_n f \xrightarrow{L^p} f$

ex h_5



and $\mathcal{F}_3 = \{[0,1/4), [1/4,1/2), [1/2,1)\}$.

$\mathcal{F}_6 = \{[0,1/8), [1/8,1/4), [1/4,3/8), [3/8,1/2), [1/2,3/4), [3/4,1)\}$.

note $\boxed{\mathbb{E}_3 h_5 = 0}$ and $\boxed{\mathbb{E}_6 h_5 = h_5}$

In gen

$\mathbb{E}_m h_n = 0 \quad m<n$

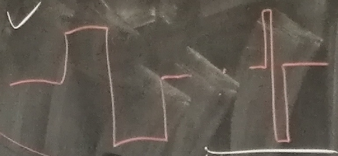
$\mathbb{E}_m h_n = h_n \quad m\geq n$

Now from 1.17 $(h_n)_{n=1}^\infty$ basis □

Remark $h_n^*(f) = \frac{1}{\|h_n\|_p^p}\int_{[0,1]} f(t)\,h_n(t)\,dt$

$m=n \quad h_n^*(h_n) = 1$ ✓

$m\neq n \quad h_n^*(h_m) = 0$



$f\in L_p$: $\sum_{n=1}^{\infty} h_n^*(f)\,h_n$.

$p>1$ it is uncondn

Observe. $p=2$, then $(h_n/\|h_n\|_2)_n$ is o.n. basis

↳ uncondition. But then Haar basis also unc

Lem. 6.15 Suppose $p>2$ tells us

$$\frac{p^{p-2}}{(p-1)^{p-1}} < 1$$

Lem 6.16.

$$\varphi(x,y) = (x+y)^{p-1}\cdot[(p-1)x - y]$$

$x,y\geq 0$

(a) $\frac{(p-1)^{p-1}}{p^{p-2}}\varphi(x,y) \le (p-1)^p x^p - y^p$

$x, y \ge 0.$

(b) $\forall x, y, a \in \mathbb{R}$ and $\varepsilon = \pm 1$

$$\varphi(|x+a|, |y+\varepsilon a|) + \varphi(|x-a|, |y-\varepsilon a|) \ge 2\varphi(|x|,|y|)$$

Proof: Straightforw...

Thm 6.17 Let $1 < p < \infty$ and $1/p + 1/q = 1$. Also $p^* = \max(p,q)$. Then $(h_n)_{n=1}^{\infty}$ is unconditional with the unc. constant $\le p^* - 1$ (prop 3.13).

That is

$$\left\|\sum_{n=1}^{N} \varepsilon_n a_n h_n\right\|_p \le (p^*-1)\left\|\sum_{n=1}^{N} a_n h_n\right\|_p \quad (*)$$

for $N \in \mathbb{N}$, $\{a_n\}_{n=1}^{N} \subset \mathbb{R}^N$, $\{\varepsilon_n\}_{n=1}^{N} \in \{\pm 1\}^N$

$1/p + 1/q = 1$

$(h_n)_{n=1}^\infty$

the

prop 3.13)

$\|_p$ (*)

$\}_{n=1}^N \subset \{\pm 1\}$

<u>Sketch</u> $p \geq 2$. Then $p^* = p$

and (*) reduces

$$\int_0^1 \varphi\Big(\Big|\sum_{n=1}^N a_n h_n(t)\Big|, \Big|\sum_{n=1}^N \varepsilon_n a_n h_n(t)\Big|\Big)\, dt \geq 0.$$

$N=1$. Lemma 6.1.6 (b)

with $x = y = 0$ and $\varepsilon = 1$

$$\varphi(|a_1|, |a_1|) \geq \varphi(0,0) = 0$$

$p = 2$ ✓

$1 < p < 2$ follows from duality □

<u>We conclude</u> the Haarsystem is unconditional for $p > 1$.

* What about $p = 1$?

§6.3 Properties of L_1.

<u>Prop 6.3.1</u> The Haar system is not unconditional in L_1.

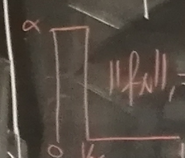
<u>Recall</u> for $A \subseteq \mathbb{N}$ the proj.

$$P_A : X \to [u_k : k \in A],\ x \mapsto \sum_{k \in A} u_k^*(x) u_k$$

<u>Recall</u> prop 3.1.5 (i–iii) we've

$(u_n)_{n=1}^\infty$ uncond. $\iff \sup_A \|P_A\| < \infty$

Now define $\boxed{f_N = 2^{1+2N}\, \mathbb{1}_{[0, 2^{-1-2N}]}}$



$$= \mathbb{1}_{[0,1]} + \sum_{j=0}^{2N} 2^j h_{[0, 2^{-j})}$$

$$P_A f_N = g_N \overset{\text{def}}{=} \sum_{j=0}^{N} 2^{2j} h_{[0, 2^{-2j})}$$

↑ for proper A

$N \to \infty$

$\|g_N\|_1 \geq \frac{N+1}{3}$

Q: does there exist a "better" system for $p=1$? No!.

Thrm 6.3.3 L_1 cannot be embedded in X Banach with uncond. basis.

↳ We do not have to look further

L_1 is "very different" in $L_{p>1}$.

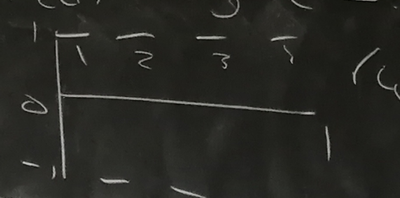
Proof thrm uses Rademacher funct.

§6.2 Averaging in Banach sp.

