FA seminar 23 on Harmonic Analysis: Lecture notes on the Orbit Method

https://www.math.leidenuniv.nl/~mdejeu/fasem_2023.html

19-06-2023

	Goal:	Method:
Rule 1:	Describe the topology of the unitary dual.	Consider the quotient topology on the space of coadjoint orbits: $\hat{G} \cong \mathfrak{g}^*/G$
Rule 2:	Construct the unitary irreducible representation π_F associated to the coadjoint orbit of $F \in \mathfrak{g}^*$.	Take a polarisation $\mathfrak{h} \subseteq \mathfrak{g}$ subordinate to F, and set $\pi_F = \mathrm{Ind}_H^G\ \rho_{F,H}$.
$\vdots$		
Rule 10:	Compute the Plancherel measure dP on $\hat{G}$.	Decompose the Lebesgue measure on $\mathfrak{g}^*$ into the Liouville measures on the coadjoint orbits.

Table 1: User's Guide to the orbit method [Kir15, p.xix].

Feel free to remark any mistakes or gaps in these notes. Any mistakes are due to me, as are any inaccuracies in claims without reference and the computed examples. The main reference for the method is [Kir15], although most preliminaries were found elsewhere.

1 Introduction

The orbit method could be described as a heuristic on where to look for unitary irreducible representations of Lie groups and their properties and how to understand them geometrically. For certain classes of Lie groups it modifies to theorems. When it works (already), it consists of some 'rules' as in table 1. We will discuss rule 1,2 and 10 for the Heisenberg group and catch a glimpse of the proof for simply-connected nilpotent groups, for which the rules are a theorem without modification. This was the type of group discussed in the seminal paper on the subject.

Contents

2 Part I: Lie Groups and Coadjoint Orbits

We start by providing some background for Lie groups, as these are the groups in the definition of the coadjoint orbits. We restrict our attention to matrices for exposition.

2.1 Lie groups

Definition 1. A **Lie group** G is a smooth (Hausdorff, second countable) manifold and a group, such that the multiplication is smooth.

This class of topological groups could be called 'rigid' in the following sense:

Proposition 1.
- *The inversion mapping is smooth (so Lie groups are topological groups) [HN11, exercise 9.4.1]*
- *Continuous morphisms between Lie groups are smooth, and so each topological group admits a unique Lie group smooth structure [HN11, thm 9.2.16]*
- *Hilbert's 5th: Locally euclidean (Hausdorff, second countable) topological groups are isomorphic to a Lie group [Tao14]*

To a Lie group G one can associate a Lie algebra $\mathfrak{g}$, realised as the tangent space at the identity, and the following subclass of Lie groups provides a convenient description of this Lie algebra[1]. Emphasis be put on the remark that this is merely an expository choice.

Definition 2. A **Matrix Lie group** is a closed subgroup $G \subset GL_n := \{A \in Mat_n(\mathbf{R}) \mid \det(A) \neq 0\} \subset \mathbf{R}^{n^2}\}$ for some natural number n.

While any subgroup of GL_n is a topological group, the closedness condition is necessary and sufficient for it to be a Lie group [HN11, prop 9.3.9].

Definition 3. The **Lie algebra** of a matrix group G is the vectorspace

$$\mathfrak{g} := \{A \in Mat_n(\mathbf{R}) \mid \forall t \in \mathbf{R} : \quad \exp(tA) \in G\}.$$

This is a vectorspace, and indeed closed under taking the commutator bracket $[A, B] := AB - BA$ [HN11, prop4.1.4].

Definition 4. A **Lie subalgebra** of a Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is a subspace closed under taking the commutator.

By definition, the exponent of a Lie algebra element $A \in \mathfrak{g}$ will lie in the Lie group G, and hence we can create the following Lie groups:

Definition 5. From a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we create the Lie group $\exp \mathfrak{h} \subset G$, generated by elements of the form $\exp A$ for $A \in \mathfrak{h}$.

These will play a role in the construction of unirreps := unitary irreducible representations of G.

Example 1 (Heisenberg)**.** As a running example we take the Heisenberg group H as it is low-dimensional enough to be convenient for calculations, nilpotent enough for the orbit method to work (without modifications, and minimizing symplectic geometry) while demonstrating interesting orbit-related behaviour of Lie group representations in general.

$$H := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbf{R} \right\} \subset GL_3$$

is a (unimodular) matrix Lie group, with Lie algebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

[1] These are some results to convince one that this expository restriction is not that bad: At least compact Lie groups and Nilpotent Lie groups are of this type. For compact Lie groups this can be shown using Peter-Weyl, and for Nilpotent groups see the references in [Kir15, p.72].

