Introduction to Algebraic Topology, Fall 2018, homework sheet IV

The deadline for this homework sheet is Friday, 30 November 2018. Please deliver your homework on paper into the physical mailbox of Stefan van der Lugt in the common room of the Snellius building. The use of LaTeX is strongly recommended; solutions that are not (easily) readable will be discarded. If delivery in the physical mailbox is not possible, then please submit your solutions by mail to `algtop2018@gmail.com`. You are allowed to cooperate, but copying is of course not allowed. You may use the material as discussed in Lectures 1-9, the material discussed in Fulton, Chapters 11–14, all material from the syllabus Topologie (including the exercises), and the homework exercises I-III. Please include references when appropriate. If you wish to use a result from the practice exercises (see weeks 1, 3, 6, 8) you are requested to include a (short) argument for that result.

Exercise 1. Let $S^1 \subset \mathbb{C}$ be the unit circle, and let $N \in \mathbb{Z}_{>0}$. Let $p\colon S^1 \to S^1$ be the covering map given by $z \mapsto z^N$. Let $\zeta = \exp(2\pi i/N)$, let $G = \boldsymbol{\mu}_N = \langle \zeta \rangle \subset \mathbb{C}^\times$ and let $a \in (\mathbb{Z}/N\mathbb{Z})^\times$.

(i) Show that the left action $G \times S^1 \to S^1$ determined by $(\zeta, z) \mapsto \zeta^a \cdot z$ defines an even action of G on S^1, and that $p\colon S^1 \to S^1$ is a quotient map for this G-action.

(ii) Write $X = S^1$. Write Y_a for S^1 equipped with the G-action as in (i). Let $a, b \in (\mathbb{Z}/N\mathbb{Z})^\times$ with $a \neq b$. Show that the G-coverings $p\colon Y_a \to X$ and $p\colon Y_b \to X$ are not isomorphic (as G-coverings!).

(iii) Fix an identification $\pi_1(X, 1) \cong \mathbb{Z}$. According to Fulton, Proposition 14.1, there is a natural bijection between the set $\mathrm{Hom}(\mathbb{Z}, G)$ and the set of G-isomorphism classes of pointed G-coverings of $(X, 1)$. Let $y \in p^{-1}(1)$. Describe the element of $\mathrm{Hom}(\mathbb{Z}, G)$ corresponding to the pointed G-covering $p\colon (Y_a, y) \to (X, 1)$.

Exercise 2. Let G be a group, and let $H \subset G$ be a subgroup. Let $N \subset G$ be the normal closure of H in G, i.e., the intersection of all normal subgroups of G that contain H. Show that the diagram with natural maps

$$\begin{tikzcd} H \arrow[r] \arrow[d] & G \arrow[d] \\ \{e\} \arrow[r] & G/N \end{tikzcd}$$

is a pushout diagram of groups.

Exercise 3. Consider the set $Y = \{(x, y) \in \mathbb{R}^2 \colon x \in \mathbb{Z} \vee y \in \mathbb{Z}\}$ with the induced topology from $\mathbb{R}^2$. Let X be a figure eight.

(i) Construct a regular covering $p\colon Y \to X$ whose automorphism group $\mathrm{Aut}(Y/X)$ is isomorphic with $\mathbb{Z}^2$. Hint: let $T = S^1 \times S^1$ be the torus. The universal covering map $u\colon \mathbb{R}^2 \to T$ is regular with automorphism group $\mathbb{Z}^2$. The figure eight is naturally a subspace of T.

(ii) Show that the abelianization (see the syllabus Algebra 1, edition 2017, p. 98) of $\mathbb{Z} * \mathbb{Z}$ is equal to $\mathbb{Z}^2$. Hint: observe that $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$ is the *direct sum* of two copies of $\mathbb{Z}$, as introduced in Exercise 46 of §9 (p. 122) of the syllabus Algebra 1, edition 2017. Try to use universal properties as much as possible!

(iii) Show that the fundamental group of Y is isomorphic with the commutator subgroup of $\mathbb{Z} * \mathbb{Z}$.