DIVISION ALGEBRAS AND TOPOLOGY: THE $(1, 2, 4, 8)$-THEOREM

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Roughly speaking, a division algebra is an algebra over a field in which division (except by zero) is always possible. Examples of division algebras over the real field $\mathbb{R}$ are the real numbers, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the Cayley octonions $\mathbb{O}$. These division algebras have dimensions $1, 2, 4, 8$, respectively. Surprisingly, every division algebra over the reals is forced to have one of these four dimensions.

While the statement of this fact is purely algebraic, its proof requires methods from (algebraic) topology.

A first step in the proof was taken by Hopf in 1940 [2]: he was able to prove, using methods from topology, that the dimension of a real division algebra over the real numbers must be a power of 2.

In 1958, Kervaire and Milnor [3] independently proved that this power of two must be either $1, 2, 4$ or 8. Their proof used a major result in algebraic topology that goes under the name of Bott periodicity.

In this project, we will prove the $(1, 2, 4, 8)$-theorem using (real) K-theory, a cohomology theory based on equivalence classes is of vector bundles.

Prerequisites: algebra, linear algebra. Knowledge of algebraic topology is an advantage, but not necessary.

References

[1] F. Hirzebruch, *Division Algebras and Topology.* In: Numbers. Graduate Texts in Mathematics (Readings in Mathematics), vol 123. Springer, New York (1991)

[2] H. Hopf, Ein topologischer Beitrag zur reellen Algebra, Comment. Math. Helv. **13** (1941), 219–239. MR0004785

[3] J. Milnor, Some consequences of a theorem of Bott, Ann. of Math. (2) **68** (1958), 444–449. MR0102805

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