

Completely bounded maps. (All C^* algebras are
 cp maps. (assumed to be unital)
 cb maps.

(Arens's extension thm.) $S \in A$
 operator system. Any $S \mapsto B(\mathcal{H})$
 cp. extends to A

(Stinespring's repr. thm.) $A \xrightarrow{\phi} B(\mathcal{H})$
 cp. Then there exists Hilbert
 space $\mathcal{K}$. repr. $A \xrightarrow{\pi} B(\mathcal{K})$ and
 $V: \mathcal{H} \rightarrow \mathcal{K}$ bounded, s.t.
 $\phi(a) = V^* \pi(a) V$

(Wittstock's extension thm.) MCA
 operator space. Any $M \xrightarrow{\phi} B(\mathcal{H})$
 cb. extends to $A \xrightarrow{\psi} B(\mathcal{H})$ with

$\|V\|_{cb} = \|\phi\|_{cb}$
 (Generalised Stinespring's repr. thm.)
 $A \xrightarrow{\phi} B(\mathcal{H})$ cb. Then there exists
 a Hilbert space $\mathcal{K}$. repr. $A \xrightarrow{\pi} B(\mathcal{K})$
 and $V_1, V_2: \mathcal{H} \rightarrow \mathcal{K}$ bounded, s.t.
 $\phi(a) = \sum_{i=1}^n \pi(a) V_i$

• Extension and repr. thms for cb maps.

Technique: 'off-diagonal embedding'

• Schur multipliers

"bimodule maps"

Lemma 8.1. $M \subset A$ operator space. Then $M_n(A)$

$\phi: M \rightarrow B$ cp. contractive $\Leftrightarrow \hat{\phi}: \bigcup_n M_n \rightarrow M_n(B)$ cp.

where $S_n = \left\{ \begin{pmatrix} \lambda & a \\ b^* & \mu \end{pmatrix} \mid \lambda, \mu \in \mathbb{C}, a, b \in M \right\}$ $\hat{\phi} \begin{pmatrix} \lambda & a \\ b^* & \mu \end{pmatrix} = \begin{pmatrix} \lambda & \phi(a) \\ \phi(b)^* & \mu \end{pmatrix}$

Proof $A \hookrightarrow B$ bounded linear $\leadsto M_n(A) \xrightarrow{\phi_n} M_n(B)$

$$\begin{array}{ccc} M_n(S_n) & \longrightarrow & M_n(M_n(B)) \\ \downarrow \cong & & \downarrow \cong \\ M_n(\bigcap_n M_n(A)) & \xrightarrow{\quad} & A \otimes M_n \\ \downarrow \cong & & \downarrow \cong \\ A \otimes M_n \otimes M_n & \xrightarrow{\quad} & M_n(M_n(A)) \end{array}$$

$$\begin{pmatrix} \square & \square & \square & \square & \vdots \\ \Delta & X & \Delta & X & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{n \times n} \longrightarrow \begin{pmatrix} \square & \square & \square & \square & \vdots \\ \Delta & \Delta & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_n$$

$$\begin{pmatrix} H & A \\ B^* & K \end{pmatrix} \longmapsto \begin{pmatrix} H & \phi_n(A) \\ \phi_n(B)^* & K \end{pmatrix}$$

$H, K \in M_n(\mathbb{C})$

If $\begin{pmatrix} H & A \\ B^* & K \end{pmatrix}$ is positive $\Rightarrow$

H, K are positive
 $A = B$

Completely bounded maps. (All C^* -algebras are assumed to be unital)

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(Wittstock's extension thm) MCA operator space. Any $M \xrightarrow{\phi} B(\mathcal{H})$ cb. extends to $A \xrightarrow{\psi} B(\mathcal{H})$ with

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 (Generalised Stinespring's repr. thm) $A \xrightarrow{\phi} B(\mathcal{H})$ cb. Then there exists a Hilbert space K . repr. $A \xrightarrow{\pi} B(K)$ and $V_1, V_2: \mathcal{H} \rightarrow K$ bounded, s.t.
 $\phi(a) = V_1^* \pi(a) V_2$

Lem. $\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix}$ is positive $\Leftrightarrow \|a\| \leq 1$

