

Final Exam – Linear Algebra and Image Processing
27 June 2012

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In all questions, justify your answer fully and show all your work. Responses that give only the numerical answer will not receive full credit.

There are 3 pages and 10 questions. Each question is worth an equal number of points.

1. Find the general solution to the system $Ax = b$, where

$$A = \begin{bmatrix} 1 & 3 & 0 & -2 & -2 \\ -2 & -6 & 0 & 4 & 3 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ -11 \\ 2 \\ -6 \end{bmatrix}.$$

Solution. Row reduce: $\text{row}_2 + 2\text{row}_1$ then rearrange:

$$[A \mid b] \rightarrow \begin{bmatrix} 1 & 3 & 0 & -2 & -2 & | & 7 \\ 0 & 0 & 0 & 0 & -1 & | & 3 \\ 0 & 0 & -1 & 2 & 1 & | & 2 \\ 0 & 1 & 1 & 0 & 1 & | & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & -2 & -2 & | & 7 \\ 0 & 1 & 1 & 0 & 1 & | & -6 \\ 0 & 0 & -1 & 2 & 1 & | & 2 \\ 0 & 0 & 0 & 0 & -1 & | & 3 \end{bmatrix}.$$

x_4 is a free variable; $x_5 = -3$; $-x_3 = 2 - 2x_4 - x_5 = 5 - 2x_4$ so $x_3 = -5 + 2x_4$; $x_2 = -6 - x_3 - x_5 = -6 - (-5) - 2x_4 - (-3) = 2 - 2x_4$; $x_1 = 7 - 3x_2 + 2x_4 + 2x_5 = 7 - 3(2 - 2x_4) + 2x_4 + 2(-3) = -5 + 8x_4$. So

$$x = \begin{bmatrix} -5 \\ 2 \\ -5 \\ 0 \\ -3 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

(Alternative: further reduce by $\text{row}_3 + \text{row}_4$, $\text{row}_2 + \text{row}_4$, $\text{row}_1 - 2\text{row}_4$ to get

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & -2 & 0 & | & 1 \\ 0 & 1 & 1 & 0 & 0 & | & -3 \\ 0 & 0 & -1 & 2 & 0 & | & 5 \\ 0 & 0 & 0 & 0 & -1 & | & 3 \end{bmatrix}$$

then $\text{row}_2 + \text{row}_3$ then $\text{row}_1 - 3\text{row}_2$ to get

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & -2 & 0 & | & 1 \\ 0 & 1 & 0 & 2 & 0 & | & 2 \\ 0 & 0 & -1 & 2 & 0 & | & 5 \\ 0 & 0 & 0 & 0 & -1 & | & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -8 & 0 & | & -5 \\ 0 & 1 & 0 & 2 & 0 & | & 2 \\ 0 & 0 & -1 & 2 & 0 & | & 5 \\ 0 & 0 & 0 & 0 & -1 & | & 3 \end{bmatrix}$$

then rearrange this system.)

2. Determine whether

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -8 \\ 7 \end{bmatrix} \right\}$$

is a linearly independent set of vectors in $\mathbb{R}^4$, and find a basis for $\text{Span } S$.

Solution. The vectors are certainly not linearly independent since they are a set of 5 vectors in $\mathbb{R}^4$. (A linearly independent set in a 4-dimensional space can contain at most 4 vectors.)

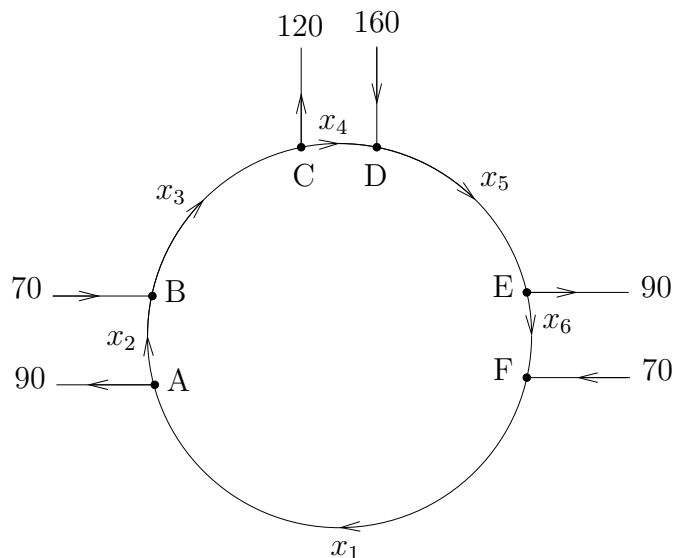
To find a basis for $\text{Span } S$, put the vectors of S into the columns of A and row-reduce:

