def Basis (e_n), TFAE

1) (e_n) unconditional
2) $\forall x$, $\sum_n e_n^*(x) e_n$ converges uncond
3) $(e_{\pi(n)})$ basis $\forall \pi$ perm. of $\mathbb{N}$
4) $\exists K \geq 1$ s.t.

$$\left\| \sum_{n=1}^{m} a_n e_n \right\| \leq K \cdot \left\| \sum_{n=1}^{m} b_n e_n \right\|$$

for all $m \geq 1$, $a_1, \dots, a_m, b_1, \dots, b_m$ s.t. $|a_k| \leq |b_k|$

def Let (x_n), (y_n) basis, TFAE

1) (x_n) and (y_n) are equiv.
2) $\forall (a_n)$ scalars $\sum_n a_n x_n$ converges $\iff$ $\sum_n a_n y_n$
3) $\exists$ bounded isom $T : [x_n] \to [y_n]$ s.t. $T x_n = y_n$
4) $\exists C \geq 1$ s.t. $\forall (a_n)$ scalars almost all 0,

$$C^{-1} \left\| \sum_n a_n x_n \right\| \leq \left\| \sum_n a_n y_n \right\| \leq C \left\| \sum_n a_n x_n \right\|$$

lemma

block

is e

def A

is c

lemma 21.1 Every normalised block basic seq of c_0, ℓ_p, $1 \le p < \infty$, is equivalent to the standard basis

def A block basic seq. (u_n) of some basis (e_n)

$$u_n = \sum_{i=p_{n-1}+1}^{p_n} a_i e_i$$

is const-coeff b.b.s if $\forall n$ $\exists c_n\ \exists A_n \subset \{p_{n-1}+1, \dots, p_n\}$.

$$u_n = c_n \sum_{i \in A_n} e_i$$

def A basis (e_n) is perfectly homogeneous if it is equiv. to every normalised const-coeff b.b.s. of (e_n)

note (e_n) a perf. hom basis, $\Rightarrow$ (e_n) unconditional

lemma 9.1.3 If (e_n) is a norm. perf. hom. basis, then $\exists C \ge 1$. $\forall$ c.c. b.b.s (u_n) $\forall (a_n)$ scalars almost all 0

$$C^{-1} \left\| \sum_n a_n e_n \right\| \le \left\| \sum_n a_n u_n \right\| \le C \left\| \sum_n a_n e_n \right\|$$

thm 9.1.8 (Zippin) If X Banach space, (e_n) is norm. perf. hom basis of X, then (e_n) is equiv. to the standard basis of c_0 or ℓ_p, $1 \le p < \infty$

proof Suppose that $\sup_N \{ \| \sum_{n=1}^{N} e_n \| \} < \infty$

So arbitrary scalars (a_n), almost all 0.

$$\| \sum_n a_n e_n \| \le K \, \| \sum_n \sup_\ell |a_\ell| \, e_n \|$$

$$\le K \sup_\ell |a_\ell| \cdot \| \sum_n e_n \|$$

$$\le K \underbrace{\sup_\ell |a_\ell|}_{= \ell_\infty \text{ norm of } (a_1, a_2, \dots)} M$$

$$\frac{1}{KM} \sup_\ell |a_\ell| = \frac{1}{KM} |a_M| = \frac{1}{KM} \| a_M e_M \|$$

$$\le \frac{1}{KM} K \, \| \sum_n a_n e_n \|$$

$$\le \| \sum_n a_n e_n \|$$

So (e_n) is equiv. to std. of c_0 $\square$

def A basis (e_n) is symmetric if (e_n) is equiv to $(e_{\pi(n)})$ $\forall \pi$ perm of $\mathbb{N}$.

lemma 9.2.2 If (e_n) is sym. basis, then $\exists D \ge 1$ s.t. $\forall$ scalars (a_n) almost all 0

$$D^{-1} \| \sum_n a_n e_n \| \le \| \sum_n a_n e_{k_n} \| \le D \, \| \sum_n a_n e_n \|$$

for all distinct nat. numbers (k_n) (almost all 0)

thm 8.3.3 ...

