Introduction to Algebraic Topology, Fall 2018, practice exercises - week 1

For $n \in \mathbb{Z}_{\geq 0}$ we put $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ and $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. Both S^n and D^n are equipped with the induced topology.

Exercise 1. The Intermediate Value Theorem implies the following: let $a \leq b$ be real numbers, let $g\colon [a,b] \to \mathbb{R}$ be a continuous map, and assume that $g(a) \leq g(b)$. Let $y \in \mathbb{R}$ and assume that $g(a) \leq y \leq g(b)$. Then there exists an $x \in [a,b]$ such that $g(x) = y$. Assume this statement from now on. Show that for each continuous map $f\colon D^1 \to D^1$ there exists an $x \in D^1$ such that $f(x) = x$.

Exercise 2. Show that for each continuous map $f\colon S^1 \to \mathbb{R}$ there exists a pair of antipodal points on S^1 that are mapped to the same point by f. One approach could be using the Intermediate Value Theorem. Another approach could be: assume that such a pair of antipodal points does not exist. Then for all $z \in S^1$ we have $f(z) - f(-z) \neq 0$. Construct a continuous surjective map $g\colon S^1 \to S^0$. Derive a contradiction.

Exercise 3. Assume the truth of the following statement (the Borsuk-Ulam theorem): for each $n \in \mathbb{Z}_{\geq 0}$, and for each continuous map $f\colon S^n \to \mathbb{R}^n$, there exists a pair of antipodal points on S^n that are mapped to the same point by f. Deduce the following result: if $m \neq n$ are distinct non-negative integers, then $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

Exercise 4. Let $n \in \mathbb{Z}_{\geq 1}$ and consider the sphere

$$S^{n-1} = \{(x_1, \ldots, x_{n+1}) \in S^n | x_{n+1} = 0\}$$

as the equator of the sphere $S^n \subset \mathbb{R}^{n+1}$. Show that there does not exist a retraction $S^1 \to S^0$. Show that there does not exist a retraction $S^2 \to S^1$.

Exercise 5. A continuous map $p\colon Y \to X$ is called *open* if for each open subset $V \subseteq Y$ the image $p(V) \subseteq X$ is an open subset of X. Show that a covering map $p\colon Y \to X$ is open.

Exercise 6. Let $p\colon Y \to X$ and $q\colon Z \to X$ be covering maps. A continuous map $r\colon Y \to Z$ is called a *morphism of coverings over* X if $p = q \circ r$. If r is a homeomorphism, we call r an *isomorphism of coverings over* X.

(i) Let I be a non-empty discrete topological space. Let X be a topological space. Show that the projection $X \times I \to X$ onto the first coordinate is a covering. A covering that is isomorphic to one of this form is called *trivial*.

(ii) Let $p\colon Y \to X$ be a covering. Let $W \subseteq X$ be a subspace of X. Show that the restriction $p|_{p^{-1}W}\colon p^{-1}W \to W$ of p to $p^{-1}W \subseteq Y$ is a covering.

A continuous map $p\colon Y \to X$ is called *locally trivial* if there exists an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of X such that for all $\alpha \in A$ the restriction $p|_{p^{-1}U_\alpha}\colon p^{-1}U_\alpha \to U_\alpha$ is a trivial covering.

(iii) Show that each covering map $p\colon Y \to X$ is locally trivial.

(iv) Conversely, let $p\colon Y \to X$ be a locally trivial continuous map. Show that $p\colon Y \to X$ is a covering map.

We have just proved that "covering map" and "locally trivial continuous map" are equivalent notions.