$$\begin{aligned} A &= \begin{bmatrix} 1 & -2 & 5 & 3 & 4 \\ 0 & 1 & -2 & 2 & 3 \\ 0 & -1 & 2 & -7 & -8 \\ 1 & 1 & -1 & 3 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 5 & 3 & 4 \\ 0 & 1 & -2 & 2 & 3 \\ 0 & -1 & 2 & -7 & -8 \\ 0 & 3 & -6 & 0 & 3 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & -2 & 5 & 3 & 4 \\ 0 & 1 & -2 & 2 & 3 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & -6 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 5 & 3 & 4 \\ 0 & 1 & -2 & 2 & 3 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Columns 1, 2, and 4 have pivots so the corresponding vectors form a basis for $\text{Span } S$ (which is $\text{Col } A$ for this matrix A). A basis is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -7 \\ 3 \end{bmatrix} \right\}.$$

3. Traffic flows around a roundabout as shown in the diagram (all figures in cars per minute).



- (a) Set up a system of equations for the flow rates $x_1, \dots, x_6$.

Solution. At each intersection point A, $\dots$, F, (flow in) = (flow out). For instance, at A, $x_1 = 90 + x_2$; at B, $x_2 + 70 = x_3$. So

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 90 \\ -70 \\ 120 \\ -160 \\ 90 \\ -70 \end{bmatrix}.$$

- (b) Do you expect your system of equations to have a unique solution?

Solution. No. Extra traffic could flow all the way around the circle without entering or leaving, and this would not affect the system of equations. This corresponds to adding $(1, 1, 1, 1, 1, 1)$ to the flow vector.

(Alternative: solve the system and see that it does not.)

- (c) Find the general solution of the system of equations.

Solution. Row reduce: add $\text{row}_1, \dots, \text{row}_5$ to row_6 to get

$$\left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & 90 \\ 0 & 1 & -1 & 0 & 0 & 0 & -70 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -160 \\ 0 & 0 & 0 & 0 & 1 & -1 & 90 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

x_6 is free, and $x_5 = 90 + x_6$, $x_4 = -160 + x_5 = -70 + x_6$, $x_3 = 120 + x_4 = 50 + x_6$, $x_2 = -70 + x_3 = -20 + x_6$, $x_1 = 90 + x_2 = 70 + x_6$. So the general solution is

$$x = \begin{bmatrix} 70 \\ -20 \\ 50 \\ -70 \\ 90 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

4. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -5 \end{bmatrix},$$

if it exists.

Solution. Row reduce $[A \mid I]$:

$$\begin{aligned} [A \mid I] &= \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -5 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -3 & 2 & 0 & 1 \end{array} \right] \\ &\rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & -1 & 10 & -2 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 11 & -2 & 1 \\ 0 & 1 & 0 & -14 & 3 & -1 \\ 0 & 0 & -1 & 10 & -2 & 1 \end{array} \right] \\ &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -17 & 4 & -1 \\ 0 & 1 & 0 & -14 & 3 & -1 \\ 0 & 0 & 1 & -10 & 2 & -1 \end{array} \right] = [I \mid A^{-1}], \end{aligned}$$

so

$$A^{-1} = \begin{bmatrix} -17 & 4 & -1 \\ -14 & 3 & -1 \\ -10 & 2 & -1 \end{bmatrix}.$$

5. For each of the following 2D transformations, find the 3×3 matrix that produces the transformation using homogeneous coordinates.

- (a) Rotating by 90 degrees (counter-clockwise) around the point $(5, -4)$. *Solution.* Translate by $(-5, 4)$, then rotate, then translate by $(5, -4)$. The matrix on the right acts first, so the matrix on the right corresponds to translation by $(-5, 4)$:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 90 & -\sin 90 & 0 \\ \sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -4 \\ 1 & 0 & -5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -9 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

(Check: $A \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$ so the rotation point does not move; and the points

$(6, -4)$, $(5, -3)$ (one unit to the right and above the rotation point) map to $(5, -3)$ and $(4, -4)$ (one unit above and to the left of the rotation point).)

- (b) Translating by $(1, 6)$, then scaling both coordinates by 2.

Solution.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 12 \\ 0 & 0 & 1 \end{bmatrix}.$$

(Check: $(x, y) \rightarrow (x, y, 1) \rightarrow (2x + 2, 2y + 12, 1) \rightarrow (2x + 2, 2y + 12)$; same as $(x, y) \rightarrow (x + 1, y + 6) \rightarrow (2(x + 1), 2(y + 6))$.)

