Ex. $B(H)_{sa}$, $M_n(\mathbb{C})_{sa}$, A_{sa} (A C^*-alg)

not an algebra under ab: $(ab)^* = b^* a^* = ba$

$a \circ b := \frac{1}{2}(ab + ba)$ comm non-associative

History .QM self-adjoint operators

1932 P. Jordan. Find algebraic setting for QM

Def A (real) Jordan algebra is a V.S with comm, non-ass bilinear map $(a,b) \mapsto a \circ b$ s.t.

$$(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$$

It is formally real if $x_1^2 + \cdots + x_n^2 = 0 \Rightarrow x_1 = \cdots = x_n = 0$

Claim. A_{sa} is a formally real Jordan algebra

Pf

$\frac{1}{4}$LHS $x(x^2y + yx^2) + (x^2y + yx^2)x = x^3y + xyx^2 + x^2yx + yx^3$

$\frac{1}{4}$RHS $x^2(yx + xy) + (yx + xy)x^2 = x^2yx + x^3y + yx^3 + xyx^2$

Material

. Geometry of state spaces of Operator Algebras (Alfsen-Shultz) prequel

. State Spaces of Operator Algebras (AS)

. Jordan Operator Algebras (Hanche-Olsen, Størmer) google "jordan operator algebras" $\to$ 6th

. A Taste of Jordan Algebras (McCrimmon) Ch 0 - google

"Freudenthal-Tits magic square"

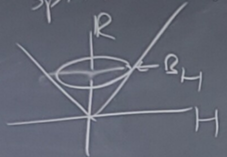
	$\mathbb{R}$	$M_3(\mathbb{R})_{sa}$	$M_3(\mathbb{C})_{sa}$	$M_3(\mathbb{H})_{sa}$	$M_3(\mathbb{O})_{sa}$
$\mathbb{R}$	0	A_1	A_2	C_3	F_4
$\mathbb{C}$	0	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	A_1	C_3	A_5	A_6	E_7
$\mathbb{O}$	G_2	F_4	A_6	E_7	E_8

Thm (Jordan, v Neumann, Wigner).

Every EJA is a direct sum of simple ones

- $M_n(\mathbb{R})_{sa}$
- $M_n(\mathbb{C})_{sa}$
- $M_n(\mathbb{H})_{sa}$ $\subseteq M_{2n}(\mathbb{C})_{sa}$
- $\mathbb{R} \oplus H$ $\quad (\lambda, x) \circ (\mu, y) = (\mu\lambda + \langle x, y \rangle, \lambda y + \mu x)$

  real $\hat{H}S$ dim $n \geq 2$ $\quad \subseteq M_{2^n}(\mathbb{C})_{sa}$



- $M_3(\mathbb{O})_{sa}$ Albert (1934) exceptional

$\rightsquigarrow$ not enough to do QM

Claim: $\{\text{squares in } A_{sa}\} = A_+$

Indeed $sp(a^2) = sp(a)^2 \geq 0$

$a \geq 0 \quad a = (\sqrt{a})^2$

Def. A Euclidean Jordan Algebra (EJA) is a fin. dim formally real Jordan algebra

It is special if it is a Jordan subalg of an associative alg. A $\quad (a \circ b = \frac{1}{2}(ab + ba))$

otherwise exceptional

Jordan: Find exceptional EJA's E_n with $\dim(E_n) \to \infty$

Ex. $B(H)_{sa}$, $M_n(\mathbb{C})_{sa}$, $\mathcal{A}_{sa}$ ($\mathcal{A}$ C^*-alg)

not an algebra under ab: $(ab)^* = b^*a^* = ba$

$a \circ b := \frac{1}{2}(ab + ba)$ comm non-associative

Def. A JB-algebra $A = (A, \circ, \| \, \|)$ is a Jordan algebra and real Banach space s.t.

$$\|a \circ b\| \leq \|a\| \|b\|$$

$$\|a^2\| = \|a\|^2 \quad (\|a^2\| = \|a^* a\| = \|a\|^2)$$

$$\|a^2\| \leq \|a^2 + b^2\|$$

Def. A JC-algebra is a JB-alg Jordan isomorphic to a closed Jordan subalg. of $B(H)_{sa}$

Convention. JB-algebras are unital

Goal. Investigate order structure of A

Def. An order unit space is an ordered normed space with closed cone and order unit, i.e.,

$$\|a\| = \|a\|_e := \inf\{\lambda > 0 : -\lambda e \leq a \leq \lambda e\}$$

Goal. Show A o.u. space (ev 1) with cone $A_+ = A^2$

Powers $a^1 = a$, $a^n = a^{n-1} \circ a$ Want $a^n = a^{n-k} \circ a^k$

- $a^2 = a \circ a$, $a^3 = a^2 \circ a = a \circ a^2$, $(a^2 \circ a) \circ a \overset{?}{=} a^2 \circ a^2$

