COARSE GEOMETRY OF METRIC SPACES

YEONGCHYUAN CHUNG

Coarse (or large scale) geometry is the study of geometric objects (often metric spaces) viewed from afar. It is a fundamental tool in geometric group theory, and is also of interest in Banach space theory, data analysis, computer science, and has applications in topology and differential geometry.

Given two metric spaces (X, d_X) and (Y, d_Y), one may consider isometric embeddings of X into Y, i.e., maps $f : X \to Y$ such that $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. These maps completely preserve the metric structure of X. Weakening this gives the notion of quasi-isometric embedding, which are maps $g : X \to Y$ for which there are constants $L \geq 1$ and $C \geq 0$ such that

$$L^{-1}d_X(x_1, x_2) - C \leq d_Y(g(x_1), g(x_2)) \leq Ld_X(x_1, x_2) + C.$$

As one moves further away from these spaces, the constants L and C become less and less significant, so these maps preserve the large scale geometry but ignore the small scale details of X. If the image of X is "suitably dense" in Y, then one may identify the two spaces up to quasi-isometry. For example, the map $g : \mathbb{R} \to \mathbb{Z}$ that sends each real number x to the largest integer less than or equal to x is a quasi-isometric embedding with $L = C = 1$.

Further weakening the definitions results in the notions of coarse embedding and coarse equivalence.

There are properties of metric spaces that are invariant under coarse equivalence, and this project will provide the interested student with an opportunity to learn about a few of these properties (asymptotic dimension, property A, coarse embeddability into Hilbert space) and the relations between them, based on selected parts of [1]. Asymptotic dimension is a large scale analog of the classical covering/topological dimension, while property A is a large scale (and weaker) analog of the classical notion of amenability and was introduced by Yu in 2000 to provide a sufficient condition for coarse embeddability into Hilbert space.

The notion of amenability of groups was introduced by von Neumann in the 1920's, and for finitely generated groups it can be defined in terms of an isoperimetric condition, comparing the "volume" of a set with the "area" of the boundary. Precisely, a finitely generated group G is amenable if for every $R > 0$ and $\varepsilon > 0$ there is a finite subset $F \subseteq G$ such that $\frac{|\{g \in G \setminus F : d(g,F) \leq R\}|}{|F|} \leq \varepsilon$. Here, the metric d on G comes from its Cayley graph, and the isoperimetric condition describes amenability in terms of existence of finite sets with small R-boundary. For example, the additive group of integers can be shown to be amenable by taking F to be $[-nR, nR] \cap \mathbb{Z}$ for $\varepsilon = \frac{1}{n}$.

Property A for a discrete metric space X can be defined by a similar isoperimetric condition involving finite subsets of $X \times \mathbb{N}$, which allows us to count points of X with multiplicity. Moreover, all amenable finitely generated groups have property A while there are also non-amenable finitely generated groups with property A.

Depending on the interests of the student, there are a few options for further study:

Property A and coarse embeddability into Hilbert space. A discrete metric space with property A coarsely embeds into Hilbert space. Study the proof of this result, and try to modify the definition of property A to get a sufficient condition for coarse embeddability into an L^p space.

Equivalent definitions of property A. There are many equivalent definitions of property A (just as there are many equivalent definitions of the classical notion of amenability). A refined version of property A that comes with certain quantitative estimates was introduced in [2]. Does it also have similar equivalent definitions?

Decomposition complexity. Decomposition complexity (also found in [1]) is a generalization of asymptotic dimension. Roughly speaking, it measures the difficulty of decomposing a metric space into uniformly bounded pieces that are well-separated from one another. Study this notion and compare it with variants that have appeared in the literature.

Obstructions to coarse embeddability. Expander graphs are metric spaces that do not coarsely embed into Hilbert space (and into L^p space), and were found to be a source of counterexamples to some conjectures in noncommutative geometry. Study the reason for the failure to coarsely embed, along with the recently introduced notion of asymptotic expanders (which also fail to coarsely embed) [3].

Prerequisites

Basic theory of metric spaces. Some knowledge about finitely generated groups or Hilbert/Banach spaces may be an advantage.

References

[1] P.W. Nowak and G. Yu, *Large Scale Geometry*, EMS Textbooks in Mathematics, European Mathematical Society, 2012.
[2] R. Tessera, *Quantitative property A, Poincaré inequalities, L^p-compression and L^p-distortion for metric measure spaces*, Geom. Dedicata **136** (2008), 203-220.
[3] A. Khukhro, K. Li, F. Vigolo, and J. Zhang, *On the structure of asymptotic expanders* (2019). arXiv:1910.13320.

E-mail address: y.c.chung@math.leidenuniv.nl