Introduction to Algebraic Topology, Fall 2018, homework sheet V

The deadline for this homework sheet is Friday, 21 December 2018. Please deliver your homework on paper into the physical mailbox of Stefan van der Lugt in the common room of the Snellius building. The use of LaTeX is strongly recommended; solutions that are not (easily) readable will be discarded. If delivery in the physical mailbox is not possible, then please submit your solutions by mail to `algtop2018@gmail.com`. You are allowed to cooperate, but copying is of course not allowed. You may use the material as discussed in Lectures 1-12, the material discussed in Fulton, Chapters 11–14, the material discussed in Looijenga's syllabus up to and including Corollary 3.2, all material from the syllabus Topologie (including the exercises), and the homework exercises I-IV. Please include references when appropriate. If you wish to use a result from the practice exercises (see weeks 1, 3, 6, 8, 11) you are requested to include a (short) argument for that result. You may quote Problem 2 from Looijenga's syllabus without giving a proof.

Exercise 1. Let X be a path-connected topological space. Show that the map $\deg\colon C_0(X) \to \mathbb{Z}$ given by sending a 0-chain $\sum_{x\in X} a_x \cdot x$ to $\sum_{x\in X} a_x$ induces an isomorphism $H_0(X) \xrightarrow{\sim} \mathbb{Z}$.

Exercise 2. Let X be a topological space. Let $A \subset X$ be a subspace with inclusion map $i\colon A \to X$. A retraction $r\colon X \to A$ is called a *deformation retraction* if there exists a homotopy between the continuous map $i \circ r\colon X \to X$ and id_X.

(i) Let $r\colon X \to A$ be a deformation retraction and let $p \in \mathbb{Z}$. Show that the maps $r_*\colon H_p(X) \to H_p(A)$ and $i_*\colon H_p(A) \to H_p(X)$ are isomorphisms, and each other's inverse.

Let $n \in \mathbb{Z}_{\geq 0}$.

(ii) Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere. Construct a deformation retraction $r\colon \mathbb{R}^{n+1} \setminus \{0\} \to S^n$.

(iii) Compute the homology groups of $\mathbb{R}^{n+1} \setminus \{0\}$.

Exercise 3. Let $n \in \mathbb{Z}_{\geq 1}$ and let $x_1, \ldots, x_n$ be points in $\mathbb{R}^2$ whose mutual distance is at least 3. For $\epsilon \in \mathbb{R}_{>0}$ we denote by $B(x_i, \epsilon)$ the open disk and by $\overline{B}(x_i, \epsilon)$ the closed disk in $\mathbb{R}^2$ with center x_i and radius ϵ. Let $X = \mathbb{R}^2 \setminus \big(\overline{B}(x_1, 1/2) \cup \ldots \cup \overline{B}(x_n, 1/2)\big)$.

(i) Show that for the homology groups of X we have:

$$H_0(X) \cong \mathbb{Z}\,, \quad H_1(X) \cong \mathbb{Z}^n\,, \quad H_p(X) = 0 \quad \text{if } p > 1\,.$$

Let $A = X \cap (B(x_1, 1) \cup \ldots \cup B(x_n, 1))$.

(ii) Show that for the homology groups of A we have:

$$H_0(A) \cong \mathbb{Z}^n\,, \quad H_1(A) \cong \mathbb{Z}^n\,, \quad H_p(A) = 0 \quad \text{if } p > 1\,.$$

Let X^+, X^- be two copies of X, and let M be the topological space obtained by glueing X^+ and X^- along the identity map $\mathrm{id}\colon A \to A$.

(iii) Compute the homology groups of M.

(iv) Let $r = r(n)$ be the rank of $H_1(M)$. Draw r loops on M whose classes in $H_1(M)$ are $\mathbb{Z}$-linearly independent.