Conventions

- X and Y are real Banach spaces
- (S, Σ, μ) is σ-finite measure space
- $(\Omega, \mathbb{P})$ is a probability space
- $T: X \to Y$ is bounded linear operator

We say $T: X \to Y$ factorizes through E if $\exists\, S, R$ s.t. $T = RS$

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "S"'] & & Y \\ & E \arrow[ur, "R"'] & \end{tikzcd}$$

- If $I_X: X \to X$ factorizes through E, then X is isomorphic to a complemented subspace of E
- If T factorizes through a reflexive E, then T is weakly compact

Maurey – Nikishin factorization

Def a function $f \in L^0(S, \mu)$ is called a density if $f > 0$ and $\int_S f \, d\mu = 1$. Note that $h\,d\mu$ is a probability measure. In particular $L^q(S, h\,d\mu) \overset{c}{\hookrightarrow} L^p(S, h\,d\mu)$ for $1 \le p \le q$

Theorem

Let T: X

a) $\Big(\int_S |T$

b) $\Big\|\Big(\sum_{k=1}^{n} |T$

Moreover C

Theorem 7.12

Let $T: X \to L^p(S,\mu)$. Then TFAE $\quad 1 \le p < q \quad \exists h \in L^1$

a) $\left(\int_S |Tx|^q h^{1-\frac{q}{p}} \, d\mu \right)^{1/q} \le C_1 \|x\| \quad \forall x \in X$

b) $\left\| \left(\sum_{k=1}^n |Tx_k|^q \right)^{1/q} \right\|_{L^p} \le C_2 \left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \quad \forall x_1 \dots x_n \in X$

Moreover $C_1 \simeq C_2$

Note

a) states

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "h^{-\frac{1}{p}} \circ T"'] & L^p(S,\mu) \\ L^q(S, h\,d\mu) \arrow[r, hook] & L^p(S, h\,d\mu) \arrow[u, "h^{1/p}"'] \end{tikzcd}$$

b) is more restrictive than boundedness of

Dual Form

$X \quad$ X q-convex Banach function space

Let $T: \cancel{L^p(S,\mu)}\ X \to Y \quad 1 \le q < p$. TFAE $\quad \exists h \in (X^q)^*$

a) $\|Tx\| \le C_1 \left(\int_S |x|^q h^{\cancel{1-\frac{q}{p}}} \, d\mu \right)^{1/q} \quad x \in \cancel{L^p(S,d\mu)}\ X$

b) $\left(\sum \|Tx_k\|^q \right)^{1/q} \le C_2 \left\| \left(\sum |x_k|^q \right)^{1/q} \right\|_{L^p(S,d\mu)} \quad x_1 \dots x_n \in \cancel{L^p(S,\mu)}\ X$

Theorem 7.12

Let $T : X \to L^p(S,\mu)$. Then TFAE $1 \le p < q$ $\exists h \in L^1$

a) $\left(\int_S |Tx|^q \, h^{1-\frac{q}{p}} \, d\mu \right)^{1/q} \le C_1 \|x\| \quad \forall x \in X$

b) $\left\| \left(\sum_{k=1}^n |T x_k|^q \right)^{1/q} \right\|_{L^p} \le C_2 \left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \quad \forall x_1, \dots, x_n \in X$

Moreover $C_1 \simeq C_2$

Note

a) states

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[d, "h^{-\frac{1}{p}} \circ T"'] & & L^p(S,\mu) \\ L^q(S, h\,d\mu) \arrow[rr, hook] & & L^p(S, h\,d\mu) \arrow[u, "h^{1/p}"'] \end{tikzcd}$$

b) is more restrictive than boundedness of

Dual Form

Let $T : L^p(S,\mu) \to Y$ $1 \le q < p$. TFAE

a) $\|Tx\| \le C_1 \left(\int_S |x|^q \, h^{1-\frac{q}{p}} \, d\mu \right)^{1/q} \quad x \in L^p(S, d\mu)$

b) $\left(\sum \|T x_k\|^q \right)^{1/q} \le C_2 \left\| \left(\sum |x_k|^q \right)^{1/q} \right\|_{L^p(S, d\mu)} \quad x_1, \dots, x_n \in L^p(S,\mu)$

$$h \in L^1 \iff h^{\frac{p-q}{p}} \in L^{\frac{p}{p-q}} = L^{\left(\frac{p}{q}\right)'}$$