Also, by **Ado's theorem**, every finite dimensional Lie algebra can be represented as some matrices with the commutator as Lie bracket [HN11, thm 7.4.11]. As a corollary, we get **Lie's third theorem**, asserting that every finite dimensional Lie algebra is in fact the tangent space at the identity of some Lie group [HN11, thm 9.4.11].

as one can compute easily since the exponential truncates after second order. It is generated by matrices

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

satisfying the relation $[X, Y] = Z$.

The Heisenberg group is an example of a **Nilpotent group**: a (Lie group isomorphic to a) matrix Lie group consisting of matrices that are upper-triangular with 1's on the diagonal. Astoundingly:

Proposition 2. *[Kir15, p.72] For simply-connected Nilpotent groups G we have:*

- *The exponential map is a diffeomorphism* $\exp : \mathfrak{g} \longrightarrow G$
- *G is unimodular, and the Lebesgue measure on its Lie algebra induces a biinvariant Haar measure via* $\exp$

2.2 Orbits

A Lie group acts on its Lie algebra's dual, introducing the orbits of the orbit method. Although we minimize the discussion of symplectic geometry aspects in this exposition, it might be good to mention that these coadjoint orbits admit a natural symplectic form[2] [Kir15, chapter 1.2], appearing e.g. in the formulas of the 10 rules.

Definition 6. The **Adjoint representation** of a matrix Lie group G,

$$Ad : G \longrightarrow GL(\mathfrak{g}), g \mapsto Ad_g$$

is defined by

$$Ad_g(A) := gAg^{-1}.$$

It is a real representation of the topological group G.
The **Coadjoint representation** is then the contragradient representation on $\mathfrak{g}^* := hom(\mathfrak{g}, \mathbf{R})$:

$$Ad^* : G \longrightarrow GL(\mathfrak{g}^*),$$

$$Ad^*_g(F) := F \circ Ad_{g^{-1}},$$

precomposing a functional F with the adjoint representation.

By rigidity, these are indeed smooth maps.

As a tangent, the modular function on a Lie group can be computed as follows:

Proposition 3. *[HN11, def 10.4.9] The modular function of a matrix Lie group satisfies:*

$$\Delta(g) = |\det Ad_g|.$$

Exercise 1. Understand why this formula equals 1 for your favourite unimodular group, which is $SU(2)$.

Let $\Omega_F \in \mathfrak{g}^*/G$ denote the orbit of the functional F, it is a topological space by the subspace topology in $\mathfrak{g}^*$ with the compact-open topology. For connected nilpotent groups the orbits are cut out by polynomials [94, lem 7.11]. For simply-connected nilpotent groups, the orbits are simply-connected too [Kir15, p.72].
We introduce one last notion before we look at some important representations[3]:

Definition 7. Given a Lie algebra $\mathfrak{g}$ and a functional F on it, a **Polarisation** for F is a maximal Lie subalgebra $\mathfrak{h}$ such that

$$F([\mathfrak{h}, \mathfrak{h}]) = 0.$$

[2] In fact, coadjoint orbits are the symplectic leaves of the Poisson manifold $\mathfrak{g}^*$, and in some sense universal examples in the Poisson context, in that homogeneous Poisson G-manifolds naturally cover coadjoint G-orbits. [Kir15, p.17]

[3] It corresponds to (G-invariant) **real polarisations**, i.e. integrable distributions with Lagrangian fibres over the symplectic manifolds, in the context of geometric quantisation. [Kir15, p.28]

The maximality amounts to the formula

$$2dim\mathfrak{h} = dim\mathfrak{g} + dimStab_F.$$

Here $Stab_F$ denotes the stabiliser Lie group of F.

Given a polarisation, one can construct a 1-dimensional unirrep of $\exp(\mathfrak{h})$ given by:

$$\rho_{F,\mathfrak{h}}(e^X) := e^{2\pi i F(X)},$$

acting by multiplication on $\mathbf{C}$. An idea of the orbit method is that these unirreps induce the (classes of) unirreps, as in rule 2 of 1.

Lemma 1. *[94, lem 7.7] These unirreps are well-defined.*

Proof. (sketch) The point is that we have the BCH formula for the product $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\text{ higher commutators}}$, so $\rho_{F,\mathfrak{h}}$ need *a priori* not be a homomorphism. However F vanishes on commutators, so that

$$\rho_{F,\mathfrak{h}}(e^X e^Y) = \rho_{F,\mathfrak{h}}(e^{X+Y+\frac{1}{2}[X,Y]+\text{ higher commutators}}) = e^{2\pi i F(X+Y)} = \rho_{F,\mathfrak{h}}(e^X)\rho_{F,\mathfrak{h}}(e^Y).$$

□

The maximality condition for polarisations further ensures that they induce irreducible representations [94, thm 7.2].

