Introduction to Algebraic Topology, Fall 2018, homework sheet II

The deadline for this homework sheet is Friday, 19 October 2018. Please deliver your homework on paper into the physical mailbox of Stefan van der Lugt in the common room of the Snellius building. The use of LaTeX is strongly recommended; solutions that are not (easily) readable will be discarded. If delivery in the physical mailbox is not possible, then please submit your solutions by mail to `algtop2018@gmail.com`. You are allowed to cooperate, but copying is of course not allowed. You may use the material as discussed in Lectures 1-4, the material discussed in Fulton, Chapters 11, 12, Sections 13a, b, and all material from the syllabus Topologie (including the exercises). Please include references when appropriate. If you wish to use a result from the practice exercises of week 1 or 3, you are requested to include a (short) argument for that result.

Exercise 1. Let $p\colon Y \to X$ be the 3-sheeted covering described in Exercise 11.14 of Fulton's book. In particular X consists of two circles A, B joined at a point P.

(i) Pick a small open subset $U \subset A \setminus \{P\}$ homeomorphic to an open interval. Describe (or better, draw) the inverse image of U in Y.

(ii) Let γ be the path in X starting and ending at P, that goes first around the circle A counterclockwise, then around B counterclockwise, then around A clockwise, then around B clockwise. Describe the three lifts of γ in Y.

(iii) Show that γ is not path-homotopic to a constant path in X.

(iv) Show that $p\colon Y \to X$ is not a G-covering.

Exercise 2. Let $Y = \mathbb{R}^2$. Let G be the subgroup of the group $\mathrm{Aut}(Y)$ of self-homeomorphisms of Y generated by the maps $(x, y) \mapsto (x + 1, y)$ and $(x, y) \mapsto (-x, y + 1)$.

(i) Show that the group G is not abelian.

(ii) Show that the projection map $p\colon Y \to K = Y/G$ is a G-covering. (Fun fact: the space K is homeomorphic with the *Klein bottle*.)

(iii) Let $H \subset G$ be the subgroup generated by the maps $(x, y) \mapsto (x + 1, y)$ and $(x, y) \mapsto (x, y+2)$. Show that H is a normal subgroup of G and exhibit an isomorphism of groups $G/H \xrightarrow{\sim} \boldsymbol{\mu}_2 = \{-1, +1\}$.

(iv) Let $T = S^1 \times S^1$. Exhibit a homeomorphism $Y/H \xrightarrow{\sim} T$.

(v) Exhibit a $\boldsymbol{\mu}_2$-covering $q\colon T \to K$.

Exercise 3. (i) Show that a continuous map $f\colon S^1 \to S^1$ that is not surjective has degree zero.

(ii) Give an example of a surjective continuous map $f\colon S^1 \to S^1$ whose degree is zero.