Wlog: H, K are invertible; otherwise replace by $H + \epsilon 1_{n \times n}$ $K + \epsilon 1_{m \times m}$

$$\begin{pmatrix} 1 & H^{-1/2} A K^{-1/2} \\ -H^{-1/2} A^* K^{-1/2} & 1 \end{pmatrix} = \begin{pmatrix} H^{-1/2} & 0 \\ 0 & K^{-1/2} \end{pmatrix} \begin{pmatrix} H & A \\ A^* & K \end{pmatrix} \begin{pmatrix} H^{-1/2} & 0 \\ 0 & K^{-1/2} \end{pmatrix}$$

$$\Rightarrow \|H^{-1/2} A K^{-1/2}\| \leq 1$$

$$\begin{pmatrix} H & \phi_n(A) \\ \phi_n(A)^* & K \end{pmatrix} = \begin{pmatrix} H^{1/2} & 0 \\ 0 & K^{1/2} \end{pmatrix} \begin{pmatrix} 1 & H^{-1/2} \phi_n(A) K^{-1/2} \\ K^{-1/2} \phi_n(A)^* H^{-1/2} & 1 \end{pmatrix} \begin{pmatrix} H^{1/2} & 0 \\ 0 & K^{1/2} \end{pmatrix}$$

Lem 8.1. MCA operator space. Then $M_n(A)$

$\phi: M \rightarrow B$ cp. contractive $\Leftrightarrow \hat{\phi}: \bigcup_n M_n \rightarrow M_n(B)$ cp.

where $S_n = \left\{ \begin{pmatrix} \lambda & a \\ b^* & \mu \end{pmatrix} \mid \lambda, \mu \in \mathbb{C} \right\}$ $\hat{\phi} \begin{pmatrix} \lambda & a \\ b^* & \mu \end{pmatrix} = \begin{pmatrix} \lambda & \phi(a) \\ \phi(b)^* & \mu \end{pmatrix}$

Proof $A \xrightarrow{\phi} B$ bounded linear. $\leadsto M_n(A) \xrightarrow{\phi_n} M_n(B)$
 $M_n(S_n) \rightarrow M_n(M_n(B))$
 $M_n(\hat{M}_2(A)) \xrightarrow{\hat{\phi}} M_n(M_n(B))$
 $A \otimes M_n \xrightarrow{\phi \otimes id} B \otimes M_n$
 $M_n(A) \xrightarrow{\phi_n} M_n(B)$

$$\begin{pmatrix} \square & \square & \square & \square \\ \Delta & X & \Delta & X \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{n \times n} \rightarrow \begin{pmatrix} \square & \square & \square & \square \\ \Delta & X & \Delta & X \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{2n \times 2n}$$

$$\begin{pmatrix} H & A \\ B^* & K \end{pmatrix} \mapsto \begin{pmatrix} H & \phi_n(A) \\ \phi_n(B)^* & K \end{pmatrix}$$

$H, K \in M_n(\mathbb{C})$
 If $\begin{pmatrix} H & A \\ B^* & K \end{pmatrix}$ is positive $\Rightarrow H, K$ are positive
 $A = B$

$$H^{-1/2} \phi_n(A) K^{-1/2} = \phi_n(H^{-1/2} A K^{-1/2})$$

$$\phi_n := \phi \otimes \text{id} : A \otimes M_n \rightarrow B \otimes M_n$$

$$\phi_n \text{ cpl. contractive} \Rightarrow \|\phi_n(H^{-1/2} A K^{-1/2})\| \leq 1$$

$$\underline{\text{Len}} \Rightarrow \begin{pmatrix} 1 & \phi_n \\ \phi_n^* & K \end{pmatrix} \text{ is positive. } A$$

Then (Wittstock's extension thm) $\bigcup A \xrightarrow{\phi} B(\mathcal{H})$ ch.