Luckily for us, polarisations exist for any functional for nilpotent Lie groups [Wal77]. There are some theorems and algorithms to determine polarisations as mentioned in [94], converting a part of the problem of finding representations to one of Lie algebras.

Consider the following examples of orbits, which we portray as orbits inside $\mathbf{R}^3$.

Example 2 (Heisenberg)**.** Recall that the vectorspace of matrices has a non-degenerate bilinear form by tracing the product of two matrices, and compute that then $\mathfrak{h}^\perp$ consists of upper-triangular matrices. Note that

$$Mat_3\mathbf{R}/\mathfrak{h}^\perp \cong \mathfrak{h}^*,$$
$$A + \mathfrak{h}^\perp \mapsto tr(A\_).$$

This isomorphism intertwines the coadjoint action with the induced adjoint action on the quotient:

$$Ad_g^*(tr(A\_)) = tr(Ag^{-1}\_g) = tr(gAg^{-1}\_) = tr(Ad_g(A)\_) \cong Ad_gA + \mathfrak{h}^\perp.$$

Let $X = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$, $Z = \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$, $Y = \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}$ be a basis of $\mathfrak{h}$, and X^*, Y^*, Z^* the dual basis. Then the isomorphism is generated by

$$\mathfrak{h}^* \longrightarrow Mat_3\mathbf{R}/\mathfrak{h}^\perp \cong \mathbf{R}^3$$

$$X^* \mapsto \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix} \mapsto (1,0,0)$$
$$Y^* \mapsto \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix} \mapsto (0,1,0)$$
$$Z^* \mapsto \begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix} \mapsto (0,0,1)$$

Here the square brackets remind us to consider the matrices up to upper-diagonal elements. We thus compute the orbit of (x,y,z) by conjugation of a general element $\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix}$ as the collection strict lower-triangular entrees of

$$\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}\begin{bmatrix}0 & 0 & 0\\ x & 0 & 0\\ z & y & 0\end{bmatrix}\begin{pmatrix}1 & -a & (ab-c)\\ 0 & 1 & -b\\ 0 & 0 & 1\end{pmatrix} = \begin{bmatrix}ax+cz & cy-a\,(ax+cz) & -cby-(c-ab)\,(ax+cz)\\ bz+x & by-a\,(bz+x) & -b^2y-(bz+x)\,(c-ab)\\ z & y-az & -by-z\,(c-ab)\end{bmatrix}$$
$$= \begin{bmatrix}0 & 0 & 0\\ bz+x & 0 & 0\\ z & y-az & 0\end{bmatrix}.$$

Thus

$$\Omega_{(x,y,z)} = \{(x+bz, y-az, z) \mid a, b \in \mathbf{R}^2\}.$$

If $z = 0$, the orbit is just a point:

$$(x, y, 0) = \Omega_{(x,y,0)} =: \Omega_{x,y}.$$

If $z \neq 0$, the orbit is the plane given by the value of $z = z_0$:

$$\Omega_{(x,y,z)} =: \Omega_z.$$

(For later use we mention that the symplectic form on Ω_z is $\omega_z := \frac{1}{z} dx \wedge dy$ [Kir15, p.62]).

Example 3 (SL_2)**.** Its Lie algebra $sl_2 \cong \mathbf{R}^3$ in an intertwining way. The coadjoint orbits are the connected components of $\{(x, y, z) \mid x^2 + y^2 - z^2 = C\} \setminus \{0\}$ for some $C \in \mathbf{R}$, and the origin. For $C > 0$ we get hyperboloids, $C < 0$ gives paraboloids, and $C = 0$ gives the cone. We note that in the Iwasawa decomposition $SL_2 = ANK$ [DE14, thm 11.1.1], the stabilisers of points of the hyperboloids correspond to A, for paraboloids to K, and for the cone minus the origin to N.

Example 4 ($SU(2)$)**.** The coadjoint orbits will be concentric spheres, including the origin. We note that the stabiliser of a sphere corresponds to the maximal torus $S^1 \subset SU(2)$. Indeed $\Omega_F \cong SU(2)/Stab_F \cong S^3/S^1 \cong S^2$, referencing the Hopf-fibration in [DE14, lem 7.5.6].

