Recall: A JB-Alg

$\{abc\} := (a\circ b)\circ c + (b\circ c)\circ a - (a\circ c)\circ b$

linear in each factor

$U_a : A \to A$

$U_a b = \{aba\}$

$T_a : A \to A$

$T_a b = a\circ b$

A^*: dual of A

$T : A \to A$ bounded linear

$T^* : A^* \to A^*$ (adjoint)

$(T^* \delta)(b) = \delta(Tb) \quad \forall b \in A$ and $\delta \in A^*$

Operator commutes

$a, b \in A \quad T_a T_b = T_b T_a$

$\text{face}(a) = \{y \in A \mid 0 \le y \le \lambda a \text{ for some } \lambda \ge 0\}$

$\text{face}(p) = \text{im}^+ U_p \quad p$ proj (Lem 1.39)

Intro to JBW-Alg

Def. An ordered Banach space A is monotone complete if each bounded above increasing net $\{b_\alpha\}$ has a least upper bound (l.u.b) b in A

write $b_\alpha \nearrow b$

Def. A bdd linear functional δ on a monotone complete space A is normal if

$b_\alpha \nearrow b \Rightarrow \delta(b_\alpha) \to \delta(b)$

Def. A JBW-Alg M is a JB-alg that is monotone complete and has separating normal states K. write

$V := \text{span } K$

Remark: V.N. alg: C^*-alg whose self-adjoint parts are monotone complete and has separating normal states (A.95)

Eg. V.N Alg A. A_{sa} with usual Jordan product is a JBW-alg

Def. Let M be a JBW-alg. The σ-weak topology is gen by the dual V

$a \mapsto |\rho(a)| \quad \rho \in K$

σ-strong topology is gen by seminorm

$a \mapsto \rho(a^2)^{1/2}$

Remark. σ-weak & σ-strong topology coincide with the same in V.N-alg context.

Lem: $U_a = \frac{1}{2}(U_{\lambda 1+a} + U_{\lambda 1-a} - 2\lambda^2 I) \quad \lambda \in \mathbb{R}$

$T_a = \frac{1}{2}(U_{1+a} - U_a - I)$

Prop (2.4) (1) U_a T_a is σ-strongly and σ-weakly continuous T^*_a, U^*_a map V into V

(2) Jordan product is $\overset{\text{sep}}{}$ σ-strong & σ-weak continuous and jointly σ-strong continuous on bdd subsets

Pf (1') Assume $a \in \mathrm{Inv}(M)$

By Lem 1.23 Thm 1.25

U_a is ord isom

then $\forall$ normal state δ on M

$b \mapsto \delta(U_a b)$

pos. normal, so $U_a^* \delta$ is a multiple of normal state so $U_a^* \delta \in V$

$M^* = V$

$U_a^*(V) \subseteq V \Rightarrow U_a$ δ-weakly continuous

Suppose $b_\alpha \to 0$ δ-strongly

then by. $b_\alpha^2 \to 0$ δ-weakly

$U_a b_\alpha^2 = \{a b_\alpha^2 a\} \to 0$

So,

$\{a b_\alpha a\}^2 = \{a \{b_\alpha a^2 b_\alpha\} a\}$ (1.15)

$\leq \|a^2\| \{a b_\alpha^2 a\}$ ($\|U_a\| = \|a\|^2$)

$\to 0$ δ-weakly

So $\{a b_\alpha a\} \to 0$ δ-weakly, Thus U_α is δ-strongly continuous

(2) $a^2 \geq 0 \Rightarrow a^4 \leq \|a^2\| a^2$ (1.27)

so if $\{a_\alpha\}$ is a bdd net s.t $a_\alpha \to 0$ δ-strongly

and δ is normal state. then

$\delta(a_\alpha^4) \leq \|a_\alpha^2\| \delta(a_\alpha^2) \to 0$

square is δ-strongly cont at 0 on bdd sets

$a \circ b = \frac{1}{2}((a+b)^2 - a^2 - b^2)$

P $a_\alpha \to a$, $b_\alpha \to b$

$a \circ b - a_\alpha \circ b_\alpha = (a - a_\alpha) \circ b + (a_\alpha - a) \circ (b - b_\alpha) + a \circ (b - b_\alpha)$

$\square$

Prop (2.5) M a JBW-alg

(1) σ-strong conv $\Rightarrow$ σ-weak conv

(2) A bdd monotone net conv σ-strongly

(3) A mono net of proj conv σ-strongly to a proj

Pf. (1) $\{a_\alpha\} \subseteq M$ s.t $a_\alpha \to a$ σ-strongly

By Cauchy-Schwarz (1.53)