Then ϕ extends to $A \xrightarrow{\psi} B(\mathcal{H})$ ch. with $\|\psi\|_{cb} = \|\phi\|_{cb}$

Proof $M \xrightarrow{\phi} B(\mathcal{H}) \xrightarrow{\underline{\text{Len}}}, S_n \xrightarrow{\hat{\phi}} M_n(B(\mathcal{H})) \xrightarrow{\text{cp.}} M_n(B(\mathcal{H}))$

Averson, $M_n(A) \xrightarrow{\hat{\psi}} M_n(B(\mathcal{H}))$

Define $\psi(a) = (1,2)\text{-entry of } \hat{\psi} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$

Then $\psi : A \rightarrow B(\mathcal{H})$ extends ϕ .

$$\|\psi(a)\| \leq \|\hat{\psi} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\| \leq \|\hat{\psi}(1)\| \cdot \|a\| = \|a\|$$

$$\Rightarrow \|\psi\| = 1$$

$$\|\psi_n(A)\| \leq \|\hat{\psi}_n \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}\| \leq \|\hat{\psi}_n(1)\| \cdot \|A\| = \|A\|$$

$$\Rightarrow \|\psi\|_{cb} = 1 = \|\phi\|$$

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(Stinespring's repr. thm). $A \xrightarrow{\phi} B(\mathcal{H})$ cp. Then there exists Hilbert space K , repr. $A \xrightarrow{\pi} B(K)$ and $V: \mathcal{H} \rightarrow K$ bounded, s.t.
 $\phi(a) = V^* \pi(a) V$

cb maps.

(Wittstock's extension thm) $M_n A$ operator space. Any $M_n \phi \xrightarrow{\psi} B(\mathcal{H})$ cb. extends to $A \xrightarrow{\psi} B(\mathcal{H})$ with

$\| \psi \|_{cb} = \| \phi \|_{cb}$
 (Generalised Stinespring's repr. thm) $A \xrightarrow{\phi} B(\mathcal{H})$ cb. Then there exists a Hilbert space K , repr. $A \xrightarrow{\pi} B(K)$ and $V_1, V_2: \mathcal{H} \rightarrow K$ bounded, s.t.
 $\phi(a) = V_1^* \pi(a) V_2$

Lem $\phi: A \rightarrow B(\mathcal{H})$ cb. Then there exists $\phi_{n,2}: A \rightarrow B(\mathcal{H})$ $\phi_{n,2}$ are cp. and $\| \phi_n \| = \| \phi_2 \| = \| \phi \|$. s.t.

$$\hat{\phi} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \phi_1(a) & \phi_1(b) \\ \phi_2^*(c) & \phi_2(d) \end{pmatrix}$$

is cp. Here $\phi^*: A \rightarrow B(\mathcal{H})$. $\phi^*(a^*) = (\phi(a))^*$

Proof $A \xrightarrow{\phi} B(\mathcal{H}) \xrightarrow[\text{8.1}]{\text{Lem}} S_n \xrightarrow{\hat{\phi}} M_n(B(\mathcal{H})) \xrightarrow[\text{Averson}]{\text{Lem}} M_n(A) \xrightarrow{\hat{\phi}} M_n(B)$
 $\hat{\phi} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \hat{\phi} \begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix} = \hat{\phi} \begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix} = \begin{pmatrix} \phi_1(a) & & & \\ & \phi_1(b) & & \\ & & \phi_2(c) & \\ & & & \phi_2(d) \end{pmatrix}$

$$H^{-1/2} \phi_n(A) K^{-1/2} = \phi_n(H^{-1/2} A K^{-1/2})$$

$$\phi_n := \phi \otimes \text{id} : A \otimes M_n \rightarrow B \otimes M_n$$

$$\phi_n \text{ cpl. contractive} \Rightarrow \| \phi_n(H^{-1/2} A K^{-1/2}) \| \leq 1$$

$$\text{Lem} \Rightarrow \begin{pmatrix} \phi & \phi_n \\ \phi_n^* & \phi \end{pmatrix} \text{ is positive.}$$

(Wittstock's extension thm) $M_n \xrightarrow{\phi} B(\mathcal{H})$ cb. extends to $A \xrightarrow{\psi} B(\mathcal{H})$ cb. with $\| \psi \|_{cb} = \| \phi \|_{cb}$

Proof $M \xrightarrow{\phi} B(\mathcal{H}) \xrightarrow[\text{Averson}]{\text{Lem}}, S_n \xrightarrow{\hat{\phi}} M_n(B(\mathcal{H})) \text{ cp.}$
 $M_n(A) \xrightarrow{\hat{\psi}} M_n(B(\mathcal{H}))$

Define $\psi(a) = (1,2)\text{-entry of } \hat{\psi} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$
 Then $\psi: A \rightarrow B(\mathcal{H})$ extends ϕ .

$$\| \psi(a) \| \leq \| \hat{\psi} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \| \leq \| \hat{\psi}(1) \| \cdot \| a \| = \| a \|$$

$$\Rightarrow \| \psi \| = 1$$

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Lem $\phi: A \rightarrow B(\mathcal{H})$ cb. Then there exists $\phi_{n,2}: A \rightarrow B(\mathcal{H})$
 $\phi_{n,2}$ are cp, and $\|\phi_n\| = \|\phi_2\| = \|\phi\|$. s.t.