Now that we know what the orbits in the orbit method are, we look at some general constructions in the harmonic analysis of locally compact groups, that appear in the rules 1 and 2 in table 1.

3 Part II: Induced Representations and Topology

Consider now just locally compact groups. In this section we extend the induction procedure for compact groups in [DE14, chapter 7.4], and introduce a topology on the unitary dual.

3.1 Induced representations

The general reference for this subsection is [Kan13, chapter 2].Note that this book takes the L_x point of view of induced representations, for right actions, and assumes $f(xh) = \pi(h)^{-1}f(x)$ as the equivariance condition, which we keep for convenient theoretical reference. At the end of this subsection we translate it to the convention we have seen[4].

Let a locally compact group G, closed subgroup H and unitary representation (π, V_π) of H be given. We discuss three incarnations of induced representations[5], the first of which might be the most intuitively comparable with what we have seen for compact groups.

Recall that for compact groups, the irreducible representations are finite dimensional, continuous functions are integrable, and the modular function is 1. In general, define on H the 'mismatch' in modular functions: $\delta(h) := \sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}}$.

[4] As does [DE14] in the compact case, [Kir15] uses equivariance condition $f(hx) = \pi(h)f(x)$ in the function space, and represents by right regular representation. Then [Kir15] creates a section induced representation in solving the equation $\gamma(yH)x = h(yH, x)\gamma(yxH)$ for h and representing G by $[x \cdot f](yH) = \pi(h(yH, x))f(yxH)$ on a space $L^2(G/H, \mu_\gamma)$ for a measure μ_γ associated to the section γ, which is assumed to exists on a certain subset of G/H, and continuous.

[5] The equivariance condition on the functions can be understood as requiring the functions to be **basic** with respect to the representation π, in the sense of vector-valued sections of principal bundles, so that these equivariant functions $G \longrightarrow V$ correspond to sections of the associated vector bundle $G/H \longrightarrow G \times_\pi V$. In this sense the induction procedure can be interpreted as an 'extension of scalars' from V to (sections of) $G \times_\pi V$, and in fact for finite groups, the Frobenius reciprocity theorem (induction-restriction adjunction) is the extension-of-scalars adjunction for modules (by viewing G-representations as $\mathbf{C}[G]$-modules).

Definition 8. The **(intrinsic) Induced representation** $\mathrm{Ind}_H^G\pi$ is the completion of the space

$$F_\delta(G,\pi) := \{f \in C(G,V_\pi) \mid \text{ supp } f/H \text{ compact },\quad f(xh) = \pi(h)^{-1}f(x)\delta(h)\},$$

with innerproduct

$$\langle f,g\rangle := \int_G \psi(x)\langle f(x),g(x)\rangle_{V_\pi} dx,$$

for any $\psi \in C_c(G) : \psi^H_{suppf/H \cup suppg/H} = 1$. Represent G unitarily on this space by

$$[\mathrm{Ind}\pi(x)f]\,(y) := f(x^{-1}y).$$

The independence of such a function ψ can be shown by (even more intrinsically) equating this innerproduct with the volume of G/H with respect to a complex Radon measure depending only on f,g as in [Kan13, prop 2.20]. The unitarity of this representation is saved by the twisting by δ in the function space $F_\delta(G,\pi)$.

Recall that for subgroups such that $\delta = 1$, and only for those, there exists an *invariant* Radon measure on the quotient, and we have the Quotient Integral Formula as in [DE14, thm 1.5.3]. One can extend also this (with some work):

Definition 9. A **quasi-invariant measure** on G/H is a regular Borel measure μ, such that

$$\mu^x \gg \mu \gg \mu^x.$$

Here $\mu^x(A) := \mu(xA)$.

It is an invariant measure according to the null-sets, in that translates of null-set are null-sets still.

Proposition 4 (Quotient Integral Formula)**.** *There exists a quasi-invariant regular Borel measure*[6] *μ on G/H, such that the Radon-Nikodym derivative gives a continuous function*

$$G \times G/H \longrightarrow \mathbf{C},$$

$$(x,yH) \mapsto \frac{d\mu_x}{d\mu}(yH).$$

Furthermore, there exists a continuous, strictly-positive function ρ such that $\forall f \in C_c(G)$:

$$\int_{G/H}\int_H f(xh)dhd\mu(xH) = \int_G f(x)\rho(x)dx.$$

[Kan13, thm 1.18, cor 1.21]

Using the existence of such measures and fixing one, defines the second incarnation, where the twist by δ is now hidden in the action and we integrate over the quotient instead of G. Different choices of μ lead to unitarily isomorphic representations:

