Waring's problem

In the eighteenth century, Waring stated without proof that every natural number is a sum of at most 9 positive integral cubes and a sum of at most 19 fourth powers, and so on. The proof that a number can be written as a sum of at most 4 squares is due to Lagrange and it appeared during the same year as Waring's paper. The fact that for all $k \geq 3$ there exists a number s such that every positive integer can be written as a sum of at most s k-th powers of positive integers was proved by Hilbert through a combinatorial argument. His result is far from giving the smallest such s. However, due to work of Mahler, Beukers and many other mathematicians, the minimal value of s is essentially known now for all k.

A related question is to find the smallest number $G(k)$ such that every sufficiently large number is the sum of at most $G(k)$ k-th powers. Hardy and Littlewood obtained the first result towards answering this question in 1920s via the circle method. In particular, they give the following upper bound $G(k) \leq (k-2)2^{k-1} + 5$. The precise minimal value for $G(k)$ is only known for $k = 2$ and 4. In these cases, we have $G(2) = 4$ due to Lagrange and $G(4) = 16$ due to Davenport. Another important result is due to Linnik who proved that $G(3) \leq 7$, but it is conjectured that $G(3) = 4$. Through numerical methods, Dickson showed that every positive integer except 23 and 239 is the sum of eight cubes. Based on work of Boklan and Elkies, Siksek proved that all positive integers with 17 exceptions can be written as the sum of seven cubes. For large k, the best upper bounds are due to Wooley.

A student could choose to study the original method of Hilbert, the circle method that was originally used by Hardy and Littlewood to study $G(k)$ or some of the many generalisations of the problem.

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