Let $a \in A$
 $0 < a < 1$ $\hat{\phi}$ is cp. $\Rightarrow$

$$\hat{\phi} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} < \hat{\phi} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} < \hat{\phi} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \phi_n(a) & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

A is spanned by all such elements.

$\Rightarrow \phi_n: A \rightarrow B(\mathcal{H})$ positive $\|\phi_n\| = \|\phi_n(1)\| = 1 = \|\phi\|$

Thm 8.4 (Generalised Stinespring's representation)

$A \xrightarrow{\phi} B(\mathcal{H})$ cb. Then there exists K $\pi: A \rightarrow B(K)$

and $V_1, V_2: \mathcal{H} \rightarrow K$, s.t. $\phi(a) = V_1^* \pi(a) V_2$ for all a

Proof $A \xrightarrow{\phi} B(\mathcal{H}) \rightsquigarrow S_A \xrightarrow{\hat{\phi}} M_n(B(\mathcal{H})) \xrightarrow{\cong} B(\mathcal{H} \oplus \mathcal{H})$
 $\rightsquigarrow M_n(A) \xrightarrow{\hat{\phi}} M_n(B(\mathcal{H}))$
 $\Rightarrow \exists \hat{K}, V: \mathcal{H} \oplus \mathcal{H} \rightarrow K, M_2(A) \xrightarrow{\hat{\pi}} B(\hat{K})$ cp.

s.t. $\begin{pmatrix} \phi_n(a) & \phi(b) \\ \phi^*(a) & \phi(d) \end{pmatrix} = \hat{\phi} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = V^* \begin{pmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{pmatrix} V$

Decompose $K \hat{K} = \text{Ran}(\hat{\pi} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \oplus \text{Ran}(\hat{\pi} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$

Then $\hat{\pi}: M_2(A) \rightarrow B(\hat{K})$

$\pi: A \rightarrow B(K)$
 $\pi \otimes \text{id}: M_n(A) \rightarrow M_n(B(K))$

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 (Generalised Stinespring's repr. thm.) $A \xrightarrow{\phi} B(\mathcal{H})$ cb. Then there exists a Hilbert space $\mathcal{K}$, repr. $A \xrightarrow{\pi} B(\mathcal{K})$ and $V_1, V_2: \mathcal{H} \rightarrow \mathcal{K}$ bounded, s.t.
 $\phi(a) = V_1^* \pi(a) V_2$

Claim $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ (check $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$)

$$\Rightarrow \begin{pmatrix} \phi_1(a) & \phi(b) \\ \phi^*(c) & \phi_2(d) \end{pmatrix} = \begin{pmatrix} V_1^* & V_2^* \end{pmatrix} \begin{pmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$= \begin{pmatrix} V_1^* \pi(a) V_1 & V_1^* \pi(b) V_2 \\ V_2^* \pi(c) V_1 & V_2^* \pi(d) V_2 \end{pmatrix}$$

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leq \hat{\phi} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \leq \hat{\phi} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} \phi_1(a) & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

A is spanned by all such elements.

$A \rightarrow B(\mathcal{H})$ positive $\| \phi_1 \| = \| \phi_1(1) \| = 1 = \| \phi \|$
 (Generalised Stinespring's repr. thm.)

Proof $A \xrightarrow{\phi} B(\mathcal{H}) \rightsquigarrow S_A \xrightarrow{\hat{\phi}} M_n(B(\mathcal{H})) \xrightarrow{\hat{\phi}} M_n(A) \xrightarrow{\hat{\phi}} M_n(B(\mathcal{H}))$
 $\rightsquigarrow M_n(A) \xrightarrow{\hat{\phi}} M_n(B(\mathcal{H}))$
 $\Rightarrow \exists \mathcal{K}, V: \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{K}, M_n(A) \xrightarrow{\hat{\pi}} B(\mathcal{K})$ cp

$$\text{s.t. } \begin{pmatrix} \phi(a) & \phi(b) \\ \phi^*(c) & \phi(d) \end{pmatrix} \leq \hat{\phi} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = V^* \begin{pmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{pmatrix} V$$

Decompose $\mathcal{K} = \text{Ran}(\hat{\pi} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \oplus \text{Ran}(\hat{\pi} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$

Then $\hat{\pi}: M_n(A) \rightarrow B(\mathcal{K})$

$\pi \otimes \text{id}: M_n(A) \rightarrow M_n(B(\mathcal{K}))$

Thm 8.5 Withstark's decomposition thm.