unique

(up to

thm 8.3. ...

at leas

bases

symmetric
to
d.
s sym
s.t.
numbers

thm 8.3.3/8.3.5 c_0, ℓ_1, ℓ_2 have a unique unconditional basis (up to equiv)

thm 8.3.7 For $1<p<\infty$, $p\neq 2$, ℓ_p has at least 2 non-equiv. uncond. bases.

thm 9.3.1 (Lindenstrauss-Zippin) A Banach X has a unique uncond basis (up to equiv.) $\iff X \approx c_0, \ell_1$ or ℓ_2.

proof Let (x_n) be such a norm. uncond. basis. Then $(x_{\pi(n)})$ are bases, and they are all equiv to (x_n). So (x_n) is symm.

$\Downarrow$ It follows that

(x_n) is a perf. hom. basis.

Zippen $\Rightarrow$ (x_n) is equiv. to std. basis of $c_0, \ell_p, 1\le p<\infty$

So every uncond. basis of X equiv. to this std. basis.

thm 9.1.8 (Zippin) If X Banach space, (e_n) is norm. perf. hom. basis of X, then (e_n) is equiv. to the standard basis of c_0 or ℓ_p, $1\le p<\infty$

So the std is not from ℓ_p, $p\neq 1,2$. Then $X \approx c_0, \ell_1$, or ℓ_2

thm 9.4.2 If (e_n) is uncond. basis s.t every block basic seq. (u_n) of a perm of (e_n) spans a complemented subsp $[u_n]$

Then (e_n) equiv. to std. basis of c_0, ℓ_p, $1 \le p < \infty$

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thm (Lindenstrauss–Tzafriri, 1970). X Banach space, then every closed subsp. is complemented $\Leftrightarrow$ X is Hilbert

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thm 9.4.4 Let X be a Banach space with an unconditional basis. If every closed subsp. of X is complemented, then $X \approx \ell_2$.

---

proof Let (x_n) be an uncond. basis of X.

Then (x_n) is equiv. to std. basis of c_0, ℓ_p $1 \le p < \infty$

Suppose that $\underbrace{c_0, \ell_p}$ is ℓ_p, $p \neq 1, 2$.

Then X contains an uncond. basis (u_n) that is not eq to the std. basis of

So apply arg at start $\rightsquigarrow$ ↯

So (x_n) is equiv. with the std. basis of $\cancel{c_0}$, $\cancel{\ell_1}$, or ℓ_2.

there is closed that is not complemented $\square$

def

def. a basis is <u>conditional</u> if it is not uncond.

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examples.

- c_0 (3.1.2) summing basis

  $f_n = e_1 + \cdots + e_n$

- ℓ_1:

  $e_1,\ e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4, \ldots$

- ℓ_2 <u>thm 9.5.2</u>

  $f_{2n-1} = e_{2n-1}$

  $f_{2n} = e_{2n} + \sum_{j=1}^{n} a_j e_{2n+1-2j}$

  with $\sum_n a_n = \infty,\quad \sum_n n a_n^2 < \infty$

  eg $a_n = \frac{1}{n \log n}$

Thm 9.5.6 (Pełczyński-Singer)

X is Banach space, with a basis, then X also has a cond basis

---

proof Suppose every basis of X is uncond. Pick a uncond. basis (x_n).

Let (u_n) be a $\overset{\text{uncond}}{\text{b.b.s.}}$ of (x_n)

Then $\exists (f_n)$ basis of X with (u_n) as subseq.

Since (f_n) is uncond and (u_n) is subseq, $[u_n]$ is complemented

Can apply same reasoning to every perm. of (x_n)

Thus every b.b.s. of every perm of (x_n) spans a compl subsp

Thm 9.4.2 $\Rightarrow$ (x_n) is equiv to the std. of c_0 or ℓ_p, $1 \le p < \infty$

Then is not ℓ_p $p \neq 1, 2$

$X \approx c_0, \ell_1$ or ℓ_2 $\rightsquigarrow$ $\lightning$