Definition 10. The **(measure) Induced representation** U^π_μ is the completion of the space

$$F(G,\pi) := \{f \in C(G,V_\pi) \mid \text{ supp } f/H \text{ compact },\quad f(xh) = \pi(h)^{-1}f(g)\},$$

with innerproduct

$$\langle f,g\rangle_\mu := \int_{G/H}\langle f(x),g(x)\rangle_{V_\pi} d\mu(xH).$$

Represent G unitarily on this space by

$$\left[U^\pi_\mu(x)f\right](y) := \sqrt{\frac{d\mu_{x^{-1}}}{d\mu}(yH)}\cdot f(x^{-1}y).$$

[6][Kan13, thm 1.18] states this as only a regular Borel measure, although it seems to be induced by the Riesz-representation theorem, and thus Radon. The proof of this, as well as the isomorphism between the first and second version of induced representations, use a so-called ρ-function inducing the measure, behaving nice w.r.t. δ. It is appearing as the extra factor in the QIF.

Note that by the unitarity of π and equivariance of the functions, the map $x \mapsto \langle f(x), g(x)\rangle_{V_\pi}$ on G descends to the quotient. The Radon-Nikodym factor makes sure we get a unitary representation.

The third incarnation requires an additional (non-canonical) choice, if it exists, namely a **measurable section**[7] γ of the quotient map $q : G \longrightarrow G/H$, i.e., a measurable functions such that $q \circ \gamma = Id$; it gives a preferred choice of representative. This will realise the induced representation concretely as an L^2-space on the quotient, *if* such a section exists.

Definition 11. The **(section) Induced representation** $\mathrm{Ind}_{\gamma,\mu}\pi$ is the Hilbert space[8]

$$L^2(G/H, \mu; V_\pi) := \{f : G/H \longrightarrow V_\pi \mid f \text{ measurable}\ , \quad \int_{G/H} ||f||^2 d\mu < \infty\}/ \text{ a.e.}\ ,$$

with innerproduct

$$\langle f, g\rangle_\mu := \int_{G/H} \langle f(xH), g(xH)\rangle_{V_\pi} d\mu(xH).$$

Represent G unitarily on this space by the elegant formula[9]

$$[\mathrm{Ind}_{\gamma,\mu}\pi(x)f]\,(yH) := \pi\left(\gamma(yH)^{-1}x\gamma(x^{-1}yH)\right) \cdot \sqrt{\frac{d\mu_{x^{-1}}}{d\mu}(yH)} \cdot f(x^{-1}yH).$$

We will see an example showing how much this expression can simplify.

Theorem 1. *The three incarnations are unitarily isomorphic, requiring existence of a measurable section for the second isomorphism:* $Ind_H^G\pi \xrightarrow{\cdot\sqrt{\rho}} U_\mu^\pi \xrightarrow{\gamma^*} L^2(G/H, \mu; V_\pi)$.

The first isomorphism is multiplication with square of the ρ-function associated to the quasi-invariant measure μ, the second by precomposition with the section γ.

Intrinsic	Measure	Section (if $\exists\gamma$)
$\mathrm{Ind}_H^G\pi$	U_μ^π	$\mathrm{L}^2(G/H, \mu; V_\pi)$
$\langle f, g\rangle = \int_G \psi(x)\langle f(x), g(x)\rangle_{V_\pi} dx$	$\langle f, g\rangle_\mu = \int_{G/H}\langle f(x), g(x)\rangle_{V_\pi} d\mu(xH)$	$\langle f, g\rangle_\mu = \int_{G/H}\langle f(xH), g(xH)\rangle_{V_\pi} d\mu(xH)$
$[\mathrm{Ind}\pi(x)f]\,(y) = f(x^{-1}y)$	$\left[U_\mu^\pi(x)f\right](y) = \sqrt{\frac{d\mu_{x^{-1}}}{d\mu}(yH)} \cdot f(x^{-1}y)$	$[\mathrm{Ind}_{\gamma,\mu}\pi(x)f]\,(yH) = \pi\left(\gamma(yH)^{-1}x\gamma(x^{-1}yH)\right) \cdot \sqrt{\frac{d\mu_{x^{-1}}}{d\mu}(yH)} \cdot f(x^{-1}yH)$

Tracing back the definitions, in conventions of [Kir15] and [DE14] and our example, for H acting on the left on G, we rephrase the action of the section induced representation $L^2(G/H, \mu; V_\pi)$, assuming G/H has invariant measure:

$$[\mathrm{Ind}^{\gamma,\mu}\pi(x)f]\,(yH) := \pi\left(\gamma(yH)x\gamma(yxH)^{-1}\right) \cdot f(yxH).$$

3.2 Imprimitivity

Rule 2 of the orbit method states that the unitary dual should consist of induced representations. The following general concept for locally compact groups G aids determining if a representation is induced. See the reference [Kan13, chapter 3] for more on this, again in the left-regular convention.

