Abelian groups of bounded exponent

All groups in this project description are abelian, and they will be written additively.

Let A be an abelian group. For each $n \in \mathbf{Z}$, the map $A \to A$ sending a to na is a group homomorphism, of which the kernel is denoted by $A[n]$ and the image by nA; so if $n > 0$, then $A[n] = \{a \in A : \text{ the order of } a \text{ is finite and divides } n\}$. We write $A_{\text{tors}} = \bigcup_{n \geq 1} A[n]$; it is a subgroup of A, called the *torsion subgroup*, and it consists of all elements of A of finite order. We say that A is a *torsion group* if $A = A_{\text{tors}}$, and that it has *bounded exponent* if there exists $n \in \mathbf{Z}_{>0}$ with $A = A[n]$; the least value of n with $A = A[n]$ is then called the *exponent* of A.

The project is devoted to the structure of torsion groups, in particular in the case of groups of bounded exponents.

Let again A be an abelian group. For a prime number p we write $A[p^\infty] = \bigcup_{i \geq 0} A[p^i]$, which is a subgroup of A_{tors}, the *p-primary part* of A_{tors}; we say that A is *p-primary* if $A = A[p^\infty]$. It is a basic theorem that the group homomorphism $\bigoplus_{p \text{ prime}} A[p^\infty] \to A_{\text{tors}}$, $(a_p)_{p \text{ prime}} \mapsto \sum_p a_p$, is an isomorphism. This isomorphism reduces most questions one may ask about torsion groups to the case that the group is p-primary for some prime number p. It also reduces most questions that one may ask about groups of bounded exponents to the case in which the exponent is a power of some prime number.

Let $n \in \mathbf{Z}_{>0}$. An abelian group has exponent dividing n if and only if it can be viewed as a module over the ring $\mathbf{Z}/n\mathbf{Z}$. The most important result about groups of bounded exponent states that any $\mathbf{Z}/n\mathbf{Z}$-module can be written as the direct sum of a collection of modules each of which is *cyclic* of order dividing n.

Let us now fix a prime number p and $k \in \mathbf{Z}_{>0}$, and let A be an abelian group of exponent dividing p^k or, equivalently, a $\mathbf{Z}/p^k\mathbf{Z}$-module. Then one can write $A \cong \bigoplus_{i=1}^{k} (\mathbf{Z}/p^i\mathbf{Z})^{(a(i))}$ for certain cardinal numbers $a(1)$, $a(2)$, $\ldots$, $a(k)$, and one can in fact show that these cardinal numbers are uniquely determined by A. We call $(\mathbf{Z}/p^k\mathbf{Z})^{(a(k))}$ the *free* part of A (notation: A_{free}), and $\bigoplus_{i=1}^{k-1} (\mathbf{Z}/p^i\mathbf{Z})^{(a(i))}$ the *non-free* part (notation: A_{nf}); we have $A \cong A_{\text{nf}} \oplus A_{\text{free}}$, but this isomorphism is not necessarily unique, and A_{free} and A_{nf} are in general not uniquely defined as subgroups of A.

Let p and k be as above. Then for each abelian group A, the groups $A[p^k]$ and A/p^kA are groups of exponent dividing p^k; it is apparently true that their non-free parts $A[p^k]_{\text{nf}}$ and $(A/p^kA)_{\text{nf}}$ are isomorphic, and that an isomorphism between these two groups is induced by the homomorphism $A[p^k] \to A/p^kA$ that is the composition of the inclusion map $A[p^k] \to A$ and the canonical map $A \to A/p^kA$. It is also reasonable to expect that a converse is true: if B and C are $\mathbf{Z}/p^k\mathbf{Z}$-modules with $B_{\text{nf}} \cong C_{\text{nf}}$, then there exists an abelian group A with $B \cong A[p^k]$ and $A/p^kA \cong C$; and that the composed map $B \cong A[p^k] \subset A \to A/p^kA \cong C$ can be made equal to a given group homomorphism $g \colon B \to C$ if and only if g induces, in a sense to be made precise, an isomorphism from B_{nf} to C_{nf}. It would be nice if the student could formulate all of this rigorously and provide proofs.

Supervisors: a subset of Mima Stanojkovski, Bart de Smit, and Hendrik Lenstra.