proof of Theorem 7.12

a) ⟹ b) Take $x_1, \dots, x_n \in X$

$$\left(\int_S \left(\sum_{k=1}^n |Tx_k|^q\right)^{p/q} d\mu\right)^{1/p} =$$

$$\left(\int_S \left(\sum_{k=1}^n |Tx_k|^q h^{-\frac{q}{p}}\right)^{p/q} h \, d\mu\right)^{1/p}$$

$$\leq \left(\int_S \sum_{k=1}^n |Tx_k|^q h^{-\frac{q}{p}} \cdot h \, d\mu\right)^{1/q}$$

$$\leq C_1 \left(\sum_{k=1}^n \|x_k\|^q\right)^{1/q}$$

b) ⟹ a) Homework □

Recall from Chapter 6

- A Rademacher sequence is iid sequence $(\varepsilon_n)_{n=1}^\infty$ on $(\Omega, \mathbb{P})$ st $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = \frac{1}{2}$ $\forall n \in \mathbb{N}$

- X has type p for $1 \leq p \leq 2$ if

$$\left(\mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|^p\right)^{1/p} \leq \tau_{p,X} \left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p} \quad x_1, \dots, x_n \in X$$

and cotype $2 \leq q < \infty$ if "$\geq$" holds (with constant $c_{q,X}$)

- The p on LHS can equivalently be replaced by $\tilde{p} \in [1, \infty)$

Proposition 6.2.9 (Parallelogram rule)

Let H be a Hilbert space an $x_1 \dots x_n \in H$ Then

$$\mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|^2 = \sum_{k=1}^n \|x_k\|^2$$

Theorem 7.12

Let $T: X \to$

a) $\left(\int_S |Tx|^q\right.$

b) $\left\| \left(\sum_{k=1}^n |Tx_k\right.\right.$

Moreover $C_1 \simeq C$

$(\varepsilon_n)_{n=1}^{\infty}$

$\forall n \in \mathbb{N}$

$x_1 \ \ x_n \in X$

$C_{q,X})$

$\in [1,\infty)$

Theorem 7.12

Let $T : X \to L^p(S,\mu)$. Then TFAE $1 \leq p < q$ $\exists h \in L^1$

a) $\left(\int_S |Tx|^q h^{1-\frac{q}{p}} \, d\mu \right)^{1/q} \leq C_1 \|x\| \quad \forall x \in X$

b) $\left\| \left(\sum_{k=1}^{n} |T x_k|^q \right)^{1/q} \right\|_{L^p} \leq C_2 \left(\sum_{k=1}^{n} \|x_k\|^q \right)^{1/q} \quad \forall x_1 \dots x_n \in X$

Moreover $C_1 \simeq C_2$

i.e. $\dfrac{\|x_1+x_2\| + \|x_1-x_2\|}{2} = \|x_1\|^2 + \|x_2\|^2$

proof

$$\mathbb{E} \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|^2 = \mathbb{E} \left\langle \sum_{k=1}^{n} \varepsilon_k x_k , \sum_{j=1}^{n} \varepsilon_j x_j \right\rangle$$

$$= \sum_{j,k=1}^{n} \langle x_k, x_j \rangle \, \mathbb{E}(\varepsilon_k \varepsilon_j)$$

$$= \sum_{k=1}^{n} \|x_k\|^2$$

$\square$

Corollary

A Hilbert space has type and cotype 2, with $T_{2,H} = C_{2,H} = 1$

Theorem (Jordan von Neumann)

If X has type 2 with $\tau_{2,X} = 1$, then X is isometrically a Hilbert space

proof

Note that X also has cotype 2 with $c_{2,X} = 1$

Define $\langle x, y \rangle = \frac{\|x+y\|^2}{4} - \frac{\|x-y\|^2}{4}$

Use parallelogram rule to check that this is an inner product $\square$

Theorem 7.9.1 (Kwapien '72)