Definition 12 (SOI)**.** Let G act continuously on a LCH X. A **System Of Imprimitivity** for the action of G on X is a pair

$$(\pi : G \longrightarrow \mathcal{U}(V), P : C_0(X) \longrightarrow B(V))\,,$$

with π unitary G-representation and P a non-degenerate $*$-representation on the same Hilbert space V, such that:

$$\pi(x)P(\phi)\pi(x^{-1}) = P(L_x\phi).$$

The big theorem about SOI [Kan13, thm 3.17] for $(G, G/H)$ with H a closed subgroup, has as important corollary that

Corollary 1. The existence of a SOI for a G-representation π implies that it is induced by one of H.

This theorem is used in the background for the proofs of the orbit methods.

[7] Note that a *continuous* section would imply that the principal H-bundle $G \longrightarrow G/H$ is trivial, so that $G \cong H \times G/H$.

[8] Completeness might be proven analogous to the $V_\pi = \mathbf{C}$ case, or arguing (analogous to [DE14] for the compact case) that $L^2(G/H, V) \cong L^2(G/H) \otimes V$ [Kan13, p.74]. For our purposes we look at 1-dimensional representations V_π, and completeness is clear.

[9] Note that this expression only needs to be measureable in yH, and continuity in x follows from the second isomorphism in the theorem 1, and continuity for U_μ^π.

3.3 Fell topology

There exists a topology on the unitary dual of a locally compact group G that has properties relating to group C^*-algebras, and should relate via the orbit method to the topology on the set of orbits as in rule 1 of table 1.

Definition 13. ([Kir15, p.106]) The **Fell topology** on the space of unitary G-representations modulo equivalence, $\tilde{G}$, is generated by the following neighbourhoods of a representation π:

$$U_{K,\epsilon,x_1,\dots x_n} := \{\sigma \in \tilde{G} \mid \exists y_1, \dots y_n \in V_\sigma : \forall g \in K : \quad |\langle \pi(g)x_i, x_j\rangle - \langle \sigma(g)y_i, y_j\rangle| < \epsilon\},$$

for some compact set $K \subset G, \epsilon > 0$ and finite $x_i \in V_\pi$.

Then we equip $\hat{G}$ with the subspace topology. Note that this topology need not be Hausdorff, and thus limits need not be unique.

Proposition 5. *Induction is continuous as map* $Ind_H^G : \tilde{H} \longrightarrow \tilde{G}$ *[Kir15, p.107].*

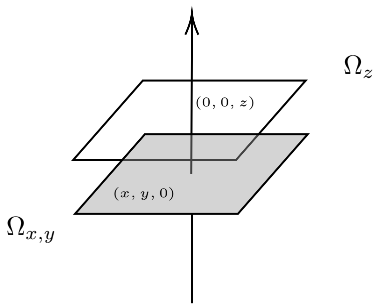
The Fell topology should make $\hat{G}$ homeomorphic to $C^*\hat{(G)}$ [Kan13, def 1.67], where the latter topology is in terms of primitive ideals of C^*-algebras [htt][10]. We list some properties:

Proposition 6.
- *For LCA groups A, the Fell topology coincides with the LCA topology on the characters. Also, the Fell topology is discrete for compact groups K [Fol16, prop 7.4]*
- *If G is second countable, TFAE: G is type I, the Fell topology is T_0*
- *If G is a connected nilpotent Lie group, the Fell topology is T_1 (points are closed)*

Now back to the orbit method.

4 Part III: Heisenberg

For simply-connected nilpotent Lie groups, [94, thm 7.2] states that rule 2 gives all unirreps, and that different points in the same orbit induce equivalent unirreps. We illustrate rules 1, 2 and 10.



Consider the Heisenberg group H, and use the basis X, Y, Z with $[X, Y] = Z$ for $\mathfrak{h}$. Every element of $H \ni (a, b, c)$ can be written as $e^{aX+bY+c'Z}$ for $c' = c - \frac{ab}{2}$. Identify a functional $xX^* + yY^* + zZ^* \in \mathfrak{h}^*$ with $(x, y, z) \in \mathbf{R}^3$ as at the end of Part I. Its orbits are planes Ω_z parameterised by $z \in \mathbf{R}^\times$, and points $\Omega_{x,y}$ parameterised by $(x, y) \in \mathbf{R}^2$.