$|\sigma(a_\alpha - a)| \leq \sigma(1)\,\sigma((a_\alpha - a)^2)^{1/2}$

so $a_\alpha \to a$ σ-wk

(2) if $a_\alpha \nearrow a$

$\forall$ normal state σ

$a - a_\alpha \geq 0 \overset{(1.27)}{\Longrightarrow} (a - a_\alpha)^2 \leq \|a - a_\alpha\| (a - a_\alpha)$

So

$\sigma((a - a_\alpha)^2) \leq \|a - a_\alpha\|\, \sigma(a - a_\alpha) \to 0$

$\Rightarrow a_\alpha \to a$ σ-strongly

Cor. $a_\alpha \nearrow a \Rightarrow U_{a_\alpha} b \to U_a b$ (σ-strongly)

$a_\alpha \searrow a \Rightarrow U_{a_\alpha} b \to U_a b$ (σ-strongly)

Cor (2.6) Every σ-weakly cont linear functional is normal

Prop (2.7) $B \subseteq M$ is σ-wk closed Jordan subalg of M. Then B has an identity and B is a JBW-alg

Pf. $\{b_\alpha\}$ is increasing net in B bdd above, let b sup.

$b_\alpha \to b$ σ-wk $\Rightarrow b \in B$

Show B has id. let $\tilde{B}$ be span B and 1. $B \subseteq \tilde{B}$

$\{V_\alpha\}$ approx id. $V_\alpha \to V \in B$

id in B $\square$

Def. M a JBW-alg. a JBW-subalg is a σ-wk closed Jordan subalg N of M

Prop (2.9) P proj in M, then $M_P = U_P M$ is a JBW-subalg of M with id P

Pf. $M_P = \{a \in M \mid U_P a = a\}$ $\square$

$\uparrow$ σ-wk cont

Cor (2.10) If P proj in M then

$M_P + M_{P'} = \operatorname{im}(U_P + U_{P'})$ is a JBW-subalg of M

$A_P \circ A_{P'} = 0$

Def. $a \in M$ is a JBW-alg. write $W(a,1)$ σ-weakly closed subalg gen 1 and a.

Prop (2.11) If B is an assoc Jordan subalg of a JBW-alg M, then $\overline{B}^{\sigma\text{-wk}}$ is an assoc JBW-alg

$\overline{B}^{\sigma\text{-wk}} \cong C_{\mathbb{R}}(X)$ ← compact Hausdorff monotone complete

Pf. Prop 1.12.

Cor (2.12) If p proj. in JBW-alg M which is oper comm with $a \in M$ p oper comm with all $W(a,1)$

Pf. By Prop 1.47 :
p oper comm with x
$\Leftrightarrow x \in M_p + M_{p'}$

So $a \in M_p + M_{p'}$ is JBW-alg (cor 2.10
$W(a,1) \subseteq M_p + M_{p'}$
all $x \in W(a,1)$ oper comm with p

Range projections

Lem (*) (A.38) X compact Hausdorff

$C_{\mathbb{R}}(X)$ monotone complete, then

$\forall a \in C_{\mathbb{R}}(X)$

$E = \overline{\{ s \in X \mid a(x) > 0 \}} \subseteq X$

is both closed and open

(1) $a \geq 0$ on E, $a \leq 0$ on $X \setminus E$

and E is the smallest closed subset

s.t $a \leq 0$ on $X \setminus E$

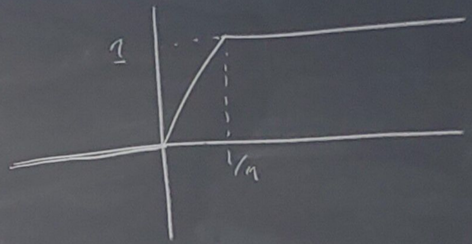
(2) E is the smallest closed subset s.t.

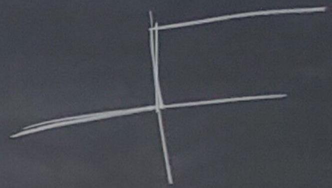
$\chi_E a^+ = a^+$

(3) χ_E is the sup in $C_{\mathbb{R}}(X)$ of an increasing seq in face(a^+)

Sketch of Pf

Consider f_n





$a \in C_{\mathbb{R}}(X)$.