If X has type 2 and cotype 2, then X is isomorphic to a Hilbert space

Back' §7.1

Theorem 7.1.4

Let $T: X \to L^p(S,\mu)$ and suppose X has type 2, Then

b) holds for $q=2$

proof

$x_1, \dots x_n \in X$

$$\left\| \left(\sum_{k=1}^n |Tx_k|^2 \right)^{1/2} \right\|_{L^p} \overset{6.2.13}{\approx} \left(\mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k T x_k \right\|_{L^p}^2 \right)^{1/2}$$

$$\leq \|T\| \left(\mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|^2 \right)^{1/2}$$

$$\leq \|T\| \tau_{2,X} \left(\sum \|x_k\|^2 \right)^{1/2}$$

$\square$

Theorem 7.12

Let $T: X \to L^p(S,\mu)$

a) $\left(\int_S |Tx|^q h^{1-\frac{q}{p}} \, d\mu \right.$

b) $\left\| \left(\sum_{k=1}^n |T x_k|^q \right)^{1/q} \right\|_{L^p(}$

Moreover $c_1 \approx c_2$

7.15

a) Every $T: L^r(S,\mu) \to$

factors through a Hil

b) If X is isomorphic to

e X has type 2, Then

$\ldots \varepsilon_k T x_k \|^2_{L^p}\Big)^{1/2}$

$\ldots \varepsilon_k x_k \|^2\Big)^{1/2}$

Theorem 7.12

Let $T: X \to L^p(S,\mu)$. Then TFAE $1 \leq p < q$ $\exists h \in L^1$

a) $\left(\int_S |Tx|^q h^{1-\frac{q}{p}} \, d\mu\right)^{1/q} \leq C_1 \|x\| \quad \forall x \in X$

b) $\left\| \left(\sum_{k=1}^n |Tx_k|^q \right)^{1/q} \right\|_{L^p(S,\mu)} \leq C_2 \left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \quad x_1, \ldots, x_n \in X$

Moreover $C_1 \approx C_2$

7.15

a) Every $T: L^r(S,\mu) \to L^p(S,\mu)$ for $1 \leq p \leq 2 \leq r < \infty$ factors through a Hilbert space

b) If X is isomorphic to a closed subspace of L^p and L^r for $1 \leq p \leq 2 \leq r$, then X is isomorphic to a Hilbert space

proof

a) Direct from 7.12 and 7.14 with $H = L^2(S, h\,d\mu)$

b)

$$\begin{tikzcd} X \arrow[r, "I_X"] \arrow[d] & X \\ L^r \arrow[r, "\tilde{I}_X"] \arrow[dr, no head] & L^p \arrow[u] \\ & H \arrow[u] \end{tikzcd}$$

Remark

Key in 7.1.5 is the connection between Rademachers and Square Sums (Theorem 6.2.13) To use 7.1.2 for $q \neq 2$ we need a replacement

Lemma 7.1.8

Let $1 \leq p < q < 2$. There exist i.i.d. $(\eta_n)_{n=1}^{\infty}$ in $L^p(\Omega)$ such that $\forall f_1, \dots, f_n \in L^p(S,\mu)$

$$\left\| \left(\sum_{k=1}^{n} |f_k|^q \right)^{1/q} \right\|_{L^p} = c \left(\mathbb{E} \left\| \sum_{k=1}^{n} \eta_k f_k \right\|_{L^p}^p \right)^{1/p}$$

We call η_n a q-stable random variable

Theorem 7.1.9

Let $T: X \to L^p(S,\mu)$ and suppose X has type r for $1 \leq p < r \leq 2$. Then T factors through L^q for all $p < q < r$

Remark If $r = 2$, T also factors through L^2. For $r < 2$ we can not take $r = q$

Section 7.3 + 7.4 Factoring through a Hilbert space

Theorem 7.4.2 (Kwapien–Maurey)

Let $T: X \to Y$ and suppose X has type 2 and Y has cotype 2. Then there is a Hilbert space H and operators $R: X \to H$, $S: H \to Y$ s.t. $T = SR$ and $\|S\| \cdot \|R\| \leq C \|T\|$

Theorem 7.3.4

For $T: X \to Y$ and $C \geq 0$ TFAE

a) There is a Hilbert space H and operators $R: X \to H$, $S: H \to Y$ s.t. $T = SR$ and $\|S\| \|R\| \leq C \|T\|$

b) For ...

then ...

proof

a) $\Rightarrow$ b)

Take x ...

ONB

$\sum_{j=1}^{m} \|$ ...

Def A function $\Omega \to \mathbb{R}$ is called a standard Gaussian if its distribution has density

$$f(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

a Gaussian sequence is an iid sequence of Gaussians $(\gamma_n)_{n\geq 1}$.