4.1 Unitary dual

The **recipe** for describing the unitary dual is as follows[11]:

- Represent some orbit Ω_F by $F \in \mathfrak{h}^*$,
- find a polarisation $\mathfrak{a}$ for F (i.e. a subalgebra with $F([\mathfrak{a}, \mathfrak{a}]) = 0$ of maximal dimension),
- take the representation $\rho_{F,\mathfrak{a}}$ of $A := exp\mathfrak{a}$, and
- realise the induced representation $\pi := \mathrm{Ind}_A^H \rho_{F,\mathfrak{a}}$ on some L^2-space.

[10] A better reference is to be found.

[11] It deviates slightly from the recipe in [Kir15] in how we compute the section induced action.

1. For the singleton orbit $\Omega_{x,y}$, the functional $F = (x, y, 0)$ vanishes on $[\mathfrak{h}, \mathfrak{h}] = Z\mathbf{R}$, so that $\mathfrak{h}$ is a polarisation. We get the unirrep

$$\pi_{x,y}(a, b, c) = \rho_{F,\mathfrak{h}}(\exp aX + bY + cZ) = e^{2\pi i(ax+by)}.$$

2. For the planes Ω_z, take the functional $F = (0, 0, z \neq 0)$. Looking at the commutators of the Lie algebra, we notice that a subalgebra containing X and Y will not be a polarisation, as the commutator Z will be present, and $F(Z) = z$. Then consider the polarisation

$$\mathfrak{a} = Y\mathbf{R} + Z\mathbf{R},$$

and write $A := exp\mathfrak{a}$. Define the continuous section

$$\gamma : H/A \longrightarrow H, \gamma(e^{aX}A) := e^{aX}.$$

Note that by nilpotency, H/A has an invariant measure μ, and we claim[12]

$$(H/A, \mu) \cong (\mathbf{R}, \lambda),$$

$$e^{aX}A \mapsto a.$$

So, the section induced representation acts on

$$\mathrm{Ind}_A^H \rho_{F,\mathfrak{a}} = L^2(H/A, \mu) \cong L^2(\mathbf{R}).$$

We now compute how it acts using BCH, $[X, Y] = Z$ and the fact that we look up to A, writing $\pi_z := \mathrm{Ind}_A^H \rho_{F,\mathfrak{a}}$:

$$\begin{aligned}
[\pi_z(e^{aX+bY+c'Z})f](e^{xX}A) &:= \rho_{F,A}(\gamma(e^{xX}A) \cdot e^{aX+bY+c'Z} \cdot \gamma(e^{xX}e^{aX+bY+c'Z}A)^{-1}) \cdot f(e^{xX}e^{aX+bY+c'Z}A) \quad (1)\\
&= \rho_{F,A}(e^{xX} \cdot e^{aX+bY+c'Z} \cdot e^{-(x+a)X}) \cdot f(e^{(x+a)X}) \quad (2)\\
&= \rho_{F,A}(e^{bY+(c+xb)Z}) \cdot f(e^{(x+a)X}) \quad (3)\\
&\cong e^{2\pi i(c+xb)z} f(x + a) \quad (4)
\end{aligned}$$

We conclude that the orbit method agrees with the Stone-von Neumann theorem [DE14, thm 10.2.1].

4.2 Topology

We reference [94]. For any simply-connected nilpotent Lie group, rule 1 holds perfectly [94, thm 7.3]. For the Heisenberg group,

$$\hat{H} \cong \mathfrak{h}^*/H \cong \mathbf{R} \cup_\sim \mathbf{R}^2,$$

$$\pi_z \mapsto \Omega_z \mapsto z,$$

$$\pi_{x,y} \mapsto \Omega_{x,y} \mapsto (x, y).$$

This space has the topology of $\mathbf{R}$, but the origin is removed, and we attach a $\mathbf{R}^2$ to it. The topology is such that that taking a sequence of planes Ω_{z_n} with $z_n \longrightarrow 0$, results in the limit-set

$$\Omega_{z_n} \longrightarrow \{\Omega_{x,y}\}_{x,y} = \mathbf{R}^2 \times 0.$$

A representation theoretic interpretation of this is the following, with $[\pi_0(a, b, c)f](x) := f(x + a)$:

Proposition 7. *Take $z \longrightarrow 0$, then $\pi_z \longrightarrow \pi_0$ in $\tilde{H}$, and so π_z converges to*

$$\int_{\mathbf{R}} \pi_{xt,yt} d\lambda(t), \quad (\forall x, y \in \mathbf{R}).$$

This means that, as z goes to zero, the corresponding class π_z degenerates into all representations in the plane of singleton orbits. The direct integral is over all lines through the origin in this degenerate plane, and each line defines an equivalent unitary representation.