$(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$

Prop 1.3 A Jordan algebra is power associative (Pf induction)

$b = 1 - a$ Define $s := \sum_{n=0}^{\infty} \beta_n b^n \in A$ Then

$$s^2 = \sum_{n=0}^{\infty} \gamma_n b^n = 1 \cdot b^0 - 1 \cdot b^1 + 0 = 1 - b = a$$

Lemma 1.77 prequel. A JB-alg

$a \in A$ TFAE. (i) $a \in A^2$ (ii) $\forall \alpha \geq \|a\|$ $\|\alpha - a\| \leq \alpha$

(iii) $\exists \alpha \geq \|a\|$ $\|\alpha - a\| \leq \alpha$

Pf. (i)$\Rightarrow$(ii) Let $a \in A^2$, assume $\|a\| \leq 1$. Let $s \in A$, $s^2 = a$

$\|1 - (1-a)\| \leq 1 \rightsquigarrow \exists t \in A : t^2 = 1 - a$ $\quad \|1\|^2 = \|1^2\| = \|1\|$

$s^2 + t^2 = a + (1-a) = 1$ so $\|1 - a\| = \|t^2\| \leq \|s^2 + t^2\| = \|1\| = 1$

Let $\alpha \geq \|a\|$, then $\|\alpha^{-1} a\| \leq 1$ so $\|1 - \alpha^{-1} a\| \leq 1$

$\|\alpha - a\| \leq \alpha$

(ii)$\Rightarrow$(iii) Trivial

(iii)$\Rightarrow$(i) suppose $\|\alpha - a\| \leq \alpha$ then $\|1 - \alpha^{-1} a\| \leq 1$

So by 1.71, $\alpha^{-1} a$ is a square $\Leftrightarrow$ a is a square

---

Cor: If $a \in A$, then $C(a, 1) = \overline{\mathbb{P}(a)}$ is associative

closed subalg gen by $a, 1$

Lemma (1.71 prequel). A JB-alg

If $\|1 - a\| \leq 1$ then $\exists s \in A : s^2 = a$.

Pf. Binomial series $(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ $\quad \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$

$\alpha = 1/2$, $x = -\lambda$

$\sqrt{1-\lambda} = \sum_{n=0}^{\infty} \underbrace{\binom{1/2}{n}(-1)^n}_{\beta_n} \lambda^n$ Conv. abs. for $\lambda = 1 \rightsquigarrow \sum_{n=0}^{\infty} |\beta_n| < \infty$

$1 - \lambda = \sqrt{1-\lambda}^{\,2} = \sum_{n=0}^{\infty} \gamma_n \lambda^n$, $\quad \gamma_n = \sum_{k=0}^{n} \beta_{n-k} \beta_k$

$\gamma_0 = 1$, $\gamma_1 = -1$, $\gamma_n = 0$ $(n \geq 2)$

Thm (1

Thm (1.78, 1.80, 1.81 prequel): A JB-alg. Then A is an ou space (ou 1) with cone of squares

- A^2 is a proper cone: let $a, b \in A^2$, $\alpha = \|a\|$, $\beta = \|b\|$ so

1.77 $\rightsquigarrow$ $\|\alpha - a\| \leq \alpha$, $\|\beta - b\| \leq \beta$, hence $\alpha + \beta \geq \|a+b\|$

$\|(\alpha+\beta) - (a+b)\| \leq \|\alpha - a\| + \|\beta - b\| \leq \alpha + \beta$. By 1.77(iii)⇒(i), $a+b \in A^2$

$\lambda \geq 0$, $a \in A^2 \Rightarrow \lambda a \in A^2$. Proper: if $a^2 + b^2 = 0$ then

$\|a\|^2 = \|a^2\| \leq \|a^2 + b^2\| = \|0\| = 0$ so $a = 0$

Closed: $A^2 = \{a \in A : \| \|a\|1 - a\| \leq \|a\|\}$

KTP: 1 is an order unit. $\|a\| \leq 1 \iff -1 \leq a \leq 1$ $\quad (B_A = [-1, 1])$

Suppose $\|a\| \leq 1$. Then $\|1 - (1-a)\| \leq 1$. By 1.77, $1 - a \in A^2 = A_+$. $a \leq 1$.