$\gamma = a_1\gamma_1 + \dots + a_n\gamma_n$ $\quad \|a\|_{\ell^2} = 1$

Def X has Gaussian type $1 \leq p \leq 2$ if

$$\left(\mathbb{E} \left\| \sum_{k=1}^{n} \gamma_k x_k \right\|^p \right)^{1/p} \leq \tau^{\gamma}_{p,X} \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p}$$

Gaussian cotype is defined analogously

Section 7.3 + 7.4 Factoring through a Hilbert Space

Theorem 7.4.2 (Kwapien-Maurey)

Let $T: X \to Y$ and suppose X has type 2 and Y has cotype 2

Then there is a Hilbert space H and operators $R: X \to H$

$S: H \to Y$ st. $T = SR$ and $\|S\| \cdot \|R\| \leq C\|T\|$

Theorem 7.3.4

For $T: X \to Y$ and $C \geq 0$ TFAE

a) There is a Hilbert space H and operators $R: X \to H$ $S: H \to Y$ st

$T = SR$ and $\|S\| \, \|R\| \leq C\|T\|$

also factors through ℓ^2 for $r<2$ we

Factoring through a Hilbert Space

...ien-Maurey)

suppose X has type 2 and Y has cotype 2,

...lbert space H and operators $R: X \to H$

$T = SR$ and $\|S\| \cdot \|R\| \leq C \|T\|$

...o TFAE

...ce H and operators $R: X \to H$ $S: H \to Y$ s.t.

$\|R\| \leq C \|T\|$

b) For all $x_1 \ldots x_n, z_1 \ldots z_m \in X$ s.t.:

$$\sum_{j=1}^m |x^*(z_j)|^2 \leq \sum_{k=1}^n |x^*(x_k)|^2 \qquad x^* \in X^*$$

then

$$\sum_{j=1}^m \|T z_j\|^2 \leq C^2 \|T\|^2 \sum_{k=1}^n \|x_k\|^2$$

<u>proof</u>

a) $\Rightarrow$ b) $T = I_X : X \to X$ Assume wlog that X is a Hilbert space

Take $x_1 \ldots x_n\ z_1 \ldots z_m \in X$ as in b) Let $(h_i)_{i=1}^{\ell}$ be a ONB of $\operatorname{span}\{x_k, z_j\}$ Then

$$\sum_{j=1}^m \|z_j\|^2 = \sum_{i=1}^{\ell} \sum_{j=1}^m \langle z_j, h_i \rangle^2 \leq \sum_{i=1}^{\ell} \sum_{k=1}^n \langle x_k, h_i \rangle^2 = \sum_{k=1}^n \|x_k\|^2$$

$\square$

ℓ^p and ℓ^r for $1 \leq p < 2 \leq r$

<u>proof</u>

a) Direct from 7.12 and 7.14 w...

b

$$\begin{tikzcd}
X \arrow[rr, "I_X"] \arrow[d] & & X \\
\ell^r \arrow[rr, "\tilde{I}_X"] \arrow[dr] & & \ell^p \arrow[u] \\
& H \arrow[ur] &
\end{tikzcd}$$

<u>Remark</u>

key in...

Rademac...

7.12

has cotype 2,

$X \to H$

$\|T\|$

$\to V$ s.t.

b) For all $x_1 \dots x_n, z_1 \dots z_m \in X$ s.t.:

$$(\circledast) \quad \sum_{j=1}^{m} |x^*(z_j)|^2 \leq \sum_{k=1}^{n} |x^*(x_k)|^2 \qquad x^* \in X^*$$

then

$$\sum_{j=1}^{m} \|Tz_j\|^2 \leq C^2 \|T\|^2 \sum_{k=1}^{n} \|x_k\|^2$$

Lemma A

If X has (co)type p, then X has Gaussian (co)type p with $\tau^{\gamma}_{p,X} \leq \tau_{p,X}$ $\quad C^{\gamma}_{p,X} \leq C_{p,X}$

proof

$$\left(\mathbb{E}_\gamma \Big\| \sum_{k=1}^{n} \gamma_k x_k \Big\|^p\right)^{1/p} = \left(\mathbb{E}_\varepsilon \mathbb{E}_\gamma \Big\| \sum_{k=1}^{n} \gamma_k \cdot \varepsilon_k x_k \Big\|^p\right)^{1/p}$$

$$\leq \tau_{p,X} \left(\mathbb{E}_\gamma \left(\sum_{k=1}^{n} \|\gamma_k x_k\|^p\right)\right)^{1/p}$$

$$= \tau_{p,X} \left(\sum_{k=1}^{n} \mathbb{E}_\gamma |\gamma_k|^p \|x_k\|^p\right)^{1/p}$$

$$= \tau_{p,X} \|\gamma\|_{L^p(\Omega)} \left(\sum_{k=1}^{n} \|x_k\|^p\right)^{1/p}$$