$\phi: A \rightarrow B(\mathcal{H})$ ch. Then there exists $\psi: A \rightarrow B(\mathcal{H})$ cp.
s.t. $\|\psi\|_{cb} \leq \|\phi\|_{cb}$.

$\psi \pm \operatorname{Re} \phi, \psi \pm i \operatorname{Im} \phi$ are cp.

So: ch maps into $B(\mathcal{H})$ are spanned by cp maps

$$\text{Here: } \operatorname{Re} \phi = \frac{1}{2}(\phi + \phi^*) \quad \operatorname{Im} \phi = \frac{1}{2i}(\phi - \phi^*)$$

$$\text{because: } \phi = \operatorname{Re} \phi + i \operatorname{Im} \phi$$

$$= \frac{1}{2}(\psi + \operatorname{Re} \phi) + \frac{1}{2}(-\psi + \operatorname{Re} \phi) + \frac{1}{2i}(\psi + i \operatorname{Im} \phi) + \frac{1}{2i}(-\psi + i \operatorname{Im} \phi)$$

Proof $\phi = \frac{1}{2}(\phi_1 + \phi_2)$

$$\operatorname{Re} \phi + i \operatorname{Im} \phi = (V_1 + V_2)^* \pi(V_1 + V_2)$$

Remark This holds for (and only for) injective C^* -alg

Smith, 1983: $C([0,1])$ does not satisfy this property

Thm 8.5 Withstock's decomposition thm.

$\phi: A \rightarrow B(\mathcal{H})$ ch. Then there exists $\psi: A \rightarrow B(\mathcal{H})$ cp. s.t. $\|\psi\|_{cb} \leq \|\phi\|_{cb}$.

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So: ch. maps into $B(\mathcal{H})$ are spanned by cp. maps

Here: $\operatorname{Re} \phi = \frac{1}{2}(\phi + \phi^*)$ $i \operatorname{Im} \phi = \frac{1}{2i}(\phi - \phi^*)$

because: $\phi = \operatorname{Re} \phi + i \operatorname{Im} \phi$

$$= \frac{1}{2}(\psi + \operatorname{Re} \phi) + \frac{1}{2}(-\psi + \operatorname{Re} \phi) + \frac{1}{-2i}(\psi + i \operatorname{Im} \phi) + \frac{1}{2i}(\psi + i \operatorname{Im} \phi)$$

Proof $\psi = \frac{1}{2}(\phi_1 + \phi_2)$

$$\operatorname{Re} \phi + i \operatorname{Im} \phi = (V_1 + V_2)^* \pi(V_1 + V_2)$$

Remark This holds for (and only for) injective C^* -alg

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Schur multipliers

finite: $A \in M_n$ $S_A: M_n \rightarrow M_n$ $(B_{ij}) \mapsto (A_{ij} B_{ij})$

infinite: $A \in M_\infty$ $S_A: B(l^2) \rightarrow B(l^2)$ bounded

Smith: "matrically-normed C^* -alg" and "bimodule maps"

Bimodule maps $C \subseteq A$ C^* -subalgebra. Then A is a C -bimodule. If C is a C^* -subalg. of both A and B . Then $\phi: A \rightarrow B$ is a C -bimodule map $\forall$
 $C \phi(a) C' = \phi(a C')$

Def $C \subseteq A$ C^* -subalg. A is matrically-normed as a C -bimodule, if
 for $(a_{ij}) \in M_n(A)$:

$$\|(a_{ij})\| = \sup \left\{ \left\| \sum_{i,j=1}^n x_i a_{ij} y_j \right\| \mid \begin{matrix} x_i, y_j \in C \\ \|\sum_{i=1}^n x_i x_i^*\| \leq 1 \\ \|\sum_{j=1}^n y_j^* y_j\| \leq 1 \end{matrix} \right\}$$

Consider A^n as a Hilbert A -module
 $M_n(A) = \text{End}_A(A^n)$. Then the equation is just

$$\|a\| = \sup_{\substack{x, y \in C^n \\ \|x\| \leq 1 \\ \|y\| \leq 1}} \|\langle x, ay \rangle\|$$

Thm $C \subseteq A, B$ be C^* -subalgebra. s.t. B is matricially-normed

Then any bounded bimodule map is cb.

