Introduction to Algebraic Topology, Fall 2018, homework sheet I

The deadline for this homework sheet is Friday, 28 September 2018. Please deliver your homework on paper into the physical mailbox of Stefan van der Lugt in the common room of the Snellius building. The use of LaTeXis strongly recommended; solutions that are not (easily) readable will be discarded. If delivery in the physical mailbox is not possible, then please submit your solutions by mail to `algtop2018@gmail.com`. You are allowed to cooperate, but copying is of course not allowed. You may use the material as discussed in Lectures 1-2, the material discussed in Fulton, Sections 11a, b, c, d, and all material from the syllabus Topologie (including the exercises). Please include references when appropriate. If you wish to use a result from the practice exercises of week 1, you are requested to include a (short) argument for that result.

Exercise 1. (i) Let $p_1\colon Y_1 \to X_1$ and $p_2\colon Y_2 \to X_2$ be covering maps. Show that the map $Y_1 \times Y_2 \to X_1 \times X_2$ given by $(y_1, y_2) \mapsto (p_1(y_1), p_2(y_2))$ is a covering map.

(ii) Exhibit a homeomorphism $\mathbb{R}_{>0} \times S^1 \xrightarrow{\sim} \mathbb{C}^*$. Hint: polar coordinates.

(iii) Show that the map $\mathbb{R}^2 \to \mathbb{R}_{>0} \times S^1$ given by $(x, y) \mapsto (\exp(x), \exp(iy))$ is a covering map.

(iv) Show that the map $\mathbb{C} \to \mathbb{C}^*$ given by $z \mapsto \exp(z)$ is a covering map.

Exercise 2. Let $p\colon Y \to X$ be a covering map, and let $x \in X$. Write $Y_x = p^{-1}\{x\}$. The monodromy action is a natural right action of the fundamental group $\pi_1(X, x)$ on the set Y_x. Let $y, y' \in Y_x$. Assume that Y is path connected.

(i) Show that the stabilizers Stab_y resp. $\mathrm{Stab}_{y'}$ of y resp. y' for the monodromy action are conjugate subgroups of $\pi_1(X, x)$.

(ii) Show that the natural map $p_*\colon \pi_1(Y, y) \to \pi_1(X, x)$ is injective, and that the image $p_*(\pi_1(Y, y)) \subseteq \pi_1(X, x)$ is equal to Stab_y.

(iii) Assume that X is simply connected. Show that the map $p\colon Y \to X$ is a homeomorphism.

Exercise 3. Let $q\colon Z \to X$ and $p\colon Y \to X$ be covering maps. Let $f\colon Z \to Y$ be a morphism of coverings (cf. Exercise 6 of the practice exercises of week 1). Let $x \in X$ and put $Z_x = q^{-1}\{x\}$ and $Y_x = p^{-1}\{x\}$.

(i) Show that the map $f|_{Z_x}\colon Z_x \to Y_x$ is compatible with the monodromy action of $\pi_1(X, x)$. More precisely, show that for all $\alpha \in \pi_1(X, x)$ and all $z \in Z_x$ the equality $f(z \cdot \alpha) = f(z) \cdot \alpha$ holds.

(ii) Let $\varphi \in \mathrm{Aut}(Y/X)$ be an automorphism of the covering map $p\colon Y \to X$. Let $y, y' \in Y_x$ and assume that $\varphi(y) = y'$. Show that the stabilizers Stab_y resp. $\mathrm{Stab}_{y'}$ of y resp. y' for the monodromy action of $\pi_1(X, x)$ on Y_x are equal in $\pi_1(X, x)$.

Exercise 4. Let Y be a Hausdorff topological space and let $G \subseteq \mathrm{Aut}(Y)$ be a finite subgroup. Assume that the action of G on Y is free[1]. Show that the action of G on Y is even.

[1]Recall that the action of G on Y is said to be free if for all $g \in G$ and all $y \in Y$ the implication $g \cdot y = y \Rightarrow g = \mathrm{id}$ holds.