

EERSTE DEELTENTAMEN LA EN BV MAART 2012, ANTWOORDEN

Each question was worth 18 points.

Question 1. Consider the following system of linear equations.

$$\begin{aligned}x_1 + kx_2 &= -2 \\ 3x_1 - 9x_2 &= h\end{aligned}$$

Choose a value for h and a value for k , such that

- (a) **(6 points)** the system is inconsistent.
- (b) **(6 points)** the system has infinitely many solutions.
- (c) **(6 points)** The system has a unique solution.

Answer: The associated augmented matrix is reduced as follows.

$$\begin{bmatrix} 1 & k & -2 \\ 3 & -9 & h \end{bmatrix} \sim \begin{bmatrix} 1 & k & -2 \\ 0 & -9 - 3k & h + 6 \end{bmatrix}$$

(a) For an inconsistent system we need the bottom line to look like $00x$ with $x \neq 0$, since this would imply that we have to solve $0 = x$ and this is not true for any choice of x_1 and x_2 . We choose k , such that $-9 - 3k = 0$, so $k = -3$, and h such that $h + 6 \neq 0$, so for example $h = 1$.

(b) For the system to have infinitely many solutions, we need a free variable and this is achieved by making the last line look like 000 . Therefore we choose $k = -3$ and $h = -6$.

(c) To get a unique solution we need the last line to look like $0xy$, where $x \neq 0$ and y can be anything. Any value $k \neq -3$ suffices.

Question 2 (18 points). Give all solutions of the equation $A\mathbf{x} = \mathbf{0}$ in the parametric vector form if the matrix A is given by:

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 & 0 & -3 \\ 0 & 0 & 2 & 0 & 4 & 10 \\ 0 & 0 & -4 & 1 & 0 & -12 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Answer: We first reduce the matrix to reduced echelon form.

$$\begin{aligned} \begin{bmatrix} 1 & 3 & 2 & -1 & 0 & -3 \\ 0 & 0 & 2 & 0 & 4 & 10 \\ 0 & 0 & -4 & 1 & 0 & -12 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} &\sim \begin{bmatrix} 1 & 3 & 2 & -1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 0 & 1 & 8 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 & -1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 3 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

You can also put a column of zeros at the end of the matrix to make it an augmented matrix and keep that column there throughout the reduction procedure if you wish. The result of the

reduction translates to

$$\begin{aligned}x_1 + 3x_2 - 5x_6 &= 0, \\x_3 + 5x_6 &= 0, \\x_4 + 8x_6 &= 0, \\x_5 &= 0.\end{aligned}$$

The solution set is

$$\mathbf{x} = \begin{bmatrix} -3x_2 + 5x_6 \\ x_2 \text{ is free} \\ -5x_6 \\ -8x_6 \\ 0 \\ x_6 \text{ is free} \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 5 \\ 0 \\ -5 \\ -8 \\ 0 \\ 1 \end{bmatrix}, \quad t, u \in \mathbb{R}.$$

Question 3. In (a) (**5 points**), (b) (**5 points**), (c) (**4 points**) en (d) (**4 points**) indicate if the given set of vectors is linearly dependent or not and explain your answer.

$$\begin{aligned}\text{(a)} \quad & \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \right\}, & \text{(b)} \quad & \left\{ \begin{bmatrix} 3 \\ -6 \\ 15 \\ 9 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -10 \\ 6 \end{bmatrix} \right\} \\ \text{(c)} \quad & \left\{ \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}, & \text{(d)} \quad & \left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 4 \\ 0 \\ -6 \end{bmatrix} \right\}\end{aligned}$$

Answer: (a) Dependent. We use Theorem 8 from 1.7. This says that a set with more vectors than entries in each vector is linearly dependent.

(b) Independent. We have shown that a set of two vectors is linearly dependent if and only if one is a multiple of the other and this is not the case here.

(c) Dependent. Theorem 9 from 1.7 says that a set of vectors containing the null vector is always linearly dependent.

(d) Dependent. Note that $-2\mathbf{v}_1 = \mathbf{v}_3$, so we have the linear dependence relation $2\mathbf{v}_1 + 0\mathbf{v}_2 + \mathbf{v}_3 = 0$.

Question 4. Consider the linear transformation $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ given by

$$T(x_1, x_2, x_3, x_4) = (2x_1, x_2 + x_3 + x_4, x_1 - x_3 + x_4).$$

(a) (**3 points**) What is the image under T of the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$?

(b) (**5 points**) What is the standard matrix for T .

(c) (**5 points**) Is T onto? Explain your answer.

(d) (**5 points**) Is T one-to-one? Explain your answer.

Answer: (a) $T(\mathbf{x}) = (2 \cdot 1, 2 + 0 + 1, 1 - 0 + 1) = (2, 3, 2)$.

(b) $T(\mathbf{e}_1) = (2, 0, 1)$, $T(\mathbf{e}_2) = (0, 1, 0)$, $T(\mathbf{e}_3) = (0, 1, -1)$ and $T(\mathbf{e}_4) = (0, 1, 1)$. By Theorem 10 we then get

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}.$$

For (c) and (d) we first reduce the augmented matrix to reduced echelon form.

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 4 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 2 \end{bmatrix}.$$

(c) Each row of A contains a pivot. By Theorem 4 this implies that the columns of A span $\mathbb{R}^3$. Then Theorem 12 says that T is onto.

(d) Since there are more columns in A than there are entries in each column, Theorem 8 says that the columns of A are linearly dependent. Then Theorem 12 says that T is not one-to-one.

Question 5. (a) (9 points) Invert the matrix

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 0 & 3 & 3 \\ -3 & 6 & 10 \end{bmatrix}.$$

(b) (3 points) Use the result from (a) to find a vector $\mathbf{x}$ such that $A\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$.

(c) (6 points) Is the matrix $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ invertible? Explain your answer.

Answer: (a) We use the algorithm from the book to find the inverse of A .

$$\begin{aligned} & \begin{bmatrix} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 1 & 0 \\ -3 & 6 & 10 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & -2 & 3 & 0 & 1 \end{bmatrix} \\ & \sim \begin{bmatrix} 1 & -2 & 0 & -5 & 0 & -2 \\ 0 & 1 & 0 & 3/2 & 1/3 & 1/2 \\ 0 & 0 & 1 & -3/2 & 0 & -1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & 2/3 & -1 \\ 0 & 1 & 0 & 3/2 & 1/3 & 1/2 \\ 0 & 0 & 1 & -3/2 & 0 & -1/2 \end{bmatrix}. \end{aligned}$$

So,

$$A^{-1} = \begin{bmatrix} -2 & 2/3 & -1 \\ 3/2 & 1/3 & 1/2 \\ -3/2 & 0 & -1/2 \end{bmatrix}.$$

(b) To find $\mathbf{x}$ we use A^{-1} :

$$\mathbf{x} = \begin{bmatrix} -2 & 2/3 & -1 \\ 3/2 & 1/3 & 1/2 \\ -3/2 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

(c) This matrix is not invertible. We calculate the determinant: $2 \cdot 3 - 1 \cdot 6 = 0$ and then Theorem 4 from Chapter 2 says that the matrix is not invertible.