Proof. $\|\phi_n(a_{ij})\| = \sup_{\|x\| \leq 1} \|\sum_{ij} \phi(a_{ij}) y_j\|$
 $= \sup_{\|x\| \leq 1} \|\phi(\sum_{ij} x_i a_{ij} y_j)\|$
 $\leq \|\phi\| \cdot \|a_{ij}\|$

$\Rightarrow \|\phi_n\| \leq \|\phi\|$

Why useful?

diagonal matrices in M_n

Lem M_n is matricially-normed as a $1D_n$ -bimodule

$S_A: M_n \rightarrow M_n$ is a $1D_n$ -bimodule map.

Thm $A \in M_n$. TFAE:

- (1) $S_A: M_n \rightarrow M_n$ is contractive
- (2) $S_A: M_n \rightarrow M_n$ is cpl. contractive
- (3) There exists a Hilbert space $\mathcal{H}$, $x_1, \dots, x_n \in \mathcal{H}$ s.t.
 $\|x_i\| \leq 1, \|y_j\| \leq 1, A_{ij} = \langle x_i, y_j \rangle$

Proof. (1) $\Rightarrow$ (2) by Lemma

(2) $\Rightarrow$ (3): Use the generalised Schur's repr. thm

$S_A(B) = V_1^* \pi(B) V_2 \quad V_1, V_2: \mathbb{C}^n \rightarrow \mathcal{H}$

If $B = E_{ij} \quad S_A(E_{ij}) = A_{ij}$

$$\begin{aligned}
 A_{ij} &= \langle e_i, A_{ij} e_j \rangle \\
 &= \langle e_i, S_A(E_{ij}) e_j \rangle \\
 &= \langle U_1 e_i, \pi(\xi_i \xi_j^*) U_2 e_j \rangle \\
 &= \langle \pi(\xi_{i1}) U_1 e_i, \pi(\xi_{j1}) U_2 e_j \rangle
 \end{aligned}$$

$$(3) \Rightarrow (1) \sup_{\alpha, \beta} \left| \langle \sum_i \alpha_i S_A(B) \beta_j \rangle \right| \quad \mathbb{C}^n \otimes \mathcal{H}$$

Then $C \subseteq A, B$ be C^* -subalgebra. s.t. B is unital-normed

Then any bounded bimodule map is cb.

Proof. $\|\phi(a_{ij})\| = \sup_{\|x\| \leq 1, \|y\| \leq 1} \|\sum_j x_j \phi(a_{ij}) y_j\|$
 $= \sup_{\|x\| \leq 1, \|y\| \leq 1} \|\phi(\sum_j x_j a_{ij} y_j)\|$
 $\leq \|\phi\| \cdot \|a_{ij}\|$

$\|\phi\|$
 ul?

diagonal matrices in

Then $A \in M_n$. TFAE:

(1) $S_A: M_n \rightarrow M_n$ is contractive

(2) $S_A: M_n \rightarrow M_n$ is cpl. contractive

(3) There exists a Hilbert space $\mathcal{H}$, $x_1, \dots, x_n \in \mathcal{H}$, $y_1, \dots, y_n \in \mathcal{H}$

$$\|x_i\| \leq 1, \|y_j\| \leq 1, A_{ij} = \langle x_i, y_j \rangle$$

Proof. (1) $\Rightarrow$ (2) by Lemma

(2) $\Rightarrow$ (3): Use the generalized Stinespring's repr

$$S_A(B) = U_1^* \pi(B) U_2 \quad U_1, U_2: \mathbb{C}^n \rightarrow \mathcal{H}$$

If $B = E_{ij}$, $S_A(E_{ij}) = A_{ij}$

Then any bounded bimodule map is $\in \mathcal{CB}_*$.