(c) Scaling both coordinates by 2, then translating by $(1, 6)$. *Solution.*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{bmatrix}.$$

(This is not the same as in (b), because the translation vector $(1, 6)$ does not get scaled by 2.)

6. Find the determinant of the matrix

$$A = \begin{bmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 5 & 0 \end{bmatrix}.$$

Solution. Expand by cofactors along the second column, then along the last column, then along the middle column:

$$\begin{aligned} \det A &= \begin{vmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 5 & 0 \end{vmatrix} = (-1)^{1+2}(1) \begin{vmatrix} 9 & 9 & 9 & 2 \\ 4 & 0 & 5 & 0 \\ 9 & 3 & 9 & 0 \\ 6 & 0 & 5 & 0 \end{vmatrix} \\ &= -(-1)^{1+4}(2) \begin{vmatrix} 4 & 0 & 5 \\ 9 & 3 & 9 \\ 6 & 0 & 5 \end{vmatrix} = 2(-1)^{2+2}(3) \begin{vmatrix} 4 & 5 \\ 6 & 5 \end{vmatrix} \\ &= 6[(4)(5) - (6)(5)] = 6(-10) = -60. \end{aligned}$$

7. (a) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}.$$

Solution.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 4 - \lambda & 1 & 1 \\ 1 & 4 - \lambda & 1 \\ 1 & 1 & 4 - \lambda \end{vmatrix} \\ &= (4 - \lambda) \begin{vmatrix} 4 - \lambda & 1 \\ 1 & 4 - \lambda \end{vmatrix} - (1) \begin{vmatrix} 1 & 1 \\ 1 & 4 - \lambda \end{vmatrix} + (1) \begin{vmatrix} 1 & 4 - \lambda \\ 1 & 1 \end{vmatrix} \\ &= (4 - \lambda)[(4 - \lambda)^2 - 1] - [(4 - \lambda) - 1] + [1 - (4 - \lambda)] \\ &= (4 - \lambda)[16 - 8\lambda + \lambda^2 - 1] - 6 + 2\lambda = (4 - \lambda)[\lambda^2 - 8\lambda + 15] - 6 + 2\lambda \\ &= 4\lambda^2 - 32\lambda + 60 - \lambda^3 + 8\lambda^2 - 15\lambda - 6 + 2\lambda \\ &= -[\lambda^3 - 12\lambda^2 + 45\lambda - 54]. \end{aligned}$$

This is 0 when $\lambda = 3$ so we can factor as $\det(A - \lambda I) = -(\lambda - 3)[\lambda^2 - 9\lambda - 18] = -(\lambda - 3)(\lambda - 3)(\lambda - 6)$. So the eigenvalues are $\lambda = 3$ (multiplicity 2) and $\lambda = 6$.

- (b) For each of the eigenvalues in part (a), compute the dimension of the corresponding eigenspace and find a basis for the eigenspace.

Solution. For $\lambda = 3$,

$$A - 3I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So the eigenspace is 2-dimensional ($\text{rank}(A - 3I) = 1$ so $\dim \text{Nul}(A - 3I) = 3 - 1 =$

2) and the vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are a basis.

For $\lambda = 6$,

$$A - 6I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1.5 & 1.5 \\ 0 & 1.5 & -1.5 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1.5 & 1.5 \\ 0 & 0 & 0 \end{bmatrix}.$$

So the eigenspace is 1-dimensional ($\text{rank}(A - 6I) = 2$ so $\dim \text{Nul}(A - 6I) = 3 - 2 =$

1). Solve to get $x_2 = x_3$ and $-2x_1 = -x_2 - x_3 = -2x_3$. So $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis.

- (c) Is A diagonalizable? If so, write down matrices P and D for which D is diagonal and $A = PDP^{-1}$. (You do not need to compute P^{-1} .)

Solution. Yes, since A has 3 linearly independent eigenvectors. D is the diagonal matrix of eigenvalues and the columns of P are the eigenvectors in the same order:

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

8. (a) Let A be an $m \times n$ matrix with independent columns. State the properties of the matrices Q and R (including the sizes of Q and R) for the QR -decomposition $A = QR$.

Solution. Q is $m \times n$ with orthonormal columns. R is $n \times n$ upper triangular. Also, since A has independent columns, R is invertible (i.e., all non-zero entries on the diagonal).

