

Final Exam – Linear Algebra and Image Processing
27 June 2012

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In all questions, justify your answer fully and show all your work. Responses that give only the numerical answer will not receive full credit.

There are 3 pages and 10 questions. Each question is worth an equal number of points.

1. Find the general solution to the system $Ax = b$, where

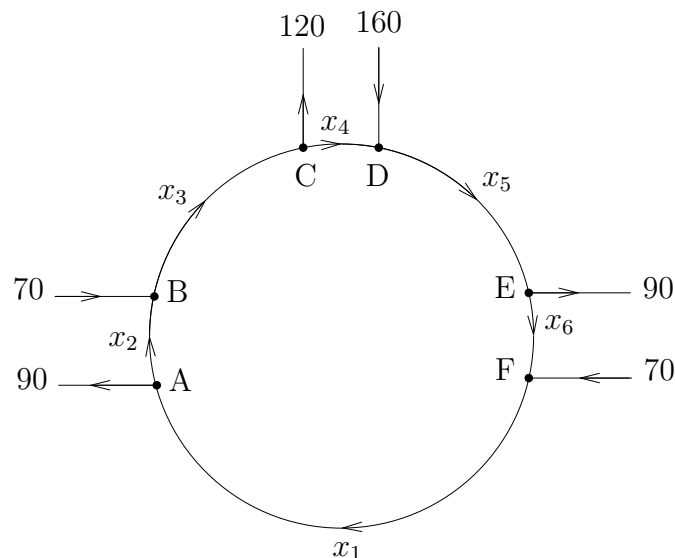
$$A = \begin{bmatrix} 1 & 3 & 0 & -2 & -2 \\ -2 & -6 & 0 & 4 & 3 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ -11 \\ 2 \\ -6 \end{bmatrix}.$$

2. Determine whether

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -8 \\ 7 \end{bmatrix} \right\}$$

is a linearly independent set of vectors in $\mathbb{R}^4$, and find a basis for $\text{Span } S$.

3. Traffic flows around a roundabout as shown in the diagram (all figures in cars per minute).



- (a) Set up a system of equations for the flow rates $x_1, \dots, x_6$.

- (b) Do you expect your system of equations to have a unique solution?
- (c) Find the general solution of the system of equations.

4. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -5 \end{bmatrix},$$

if it exists.

5. For each of the following 2D transformations, find the 3×3 matrix that produces the transformation using homogeneous coordinates.

- (a) Rotating by 90 degrees (counter-clockwise) around the point $(5, -4)$.
- (b) Translating by $(1, 6)$, then scaling both coordinates by 2.
- (c) Scaling both coordinates by 2, then translating by $(1, 6)$.

6. Find the determinant of the matrix

$$A = \begin{bmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 5 & 0 \end{bmatrix}.$$

7. (a) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}.$$

- (b) For each of the eigenvalues in part (a), compute the dimension of the corresponding eigenspace and find a basis for the eigenspace.
- (c) Is A diagonalizable? If so, write down matrices P and D for which D is diagonal and $A = PDP^{-1}$. (You do not need to compute P^{-1} .)

8. (a) Let A be an $m \times n$ matrix with independent columns. State the properties of the matrices Q and R (including the sizes of Q and R) for the QR -decomposition $A = QR$.

(b) Find the QR -decomposition of

$$A = \begin{bmatrix} -1 & 4 \\ 1 & -2 \\ 1 & 4 \\ -1 & 2 \end{bmatrix}.$$

9. In each part, either (i) give an example of a matrix A with the specified property; or else (ii) explain why it is impossible for A to have the specified property:

(a) a matrix A such that $A^T A \neq A A^T$

(b) a 3×3 matrix A with two linearly independent rows and two free variables

(c) a 3×4 matrix A with two linearly independent rows and two free variables

(d) a 2×3 matrix A such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (for some other matrix B)

(e) a 3×2 matrix A such that $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (for some other matrix B)

10. The matrix A has the singular value decomposition $A = U\Sigma V^T$ with

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) What are the sizes of the matrices A , U and V ?

(b) State the value of $\text{rank } A$, and explain briefly why this value is correct.

(c) Suppose U and V are written in column form as

$$U = \begin{bmatrix} | & | & \cdots & | \\ u_1 & u_2 & \cdots & u_p \\ | & | & \cdots & | \end{bmatrix}, \quad V = \begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_q \\ | & | & \cdots & | \end{bmatrix},$$

where p and q are the sizes from part (a). In terms of $u_1, \dots, u_p$ and/or $v_1, \dots, v_q$, give bases for $\text{Col } A$ and $\text{Nul } A$.