Def The Rademacher functions $(r_k)_{k=1}^{\infty}$ on $[0,1]$ are

$$r_k(t) = \mathrm{sgn}(\sin 2^k \pi t)$$

Visual:



Remark $r_{k+1} = \sum_{s=1}^{2^k} h_{2^k+s}$ a.e. $\forall k$

Morea

Averag

$\varepsilon_i = \pm 1$

$\frac{1}{2^n}$

$=$

This gi

(t)

$\forall h$

Moreover, $\{x_i\}_{i=1}^n \subset X$

$$\underset{\varepsilon_i = \pm 1}{\text{Average}} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\| = \frac{1}{2^n} \sum_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|$$

$$= \int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\| dt$$

This yields the Khintchine's inequalities

Thrm 6.2.12 $\forall p \ \exists A_p, B_p > 0$ s.t.

$$A_p \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2} \le \left\| \sum_{i=1}^n a_i r_i \right\|_p \le B_p \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2}$$

$\forall n \in \mathbb{N}$ and $(a_i)_{i=1}^n \subset \mathbb{R}^n$ arbitrary.

Remarks
- $1 \le p < 2 \Rightarrow B_p = 1$
- $p > 2 \Rightarrow A_p = 1$
- $p = 2$: $(r_k)_k$ o.n. basis and therefore $A_p = B_p = 1$

Generalisation of KI in $L_p(\mu)$

Thrm 6.2.13 $\forall p$ and same A_p, B_p as in 6.2.12 we have $\forall \{f_i\}_{i=1}^n \subset L_p(\mu)$:

$$A_p \left\| \left(\sum_{i=1}^n |f_i|^2 \right)^{1/2} \right\|_p \le \left(\mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i f_i \right\|_p^p \right)^{1/p} \le B_p \left\| \left(\sum_{i=1}^n |f_i|^2 \right)^{1/2} \right\|_p$$

Proof From 6.2.2 we know

$$A_p \Big(\sum_{i=1}^{n} |f_i(\omega)|^2 \Big)^{1/2} \le \Big(\mathbb{E}_\varepsilon \Big| \sum_{i=1}^{n} \varepsilon_i f_i(\omega) \Big|^p \Big)^{1/p} \quad (*)$$

$\omega \in \Omega$. Now use Fubini:

$$A_p^p \Big\| \Big(\sum_{i=1}^{n} |f_i|^2 \Big)^{1/2} \Big\|_p^p \overset{(*)}{\le} \int_\Omega \mathbb{E}_\varepsilon \Big| \sum_{i=1}^{n} \varepsilon_i f_i(\omega) \Big|^p d\mu(\omega)$$

$$\overset{\text{Fubini}}{=} \mathbb{E} \Big(\int_\Omega | \ldots |^p \, d\mu \Big)$$

$$= \mathbb{E} \Big\| \sum_{i=1}^{n} \varepsilon_i f_i \Big\|_p^p \qquad \square$$

I had to do earlier;

Def A rademacher sequence is a seq. $(\varepsilon_n)_{n=1}^{\infty}$ of pairwise independent random variables s.t.

$\mathbb{P}(\varepsilon_n = -1) = \mathbb{P}(\varepsilon_n = 1) = 1/2$.

L_1 | $L_{1<p<2}$ | L_2 | $L_{2<p<\infty}$

discontinuity

Def 6.2.10 X ...

for ...

s.t. $(\mathbb{E} \ldots$

for all

cotype ...

$\exists C > 0$ s...

ence

pairwise

ex s.t

$< p < \infty$

Def 6.2.10 X Banach is of <u>type p</u> for $1 \le p \le 2$ if $\exists C > 0$ s.t.

$$\left(\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i x_i\right\|^p\right)^{1/p} \le C \cdot \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}$$

for all $\{x_i\}_{i=1}^{n} \subset X$. Similarly <u>cotype q</u>, $2 \le q < \infty$, if $\exists C > 0$ s.t.

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \le C \left(\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i x_i\right\|^q\right)^{1/q}$$

Thm 6.2.14

(a) $1 \le p < 2 \Rightarrow L_p(\mu)$ has type p and cotype 2

(b) $p > 2 \Rightarrow L_q(\mu)$ has type 2 and cotype q.

§6.4 Subspaces of $L_p = L_p[0,1]$

* $\ell_1 \not\simeq L_1$
* $\ell_p \not\simeq L_p$ iff $p \ne 2$

"prime" $\longrightarrow$ prop 6.4.2 / ℓ_2 embeds L_p

Assume $\ell_p \simeq L_p$ $(p \ne 2)$

$$L_2 \simeq \ell_2 \underset{\substack{\text{prime} \\ + \\ \text{prop 6.4.2}}}{\simeq} \ell_p \simeq L_p \quad \lightning$$

<u>prop 6.4.2</u> ℓ_2 embeds in L_p $\forall p$

Also ℓ_2 embeds <u>complementably</u> in L_p iff $1 < p < \infty$.

$$* \quad P : f \longmapsto \sum_{n=1}^{\infty} \left(\int_0^1 f(t) r_n(t)\, dt\right)$$

Prop 6.4.3 If ℓ_q embeds in L_p
then we must have $p \le q \le 2$
or $2 \le q \le p$

Proof "follows" from thrm 6.2.14

Take a look at thrm 6.4.8

Thrm 6.4.8 (isometrically embedding)
(i) $1 \le p \le 2$, ℓ_q embeds in L_p
iff $p \le q \le 2$.
(ii) $2 < p < \infty$, ℓ_q embeds into L_p iff $q=2$ or $q=p$

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$\mathbb{P}(\varepsilon_n =$

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