[12] Using QIF and exp preserving measures.

Proof. WLOG we prove that

$$(\pi_0, L^2(\mathbf{R})) \cong \left(\int_{\mathbf{R}} \pi_{x,0} d\lambda(x), \int_{\mathbf{R}} \mathbf{C} d\lambda(x) \right).$$

Explicitly, we have the Hilbert bundle

$$\sqcup_x (\mathbf{C}, \pi_{x,0}) \longrightarrow (\mathbf{R}, \lambda),$$

with quasi-ONB $\{1\}$. Then by [DE14, p.159], $\int_{\mathbf{R}} \mathbf{C} d\lambda(x) \cong L^2(\mathbf{R})$ and the direct integral representation acts by

$$\left[\int_{\mathbf{R}} \pi_{x,0} d\lambda(x)(a,b,c) \cdot f \right](y) = \pi_{y,0}(a,b,c) f(y) = e^{2\pi i a y} f(y).$$

The isomorphism is then the Fourier transform, mapping translations to phases:

$$(\pi_0, L^2(\mathbf{R})) \cong \left(\int_{\mathbf{R}} \pi_{x,0} d\lambda(x), L^2(\mathbf{R}) \right).$$

□

4.3 Plancherel

Using rule 10 via the formula

$$\int_{\mathfrak{h}^*} f(F) dF = \int_{\hat{H}} \left[\int_{\Omega} f(F) d\mathcal{L}_{\Omega}(F) \right] dP(\Omega)$$

in [Kir15, p.100] and the Liouville measure $\mathcal{L}_{\Omega}$ induced by the symplectic form ω on the orbits (Ω, ω), one deduces the Plancherel formula in [DE14, thm 10.3.1] for the Plancherel measure P, which is supported on the orbits of maximal dimension $(\Omega_z, \frac{1}{z} dx \wedge dy)$:

$$dP(\Omega_z) = |z| dz$$

5 Part IV: Nilpotent Proof and other Lie Groups

See blackboard and [Kir15, p.95].

Exercise 2. Compute the orbits for your favourite (non- simply-connected nilpotent) Lie group, and figure out where the orbit method breaks down and how to fix it.

References

[94] *Representation Theory and Noncommutative Harmonic Analysis I : Fundamental Concepts. Representations of Virasoro and Affine Algebras*. eng. Encyclopaedia of Mathematical Sciences, 22. Berlin, Heidelberg: Springer Berlin Heidelberg, 1994. ISBN: 9783662030028.

[DE14] Anton Deitmar and Siegfried Echterhoff. *Principles of Harmonic Analysis*. eng. Universitext. Cham: Springer, 2014. ISBN: 9783319057910.

[Fol16] G. B. Folland. *A course in abstract harmonic analysis*. eng. 2nd ed. Textbooks in mathematics (Boca Raton, Fla.) 2016. ISBN: 0-429-15469-0.

[HN11] Joachim Hilgert and Karl-Hermann Neeb. *Structure and geometry of lie groups*. eng. Springer monographs in mathematics. Springer, 2011. ISBN: 9780387847948.

[htt] Alex zorn (https://mathoverflow.net/users/35169/alex-zorn). *Topology on the Unitary Dual*. MathOverflow. URL:https://mathoverflow.net/q/134312 (version: 2013-06-21). eprint: `https://mathoverflow.net/q/134312`. URL: `https://mathoverflow.net/q/134312`.

[Kan13] Eberhard Kaniuth. *Induced representations of locally compact groups*. eng. Cambridge tracts in mathematics ; 197. New York: Cambridge University Press, 2013. ISBN: 9781139839723.

[Kir15] A. A Kirillov. *Lectures on the Orbit Method*. eng. Vol. 64. Graduate studies in mathematics. American Mathematical Society, 2015. ISBN: 0821835300.

[Tao14] Terence Tao. *Hilbert's fifth problem and related topics*. eng. Vol. 153. Graduate Studies in Mathematics. Providence, Rhode Island: American Mathematical Society, 2014. ISBN: 147041564X.

[Wal77] Nolan R Wallach. *Symplectic geometry and Fourier analysis*. eng. Lie groups. Series A, History, frontiers and applications ; vol. 5 791635074. Brookline, Mass.: Math Sci Press, 1977. ISBN: 0915692155.