Proof. $\|\phi(a_{ij})\| = \sup_j \|\sum_i \phi(a_{ij}) y_j\|$
 $= \sup_j \|\phi(\sum_i x_i a_{ij} y_j)\|$
 $\leq \|\phi\| \cdot \|a_{ij}\|$
 $\Rightarrow \|\phi\| \leq \|a\|$

$$\Rightarrow \|\phi_n\| \leq \|\phi\|$$

Why useful?

diagonal matrices in M_n

Lemma M_n is naturally viewed as a D_n -bimodule
 $S_A: M_n \rightarrow M_n$ is a D_n -bimodule map.

Thm $A \in M_\infty$ TFAE

n) $S_A: B(\ell^2) \rightarrow B(\ell^2)$ is contractive

(2) $S_A: B(l^2) \rightarrow B(l^2)$ is cpl. contractive

(3) There exists a Hilbert space $\mathcal{H}$, $\{x_i\}_{i=1}^{\infty}, \{y_j\}_{j=1}^{\infty} \subset \mathcal{H}$

$$\|x_i\| \leq 1, \|y_j\| \leq 1, A_{ij} = \langle x_i, y_j \rangle$$

Proof. $\|S_A(B)\| = \sup_n \|S_A(B)_n\|$
 $= \sup_n \|S_{A_n}(B)\|$

$$\Rightarrow \|S_A\| = \sup_n \|S_{A_n}\| \cdot \|B\|$$

$$\|S_A\|_{cb} = \sup_n \|S_{A_n}\|_{cb}$$

$$\begin{aligned}
 A_{ij} &= \langle e_i, A_{ij} e_j \rangle \\
 &= \langle e_i, S_A(E_{ij}) e_j \rangle \\
 &= \langle v_1 e_i, \pi(\xi_{n_1} \xi_{n_2}) v_2 e_j \rangle \\
 &= \langle \pi(\xi_{n_1}) v_1 e_i, \pi(\xi_{n_2}) v_2 e_j \rangle
 \end{aligned}$$

$$(3) \Rightarrow (1) \sup_{\alpha, \beta} \left| \sum_{i,j} \alpha_i S_A(B) \beta_j \right| \quad \begin{matrix} x_i \\ y_j \end{matrix} \quad \mathbb{C}^n \otimes \mathcal{H}$$

Then $A \in M_\infty$. $B = (B_{ij})$ where

$$B_{ij} = A_{ij} + A_{i+n, j+n} - A_{i, j+n} - A_{i+n, j} \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

- If:
- $A_{\infty, j} := \lim_i A_{ij}$ exist
 - $A_{i, \infty} := \lim_j A_{ij}$
 - $\lim_i \lim_j A_{ij}, \lim_j \lim_i A_{ij}$ exist
 - (B_{ij}) is absolutely summable

Then S_A is bounded and hence cb.

Then $C \subseteq A, B$ be C^* -subalgebra s.t. B is matrixly-normed

Then any bounded bimodule map is cb.

Proof. $\|\phi_n(a_{ij})\| = \sup_{i,j} \|\sum_i \phi(a_{ij}) y_j\|$

$$= \sup_{i,j} \|\phi(\sum_i x_i a_{ij} y_j)\|$$

$$\leq \|\phi\| \cdot \|a_{ij}\|$$

$$\Rightarrow \|\phi_n\| \leq \|\phi\|$$

diagonal matrices in M_n

Then $A \in M_\infty$. TFAE

- $S_A: B(\ell^2) \rightarrow B(\ell^2)$ is contractive
- $S_A: B(\ell^2) \rightarrow B(\ell^2)$ is cpl. contractive
- There exists a Hilbert space $\mathcal{H}$, $\{x_i\}_{i=1}^\infty, \{y_j\}_{j=1}^\infty \subset \mathcal{H}$

$$\|x_i\| \leq 1, \|y_j\| \leq 1, A_{ij} = \langle x_i, y_j \rangle$$

Proof. $\|S_A(B)\| = \sup_n \|S_A(B)_n\|$

$$= \sup_n \|S_{A_n}(B)\|$$

$$\Rightarrow \|S_A\| = \sup_n \|S_{A_n}\| \leq \sup_n \|S_{A_n}\| \cdot \|B\|$$

$$\|S_A\|_{cb} = \sup_n \|S_{A_n}\|_{cb}$$