Introduction to Algebraic Topology, Fall 2018, practice exercises - week 6

Exercise 1. In class we have introduced the *degree* (or *winding number*) of maps $f\colon S^1 \to S^1$, notation $\deg(f)$. Verify the following properties of the degree.

(i) $\deg(\mathrm{id}_{S^1}) = 1$.

(ii) If $f, g\colon S^1 \to S^1$ are homotopic, then $\deg(f) = \deg(g)$.

(iii) For all $n \in \mathbb{Z}$ we have $\deg(z \mapsto z^n) = n$.

(iv) Let $f, g\colon S^1 \to S^1$ be continuous maps. Then $\deg(f \circ g) = \deg(f)\deg(g)$.

(v) $\deg(-\mathrm{id}_{S^1}) = 1$. [In fact, id_{S^1} and $-\mathrm{id}_{S^1}$ are homotopic. Show that id_{S^0} and $-\mathrm{id}_{S^0}$ are *not* homotopic. Later we will see: id_{S^n} and $-\mathrm{id}_{S^n}$ are homotopic $\Leftrightarrow$ n is odd.]

Assumption: From now on our base spaces X, $X_{(\dots)}$ are connected and locally path connected.

Exercise 2. Assume that X admits a universal covering space $u\colon \tilde{X} \to X$. The aim of this exercise is to show that the classification of the coverings $p\colon Y \to X$ can be reduced to the classification of the coverings $Y \to X$ *with Y connected*.

(i) Let A be a non-empty set and for each $\alpha \in A$ let $p_\alpha\colon Y_\alpha \to X$ be a covering with Y_α connected. Let $p\colon \sqcup_{\alpha \in A} Y_\alpha \to X$ be the continuous map given by sending $y \in Y_\alpha$ to $p_\alpha(y) \in X$. Show that p is a covering.

(ii) Let $p\colon Y \to X$ be a covering and let Y_α for $\alpha \in A$ be the connected components of Y. Note that these are also the path-connected components of Y, as Y is locally path-connected (cf. Propositie 10.10 of the syllabus Topologie, Fall 2017). Show that for each $\alpha \in A$, the restriction $p|_{Y_\alpha}\colon Y_\alpha \to X$ is a covering.

Exercise 3. Let $x \in X$, and assume we are given a universal covering $u\colon (\tilde{X}, \tilde{x}) \to (X, x)$. Set $A = \mathrm{Aut}(\tilde{X}/X)$. In class we established a natural bijection between the set of isomorphism classes of connected pointed coverings of (X, x) and the set of subgroups of A, called the "Galois correspondence". The aim of this exercise is to show that the Galois correspondence is *order-preserving* in an appropriate sense.

Given two connected coverings $p\colon (Y, y) \to (X, x)$ and $p'\colon (Y', y') \to (X, x)$, we say that $(Y, y) \leq (Y', y')$ if and only if there exists a continuous map of pointed spaces $f : (Y, y) \to (Y', y')$ such that $p' \circ f = p$.

(i) Show that '$\leq$' defines a partial ordering on the set of isomorphism classes of connected pointed coverings

$$\{p\colon (Y, y) \to (X, x) \text{ covering with } Y \text{ connected}\}/\cong .$$

(ii) Show that '$\subseteq$' defines a partial ordering on the set of subgroups of A.

(iii) Show that the Galois correspondence is an isomorphism of partially ordered sets, i.e., respects the partial orderings defined in (i) and (ii).

Exercise 4. Let $u_1\colon (\tilde{X}_1, \tilde{x}_1) \to X_1$ and $u_2\colon (\tilde{X}_2, \tilde{x}_2) \to X_2$ be universal coverings.

(i) Show that the map $\tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2$ given by $(y_1, y_2) \mapsto (u_1(y_1), u_2(y_2))$ is a universal covering.

(ii) Exhibit an isomorphism of groups $\mathrm{Aut}(\tilde{X}_1 \times \tilde{X}_2 / X_1 \times X_2) \xrightarrow{\sim} \mathrm{Aut}(\tilde{X}_1/X_1) \times \mathrm{Aut}(\tilde{X}_2/X_2)$.

(iii) Exhibit an isomorphism of groups $\pi_1(\tilde{X}_1 \times \tilde{X}_2, (\tilde{x}_1, \tilde{x}_2)) \xrightarrow{\sim} \pi_1(X_1, x_1) \times \pi_1(X_2, x_2)$.