Exercises Reading course Algebraic Topology - Covering spaces, Fall 2013 - Set IIa

Exercise 1. Consider the following operations on groups G:

- forming the commutator subgroup $[G, G]$ of G;
- forming the abelianization $G^{\mathrm{ab}} = G/[G, G]$ of G;
- forming the center $Z(G)$ of G;
- taking $\mathrm{Hom}(H, G)$ where H is a fixed group;
- taking $\mathrm{Hom}(G, H)$ where H is a fixed group;
- forming the set of conjugacy classes of G;
- forming the automorphism group $\mathrm{Aut}(G)$.

Which of the above operations naturally give rise to a functor on the category of groups? Motivate your answer.

Exercise 2. Let **Top** be the category of topological spaces. Let X, Y be objects in **Top**.
(i) Recall the universal property of the product $X \times Y$ in **Top**.
(ii) Let **C** be a category. Formulate the notion of representability for a contravariant functor $F\colon \mathbf{C} \to \mathbf{Sets}$.
(ii) Show that the contravariant functor from **Top** to **Sets** that associates to each object Z of **Top** the product set $\mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y)$ is represented by the product $X \times Y$.

In the following exercises X is a topological space.

Exercise 3. Let $p\colon Y \to X$ be a cover and assume that Y is compact. Show that for each $x \in X$, the fiber $p^{-1}(x)$ is finite.

Exercise 4. Prove that each cover $p\colon Y \to X$ is an open map (i.e. the map p sends open sets to open sets).

In the following exercises we assume that X is locally connected.

Exercise 5. Let $p\colon Y \to X$ be a cover. Prove that Y is locally connected.

Exercise 6. Let $p\colon Y \to X$ be a connected cover, and suppose there exists a section s of p, that is, a map $s\colon X \to Y$ such that $p \circ s = \mathrm{id}_X$. Show that p is a homeomorphism.