$\|1 - (1+a)\| = \|-a\| \leq 1$. By 1.77, $1 + a \in A^2 = A_+$, so $-1 \leq a$

Suppose $-1 \leq a \leq 1$, so $1-a, 1+a \in A_+ = A^2$. Let $\lambda = \|1-a\|$

By 1.77(iii) $\|1 - \lambda^{-1}(1-a)\| \leq 1$. By 1.71, $\lambda^{-1}(1-a)$ is a square

$\Rightarrow \exists s \in A : s^2 = 1 - a$

Similar: $\exists t \in A$: $t^2 = 1 + a$

Now $1 - a^2 = (1-a)(1+a) = s^2 \circ t^2 = (s \circ s)(t \circ t) = (s \circ t)(s \circ t) = (s \circ t)^2$

$\|a\|^2 = \|a^2\| \leq \|a^2 + (s \circ t)^2\| = \|1\| = 1 \rightsquigarrow \|a\| \leq 1$

Lemma 1.16 prequel: $\rho : A \to \mathbb{R}$ is positive $\iff$ ρ is bdd $\|\rho\| = \rho(1)$

Def (State space): $K := \{\rho \in A^* : \rho(1) = 1 \ (= \|\rho\|)\}$

Ex: $A = C(X)$, $K = P(X)$, $A = K(H)_{sa}$, $K =$ trace 1 pos $N(H)$

Remark: $A \cong$ (dense subspace of) $A_{w^*}(K)$ $\quad \hat{a}(\rho) := \rho(a)$

a.u. space $\qquad$ affine

$A^{**} \cong A_b(K)$

A JB-alg, $a, b \in A_+$. Is $a \circ b \in A_+$?

. If A is associative. $a^2 \circ b^2 = (a \circ b)^2 \in A_+$

__Lemma 1.18 modified (prequel)__ . A o.u. space, unital algebra

$a.u = a.u$. If $ab \in A_+$ $\forall a, b \in A_+$ then pure states are multiplicative

__Pf__ Let $\rho \in \partial_e K$. $0 \le a \le 1 \to 1 - a \ge 0$. For $x \in A_+$

$ax, (1-a)x \in A_+$ so $0 \le ax \le x$. Hence $\rho_a : x \mapsto \rho(ax)$

satisfies $0 \le \rho_a \le \rho \Rightarrow$ ($\rho \in \partial_e K$) $\rho_a = \lambda \rho$

$$\rho(a)\rho(x) = \rho(a1)\rho(x) = \rho_a(1)\rho(x) = \lambda\rho(1)\rho(x) = \lambda\rho(x) = \rho_a(x) = \rho(ax)$$

$\boxed{\text{Lemma 1.18 prequel}}$: $a \in A_+ \Leftrightarrow \rho(a) \ge 0$ $\forall \rho \in K$ ($\Leftrightarrow \forall \rho \in \partial_e K$)

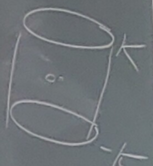
. $\|a\| = \sup_{\rho \in K} |\rho(a)|$ $\left(= \sup_{\rho \in \partial_e K} |\rho(a)| \right)$?

__Pf__. If $a \in A_+$ then $\rho(a) \ge 0$ $\forall \rho \in K$. Suppose $\rho(a) \ge 0$ $\forall \rho \in K$.

and $a \notin A_+$. By H.B. $\exists \varphi$: $\varphi(a) < 0$, $\varphi(b) \ge 0$ $\forall b \in A_+$.

So $\varphi \ge 0$, $\psi = \frac{\varphi}{\varphi(1)} \in K$ with $\psi(a) < 0$ $\lightning$

__Cor.__ A^* is a __base norm space__ $\quad B_{A^*} = co(K \cup -K)$



__Pf__. Clearly, $co(K \cup -K) \subset B_{A^*}$. Note $co(K \cup -K) = R(f)$ is w^*-cpt

$f : [0,1] \times K \times (-K) \to A^*$, $f(\lambda, \rho, \sigma) = \lambda\rho + (1-\lambda)\sigma$

Suppose $co(K \cup -K) \ne B_{A^*}$, so $\exists \rho \in B_{A^*} \setminus co(K \cup -K)$. By HB

$\exists a \in A$ with $\rho(a) > 1$ and $\sigma(a) < 1$ $\forall \sigma \in co(K \cup -K)$

Hence $|\sigma(a)| \le 1$ $\forall \sigma \in K$, so $\|a\| \le 1$. Hence $|\rho(a)| \le \|\rho\| \|a\| \le 1 \cdot 1 = 1$ $\lightning$

Thm 1.69 modified (prequel): A ou space unital algebra, ou = a.u

$\forall a,b \in A_+ : ab \in A_+$. Then $A \cong$ (dense subalg of) $C(X)$ X cpt Hsdff

Pf: $X = \{$mult. states$\} \subset K$ w^*-closed

$A \to C(X)$ $\quad \hat{a}(\rho) = \rho(a)$ By lemma, ex $K \subset X$

$a \mapsto \hat{a}$

order + norm isom Note $\widehat{ab}(\rho) = \rho(ab) = \rho(a)\rho(b) = \hat{a}(\rho)\hat{b}(\rho)$

alg. hom If $\rho \neq \sigma$ then $\exists a \in A$: $\hat{a}(\rho) = \rho(a) \neq \sigma(a) = \hat{a}(\sigma)$

So range sep. points, contains $1_X = \hat{1}$. By Stone-Weierstrass A is dense.

Cor: Associative JB-algs are $\cong C(X)$

$\Rightarrow C(G,1) \cong C(X)$