Lemma 7.4.3

Let $x_1 \dots x_n, z_1 \dots z_m$ such that $\circledast$ holds. Then

$$\mathbb{E} \Big\| \sum_{j=1}^{n} \gamma_j z_j \Big\|^2 \leq \mathbb{E} \Big\| \sum_{k=1}^{n} \gamma_k x_k \Big\|^2$$

proof

Define $V := \{ (x^*(x_k))_{k=1}^{n} : x^* \in X^* \} \subseteq \ell^2_n$

Then $A: V \to \ell^2_m$ given by $(x^*(x_k))_{k=1}^{n} \longmapsto (x^*(z_j))_{j=1}^{m}$

Then $\|A\| \le 1$, thus A exteds boundedly to

$A : \ell_n^2 \to \ell_m^2$ with $\|A\| \le 1$ Then A is given by a matrix $(a_{jk})_{j,k=1}^{m,n}$ and thus

$$x^*(z_j) = \sum_{k=1}^{n} a_{jk}\, x^*(x_k) \qquad \forall\, x^* \in X^*$$

Therefore

$$z_j = \sum_{k=1}^{n} a_{jk}\, x_k$$

We conclude

$$\mathbb{E} \left\| \sum_{j=1}^{m} \gamma_j z_j \right\|^2 = \mathbb{E} \left\| \sum_{j=1}^{m} \gamma_j \sum_{k=1}^{n} a_{jk} x_k \right\|^2$$

$$= \mathbb{E} \left\| \sum_{k=1}^{n} x_k \cdot \sum_{j=1}^{m} \gamma_j a_{jk} \right\|$$

$$\le \mathbb{E} \left\| \sum_{k=1}^{n} \gamma_k x_k \right\|^2 \qquad \square$$

Section 7.3 + 7.4 Factoring through a Hilbert Space

Theorem 7.4.2 (Kwapien–Maurey)

Let $T : X \to Y$ and suppose X has type 2 and Y has cotype 2,

Then there is a Hilbert space H and operators $R : X \to H$

$S : H \to Y$ st $T = SR$ and $\|S\| \cdot \|R\| \le C \|T\|$

Theorem 7.3.4

For $T : X \to Y$ and $C \ge 0$ TFAE

a) There is a Hilbert space H and operators $R : X \to H$ $S : H \to Y$ st

$T = SR$ and $\|S\| \|R\| \le C \|T\|$

b) For all $x_1 \dots x_n, z$

$(*)$ $\sum_{j=1}^{m} |x^*(z_j)|^2$

then

$$\sum_{j=1}^{m} \|T z_j\|^2$$

proof of Theore

It suffices to show

has cotype 2

$X \to H$

$T\|$

x st

b) For all $x_1 \dots x_n, z_1 \dots z_m \in X$ st:

$$\circledast \quad \sum_{j=1}^{m} |x^*(z_j)|^2 \leq \sum_{k=1}^{n} |x^*(x_k)|^2 \qquad x^* \in X^*$$

then

$$\sum_{j=1}^{m} \|Tz_j\|^2 \leq C^2 \|T\|^2 \sum_{k=1}^{n} \|x_k\|^2$$

<u>proof of Theorem 7.4.2</u>

It suffices to show b). So take $x_1 \dots x_n, z_1 \dots z_m \in X$ st $\circledast$ holds

$$\sum_{j=1}^{m} \|Tz_j\|^2 \leq (C_{2,Y}^{\gamma})^2 \, \mathbb{E}\Big\|\sum_{j=1}^{m} \gamma_j T z_j\Big\|^2$$

$$\leq (C_{2,Y}^{\gamma} \|T\|)^2 \, \mathbb{E}\Big\|\sum_{j=1}^{m} \gamma_j z_j\Big\|^2$$

$$\leq (C_{2,Y}^{\gamma} \|T\|)^2 \, \mathbb{E}\Big\|\sum_{k=1}^{n} \gamma_k x_k\Big\|^2$$

$$\leq (C_{2,Y}^{\gamma} \tau_{2,X} \|T\|)^2 \sum_{k=1}^{n} \|x_k\|, \text{ which proves b) with}$$

$$C = C_{2,Y}^{\gamma} \tau_{2,X}^{\gamma}$$

$\square$

<u>Lemma 7.4.3</u>

Let $x_1 \dots x_n, z_1 \dots z_m$ such that $\circledast$ holds. Then

$$\mathbb{E}\Big\|\sum_{j=1}^{n} \gamma_j z_j\Big\|^2 \leq \mathbb{E}\Big\|\sum_{k=1}^{n} \gamma_k x_k\Big\|^2$$