$\{f_n \circ a\}$ has sup b, then $b = \chi_E$

Prop (2.13) M JBW-alg $\forall a \in M$, $\exists!$ proj p

$p \in W(a,1) \cap \overline{\text{face}(a^+)}^{\sigma\text{-wk}}$

s.t

$U_p a \geq a$ and $U_p a \geq 0$

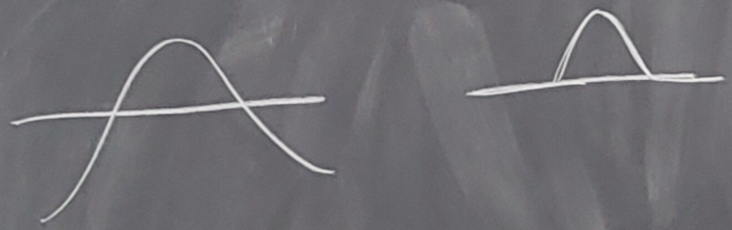
and

p is the smallest proj s.t

$U_p a^+ = a^+$

Pf. Prop. 2.11. $W(a,1) \cong C_{\mathbb{R}}(X)$

E in lem, $p = \chi_E$, $p^2 = p$



Def. M is a JBW-alg. $0 \le a \in M$, the smallest proj p s.t $U_p a = a$ is the range projection of a and denoted as $r(a)$

Prop (2.15) $0 \le a \in M$. K is the normal state sp

(1) $r(a) \in W(a,1) \cap \overline{\text{face}(a)}$

(2) $a \le \|a\| r(a)$; $a \in \text{face}(r(a))$

(3) If $\delta \in K$. then

$\delta(a) = 0 \iff \delta(r(a)) = 0$ $\qquad a \in N \subseteq M$

(4) $a \le b \Rightarrow r(a) \le r(b)$ $\qquad 1_M \in N$

(5) $r(a) \le \overset{\checkmark \text{proj}}{p = p^2} \Rightarrow U_p a = a$

Pf. (1) Def, Prop 2.13

(2) Apply $U_{r(a)}$ to $a \le \|a\| \cdot 1$

(3) from (1) $(\Rightarrow)$ and (2) $(\Leftarrow)$

(4) $a \le b \overset{(2)}{\le} \|b\| r(b)$

$\Rightarrow a \in \text{face}(r(b)) = \text{im}^+ U_{r(b)}$

$\Rightarrow U_{r(b)} a = a$

$\Rightarrow r(a) \le r(b)$

(5) $r(a) \le p \Rightarrow a \in \text{face}(r(a)) \subset \text{face}(p) = \text{im}^{+} U_p$

$\Rightarrow U_p a = a$ $\square$

Prop (2.16) Let $0 \le a, b$ in M, then

$a \perp b \iff r(a) \perp r(b)$

Pf. $r(a) \perp r(b) \overset{\text{Def}}{\Longrightarrow} U_{r(a)} r(b) = 0$

$\overset{\text{Prop 2.15(2)}}{\Longrightarrow} 0 \le U_{r(a)} b \le \|b\| \underbrace{U_{r(a)} r(b)}_{=0}$

$\Rightarrow U_{r(a)} b = 0$

$\overset{\text{Lem 1.26}}{\Longrightarrow} U_b r(a) = 0$

Hence

$0 \le U_b a \le \|a\| U_b r(a) = 0$

So. $a \perp b$

$a \perp b \Rightarrow U_a b = 0$

$\Rightarrow U_a \text{ ann face}(b)$

$\overset{\text{Prop 2.15(2)}}{\Longrightarrow} U_a r(b) = 0$

$\Rightarrow a \perp r(b)$

replace a by $r(b)$

b by a $\square$

then $r(b) \perp r(a)$

Cor. $0 \le a, b$, $a \perp b$ then $r(a) \perp b$ i.e.

$U_{r(a)} b = 0$ and $U_{r(a)'} b = b$

Cor (2.17) (1) $a \ge 0 \iff \sigma(a) \ge 0 \quad \forall \sigma \in K$

(2) $\|a\| = \sup\{ |\sigma(a)| \mid \sigma \in K \}$

Pf. (1) Suppose $\delta(a) \geq 0$. $\forall \delta \in K$.

Write $a = a^+ - a^-$

If $a^- \neq 0$ K sep points

$\exists \omega \in K$ s.t.

$\omega(a^-) \neq 0 \overset{\text{Prop 2.13 (2)}}{\Longrightarrow} \omega(r(a^-)) \neq 0$

Define

$\tau = U^*_{r(a^-)} \omega \in V$ (Prop 2.4)

then

$\tau(1) = (U^*_{r(a^-)} \omega)(1) = \omega(r(a^-)) \neq 0$

so $\tau \neq 0$. By cor 2.6 τ normal.

