Introduction to Algebraic Topology, Fall 2018, homework sheet III

The deadline for this homework sheet is Friday, 9 November 2018. Please deliver your homework on paper into the physical mailbox of Stefan van der Lugt in the common room of the Snellius building. The use of LaTeX is strongly recommended; solutions that are not (easily) readable will be discarded. If delivery in the physical mailbox is not possible, then please submit your solutions by mail to `algtop2018@gmail.com`. You are allowed to cooperate, but copying is of course not allowed. You may use the material as discussed in Lectures 1-7, the material discussed in Fulton, Chapters 11–13, Section 14a, and all material from the syllabus Topologie (including the exercises). Please include references when appropriate. If you wish to use a result from the practice exercises of week 1, 3 or 6, you are requested to include a (short) argument for that result.

Exercise 1. Let $S^1 \subset \mathbb{R}^2$ be the unit circle. Let C be the space obtained from $[0,1] \times S^1$ by contracting the subspace $\{0\} \times S^1$ to a point. More precisely, C is the quotient space $([0,1] \times S^1)/\!\sim$ for the equivalence relation $\sim$ on $[0,1] \times S^1$ given by $y \sim y' \Leftrightarrow y = y'$ or $y, y' \in \{0\} \times S^1$.

(i) Construct a homeomorphism $C \xrightarrow{\sim} D^2$, where D^2 is the closed unit disk in $\mathbb{R}^2$.

Let X be a topological space and let $f\colon S^1 \to X$ be a continuous map.

(ii) Prove the following statement: f is homotopic to a constant map if and only if f extends into a continuous map $\tilde{f}\colon D^2 \to X$.

(iii) Let $f\colon S^1 \to S^1$ be a non-surjective continuous map. Show that f has a fixed point.

Let X be a topological space which is **connected and locally path connected.**

Exercise 2. The purpose of this exercise is to show that the task of computing the automorphism group of a connected covering of X can be reduced to a purely group theoretical question. Let π be a group and let S be a right π-set. Put

$$\mathrm{Aut}_\pi(S) = \{f\colon S \to S \,\text{bijective} : \forall \alpha \in \pi \, \forall s \in S\colon f(s \cdot \alpha) = f(s) \cdot \alpha\}\,.$$

(i) Show that the set $\mathrm{Aut}_\pi(S)$ is a group, where the group operation is composition of maps.

Let $p\colon Y \to X$ be a covering, and assume that Y is connected. Fix a base point $x \in X$ and let Y_x be the fiber of x along p. Let $\pi = \pi_1(X, x)$. We know that Y_x has a natural structure of right π-set, given by the monodromy action.

(ii) Show that the assignment $\varphi \mapsto \varphi|_{Y_x}$ yields a group homomorphism

$$\mathrm{res}\colon \mathrm{Aut}(Y/X) \to \mathrm{Aut}_\pi(Y_x)\,.$$

(iii) Show that res is a group isomorphism.

Exercise 3. Assume that there exists a universal covering $u\colon \tilde{X} \to X$. Let $U \subseteq X$ be an open subset which is evenly covered by u.

(i) Show that the natural map $\pi_1(U) \to \pi_1(X)$ induced by the inclusion $U \to X$ is trivial.

(ii) Deduce from (i) that X is semi-locally simply connected, i.e. every point in X has an open neighborhood such that every loop in the neighborhood is path homotopic in X to a constant path.