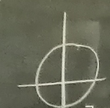
<u>proof</u>

Define $V := \{ (x^*(x_k))_{k=1}^{n} : x^* \in X^* \} \subseteq \ell_n^2$

Then $A: V \to \ell_m^2$ given by $(x^*(x_k))_{k=1}^{n} \longmapsto (x^*(z_j))_{j=1}^{m}$

For $f \in L^1(\mathbb{T};X)$ define the Fourier transform

$$\hat{f}(k) := \int_{\mathbb{T}} f(t) e^{2\pi i k t} \, dt \qquad k \in \mathbb{Z}$$


$z=1$

Theorem (Plancherel)

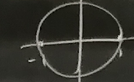
Let X be a Hilbert space The Fourier transform extends to an isometry from $L^2(\mathbb{T};X)$ to $\ell^2(\mathbb{Z};X)$

Theorem

Suppose that the Fourier transform extends to an isomorphism from $L^2(\mathbb{T};X)$ to $\ell^2(\mathbb{Z};X)$, Then X is isomorphic to a Hilbert space

proof

Let $x_1, \ldots, x_n \in X$ and let $(\varepsilon_k)_{k=1}^n$ be complex Rademacher sequence. Then



$$\int_\Omega \Big\| \sum_{k=1}^n \varepsilon_k x_k \Big\|^2 d\mathbb{P} = \int_{\mathbb{T}} \int_\Omega \Big\| \sum_{k=1}^n \varepsilon_k e^{2\pi i k t} x_k \Big\|^2 d\mathbb{P} \, dt$$

$$= \int_\Omega \int_{\mathbb{T}} \Big\| \sum_{k=1}^n e^{2\pi i k t} \varepsilon_k x_k \Big\|^2 dt \, d\mathbb{P}$$

$$\leq C \int_\Omega \sum_{k=1}^n \| \varepsilon_k x_k \|^2 \, d\mathbb{P}$$

$$= C \sum_{k=1}^n \| x_k \|^2$$

So X has type 2

$\dots x_n \in X$ and let $(\varepsilon_k)_{k=1}^{n}$ be complex Rademacher

... Then

$$\dots \varepsilon_k x_k\|^2 d\mathbb{P} = \int_{\mathbb{T}}\int_{\Omega} \|\sum_{k=1}^{n} \varepsilon_k e^{2\pi i k t} x_k\|^2 d\mathbb{P}\, dt$$

$$= \int_{\Omega}\int_{\mathbb{T}} \|\sum_{k=1}^{n} e^{2\pi i k t} \varepsilon_k x_k\|^2 dt\, d\mathbb{P}$$

$$\leq C \int_{\Omega} \sum_{k=1}^{n} \|\varepsilon_k x_k\|^2 d\mathbb{P}$$

$$= C \sum_{k=1}^{n} \|x_k\|^2$$

$\boxed{\text{... has type 2}}$

b) For all $x_1 \dots x_n, z_1 \dots z_n \in X$ s.t.:

$$\circledast \quad \sum_{j=1}^{m} |x^*(z_j)|^2 \leq \sum_{k=1}^{n} |x^*(x_k)|^2 \qquad x^* \in X^*$$

then

$$\sum_{j=1}^{m} \|Tz_j\|^2 \leq C^2 \|T\|^2 \sum_{k=1}^{n} \|x_k\|^2$$

Analogously X has cotype 2, so by theorem 7.9.1 X is isomorphic to a Hilbert space

$$\leq (C_{2,Y}\, \tau_{2,X} \|T\|)^2 \sum_{k=1}^{\dots} \dots$$

Lemma 7.9.3

Let $x_1 \dots x_n, z_1 \dots z_m$ such th...

$$\mathbb{E} \|\sum_{j=1}^{n} \gamma_j z_j\|^2 \leq \dots$$

proof

Define $V := \{ (x^*(x_k))_{k=1}^{n} \dots$

Then $A: V \to \ell^2_m$ given b...