Exercises Reading course Algebraic Topology - Covering spaces III, Fall 2013

Let X be a path connected and locally simply connected topological space, $x \in X$ a point.

Exercise 1. Assume the equivalence of categories in Theorem 2.3.4, and the construction of a connected cover $\tilde{X}_x \to X$ as in Construction 2.4.1.
(i) What is the isomorphism type of $\mathrm{Fib}_x(\tilde{X}_x)$ as a transitive $\pi_1(X,x)$-set?
(ii) Compute $\mathrm{Aut}_{\pi_1(X,x)}(\mathrm{Fib}_x(\tilde{X}_x))$.
(iii) Describe an isomorphism $\mathrm{Aut}(\tilde{X}_x/X) \xrightarrow{\sim} \pi_1(X,x)$ using the equivalence and (ii).

Exercise 2. Let $Y = \mathbb{R}^2$, let $X = S^1 \times S^1$ and consider the (universal) cover $p\colon Y \to X$ given by $(u,v) \mapsto (\exp(2\pi i u), \exp(2\pi i v))$. Let $x \in X$ be the point $(1,1)$.
(i) Describe the monodromy action of $\pi_1(X,x)$ on $F = p^{-1}(x) \subset Y$.
For integers a,b,c,d, let $q\colon X \to X$ be the map given by $(z,w) \mapsto (z^a w^b, z^c w^d)$. Assume that $ad - bc \neq 0$.
(ii) Show that $q\colon X \to X$ lifts to a map $\tilde{q}\colon Y \to Y$.
(iii) Show that q is a cover of X, and compute the number of points in a fiber of q.

Exercise 3. Take the notations from the previous exercise. Let $a \in X$ be the point $(-1,-1)$.
(i) Describe a connected cover of the connected space $X' = X \setminus \{a\}$ with fiber F over x.
(ii) Let $i_1\colon S^1 \to X'$ be the inclusion given by $y \mapsto (y,1)$ and $i_2\colon S^1 \to X'$ the inclusion given by $y \mapsto (1,y)$. Let γ be a generator of $\pi_1(S^1)$ and let $\alpha = i_{1*}(\gamma)$ and $\beta = i_{2*}(\gamma)$. Show that $\alpha\beta \neq \beta\alpha$ in $\pi_1(X',x)$ by considering the monodromy action on F.

Exercise 4. For each $n \in \mathbb{Z}_{>0}$ let $\mathbb{P}^n(\mathbb{R})$ be the n-dimensional real projective space.
(i) Exhibit a cover $S^n \to \mathbb{P}^n(\mathbb{R})$.
(ii) Compute the fundamental group of $\mathbb{P}^n(\mathbb{R})$.

Exercise 5. (Base change) Let $f\colon (Z,z) \to (X,x)$ be a map of pointed spaces, where Z is path connected and locally simply connected.
(i) Using the text in Szamuely's book between Example 2.4.12 and Construction 2.4.13, give a natural functor $F\colon \mathrm{Cov}(X) \to \mathrm{Cov}(Z)$ induced by f. Prove/verify the functoriality of your choice of F.
(ii) Describe a natural functor $G\colon \pi_1(X,x){-}\mathbf{Sets} \to \pi_1(Z,z){-}\mathbf{Sets}$. Prove/verify the functoriality of your choice of G.
(iii) Show that $\mathrm{Fib}_z \circ F \cong G \circ \mathrm{Fib}_x$.