$U_{r(a^-)}(a^+) = 0$ $a^+ \perp a^-$

But

$\tau(a) = \omega(U_{r(a^-)}(a^+ - a^-)) = -\omega(a^-) < 0$

(2) $\|\cdot\|$ is the order unit norm. by (1) K space determine the order so

$|\delta(a)| \leq 1$ $\forall \delta \in K$

$\Leftrightarrow -1 \leq a \leq 1$

$\Leftrightarrow \|a\| \leq 1$

take sup $\square$

Prop (2.18) Let p, q proj in JBW-alg M

TFAE

(1) $p \perp q$

(2) $p \circ q = 0$

(3) $p \leq q' = 1 - q$

(4) $p + q \leq 1$

(5) $U_p U_q = 0$

Pf. $(1) \Rightarrow (2)$ Lem 1.28

$(2) \Rightarrow (3)$ $p \circ q = 0$

$\rightarrow (p+q)^2 = p+q$ so

$p+q$ proj then

$p+q \leq 1$ so $p \leq 1-q$

$(3) \Leftrightarrow (4)$ ✓

$(3) \Rightarrow (5)$ If $p \leq 1-q$, then $q \leq 1-p$

so $U_p q \leq U_p(1-p) = 0$

$U_p p'$

$\forall a \geq 0$ $U_p U_q a \leq U_p U_q(\|a\| 1)$

$= \|a\| U_p q$

$= 0$

in general write $a = a^+ - a^-$

$(5) \Rightarrow (1)$ $U_p U_q = 0 \Rightarrow U_p q = 0 \Rightarrow p \perp q$ □

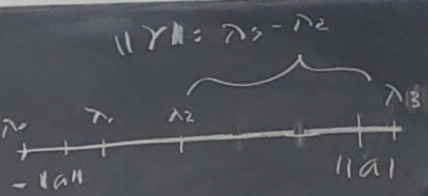
Def. M is a JBW-alg, $a \in M$.

$\gamma = \{\lambda_0, \lambda_1, \dots, \lambda_n\} \subseteq \mathbb{R}$ increasing

$\lambda_0 \leq -\|a\|$ and $\lambda_n \geq \|a\|$

define

$\|\gamma\| = \max_i \{\lambda_i - \lambda_{i-1}\}$



Thm (Spectral resolution for $C_{\mathbb{R}}(X)$). Let X be a compact Hausdorff space s.t. $C_{\mathbb{R}}(X)$ is monotone complete and let $a \in C_{\mathbb{R}}(X)$. Then $\exists!$ family of proj $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ in $C_{\mathbb{R}}(X)$ s.t.

(1) $e_\lambda a \leq \lambda e_\lambda$ and $\overset{1-e_\lambda}{e'_\lambda}\, a \geq \lambda e'_\lambda$ $\forall \lambda \in \mathbb{R}$

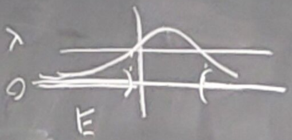
(2) $e_\lambda = 0$ for $\lambda \leq -\|a\|$ and $e_\lambda = 1$ for $\lambda > \|a\|$

(3) $e_\lambda \leq e_\mu$ for $\lambda \leq \mu$

(4) $\bigwedge_{\mu>\lambda} e_\mu = e_\lambda \quad \forall \lambda \in \mathbb{R}$

↑ greastest lower bound

$\{e_\lambda\}$ is given $e_\lambda = 1 - r((a-\lambda 1)^+)$



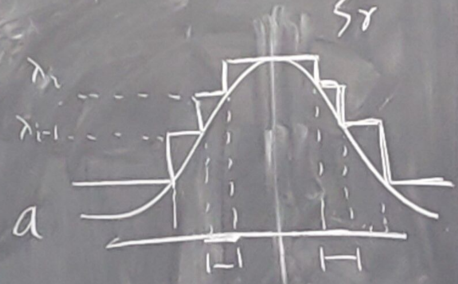
Define

$$S_\gamma = \sum_{i=1}^{n} \lambda_i (e_{\lambda_i} - e_{\lambda_{i-1}})$$

then

$$\lim_{\|\gamma\|\to 0} \|S_\gamma - a\| = 0$$

write $a = \int \lambda \, de_\lambda$



$E_i \setminus E_{i-1}$

$e_{\lambda_i} = \chi_{E_i}$

Thm (Spectral resolution for JBW-alg)

(5) each e_λ oper comm with a

Pf. $W(a,1) \cong C_{\mathbb{R}}(X)$

$W(a,1,\{e_\lambda\}) \cong C_{\mathbb{R}}(X')$

$e_\lambda \in W(a,1) \subseteq W(a,1,\{e_\lambda\})$ □

Def. $\{e_\lambda\}$ spectral projection of a

Thm above call spectral resolution for a.