- (b) Find the QR -decomposition of

$$A = \begin{bmatrix} -1 & 4 \\ 1 & -2 \\ 1 & 4 \\ -1 & 2 \end{bmatrix}.$$

Solution. Use the Gram-Schmidt process on the columns:

$$v_1 = x_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad u_1 = \frac{v_1}{\|v_1\|} = \frac{v_1}{\sqrt{1+1+1+1}} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix},$$

$$v_2 = x_2 - \frac{v_1 \cdot x_2}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 4 \\ -2 \\ 4 \\ 2 \end{bmatrix} - \frac{-4-2+4-2}{1+1+1+1} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 5 \\ 1 \end{bmatrix},$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{v_2}{\sqrt{9+1+25+1}} = \begin{bmatrix} 3/6 \\ -1/6 \\ 5/6 \\ 1/6 \end{bmatrix}.$$

$x_1 = v_1 = 2u_1$ and $x_2 = v_2 + (-1)v_1 = 6u_2 - 2u_1$ so

$$Q = \begin{bmatrix} -1/2 & 3/6 \\ 1/2 & -1/6 \\ 1/2 & 5/6 \\ -1/2 & 1/6 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & -2 \\ -0 & 6 \end{bmatrix}.$$

9. In each part, either (i) give an example of a matrix A with the specified property; or else (ii) explain why it is impossible for A to have the specified property:

- (a) a matrix A such that $A^T A \neq A A^T$

Solution. An example is $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$: $A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ whereas $A A^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. (In general, if A is not square then $A^T A$ and $A A^T$ do not even have the same shape, so any non-square matrix A is also an example.)

- (b) a 3×3 matrix A with two linearly independent rows and two free variables

Solution. This is impossible since $\text{rank } A + \dim \text{Nul } A = n = 3$: if A has two linearly independent rows then $\text{rank } A = 2$, but if A has two free variables then $\dim \text{Nul } A = 2$.

- (c) a 3×4 matrix A with two linearly independent rows and two free variables

Solution. Here $n = 4 = 2 + 2$. An example is $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with two linearly independent rows and two free variables (x_3 and x_4 are free).

- (d) a 2×3 matrix A such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (for some other matrix B)

Solution. For instance $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ have $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

- (e) a 3×2 matrix A such that $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (for some other matrix B)

Solution. This is impossible. Since A is 3×2 , $\text{rank } A \leq 2$ (there can be at most 2 pivots). Therefore $\dim \text{Col } A \leq 2$. But every column of AB is a linear

combination of the columns of A . If $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then the three linearly

independent column vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ belong to $\text{Col } A$. This means that $\dim \text{Col } A \geq 3$, which is impossible since we know $\dim \text{Col } A \leq 2$.

10. The matrix A has the singular value decomposition $A = U\Sigma V^T$ with

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- (a) What are the sizes of the matrices A , U and V ?

Solution. A and Σ are always the same shape so A is $m \times n = 4 \times 3$. U is $m \times m = 4 \times 4$ and V is $n \times n = 3 \times 3$.

- (b) State the value of $\text{rank } A$, and explain briefly why this value is correct.

Solution. $\text{rank } A = 2$ since the rank is the number of non-zero singular values (the entries on the diagonal of Σ).

- (c) Suppose U and V are written in column form as

$$U = \begin{bmatrix} | & | & \cdots & | \\ u_1 & u_2 & \cdots & u_p \\ | & | & \cdots & | \end{bmatrix}, \quad V = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_q \\ | & | & | \end{bmatrix},$$

where p and q are the sizes from part (a). In terms of $u_1, \dots, u_p$ and/or $v_1, \dots, v_q$, give bases for $\text{Col } A$ and $\text{Nul } A$.

Solution. From part (a), $p = m = 4$ and $q = n = 3$.

For $\text{Nul } A$: solve $Ax = 0$. We have $Ax = U\Sigma V^T x = 0 \iff \Sigma V^T x = 0$ since U is orthogonal, and therefore invertible. Let $y = V^T x$. Then $\Sigma y = 0 \iff y_1 = y_2 =$

0, so $y = y_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then $Ax = 0 \iff x = Vy = y_3 v_3$, so v_3 is a basis for $\text{Nul } A$.

For $\text{Col } A$: every column of A is a combination of the columns u_1 and u_2 . (Columns u_3 and u_4 get multiplied by 0 when expanding $A = U\Sigma V^T$.) u_1 and u_2 are linearly independent (since they are orthonormal) and $\dim \text{Col } A = \text{rank } A = 2$ from part (b), so $\{u_1, u_2\}$ must be a basis for $\text{Col } A$.