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MULTIPLICATIVE DIVISION ALGORITHMS ON THE INTEGERS

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Multiplicative division algorithms on the integers.

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1. Introduction.

Let $\mathbb{Z}$ denote the ring of rational integers, and let W be a totally ordered set. A function $\phi : \mathbb{Z} - \{0\} \to W$ is called a division algorithm on $\mathbb{Z}$ if

(i) the image of ϕ is a well ordered subset of W;

(ii) for every a, b $\in \mathbb{Z}$, $b \neq 0$, there exist q, r $\in \mathbb{Z}$ such that

$$a = qb + r$$

$$r = 0 \text{ or } \phi(r) < \phi(b).$$

If W is the set of positive real numbers $\mathbb{R}_+$, we call ϕ multiplicative if

$$\phi(ab) = \phi(a)\phi(b)$$

for all a, b $\in \mathbb{Z}$, $ab \neq 0$.

Theorem 1 describes all multiplicative division algorithms on $\mathbb{Z}$, thus answering a question of R.K. Dennis [1].

Theorem 1.
Let $\phi : \mathbb{Z} - \{0\} \to \mathbb{R}_+$ be a multiplicative division algorithm. Then there exist a prime number p and real numbers $A > 0$, $B \geq 0$ such that

$$\phi(a) = |a|^A \cdot a_p^B \qquad \text{for all } a \in \mathbb{Z},\ a \neq 0;$$

here a_p denotes the largest power of p dividing a. Conversely, if p is a prime and $A > 0$, $B \geq 0$ are reals, then the function ϕ defined by the above equation is a multiplicative division algorithm on $\mathbb{Z}$.
Moreover, ϕ assumes only integral values if and only if both A and p^{A+B} are positive integers.

This theorem will be deduced from the following two results.

Theorem 2.

Let W be any well ordered set, and let $\phi : \mathbb{Z} - \{0\} \to W$ be a function. Then ϕ is a division algorithm on $\mathbb{Z}$ if and only if

$$\min\{\phi(r), \phi(-s)\} < \min\{\phi(r+s), \phi(-r-s)\}$$

for all $r, s \in \mathbb{Z}$, $r > 0$, $s > 0$.

Theorem 3.

Denote by $\mathbb{N}$ the set of positive integers. Suppose $\phi : \mathbb{N} \to \mathbb{R}_+$ satisfies

$$\phi(ab) = \phi(a).\phi(b)$$

$$\phi(a+b) \geq \min\{\phi(a), \phi(b)\}$$

for all $a, b \in \mathbb{N}$. Then there exist a prime number p and nonnegative real numbers A, B such that

$$\phi(a) = a^A . a_p^{\ B}$$

for all $a \in \mathbb{N}$.

In section 5 we show how theorem 3 can be used to sharpen a certain lemma from valuation theory.

2. Proof of theorem 2.

Let W be a well ordered set, and let $\phi : \mathbb{Z} - \{0\} \to W$ be a map. If ϕ satisfies the system of inequalities indicated in theorem 2, it is clear that ϕ is a division algorithm. In fact, for $a, b \in \mathbb{Z}$, $b \neq 0$, one can find $q, r \in \mathbb{Z}$ such that

$$a = q.b + r,$$
$$r = 0 \text{ or } \phi(r) < \phi(b),$$
$$|r| < |b|.$$

Conversely, assume ϕ is a division algorithm. Consider a triple (r,s,b) of integers such that

(2.1) $r > 0, \quad s > 0, \quad r + s = |b|.$

To prove theorem 2, it suffices to show

(2.2) $\phi(r) < \phi(b)$ or $\phi(-s) < \phi(b).$

This is done with induction on $\phi(b)$. So assume the assertion is true for all triples (r',s',b') as above for which $\phi(b') < \phi(b)$.

If $\phi(-b) < \phi(b)$, the induction hypothesis, applied to the triple $(r,s,-b)$, yields $\phi(r) < \phi(-b)$ or $\phi(-s) < \phi(-b)$, and (2.2) follows. Therefore assume $\phi(-b) \geq \phi(b)$, so

(2.3) $\phi(|b|) \geq \phi(b), \qquad \phi(-|b|) \geq \phi(b).$

Now choose d in the residue class (r mod b) such that $\phi(d)$ is minimal (remark that 0 is not in this residue class, by (2.1)). Because ϕ is a division algorithm, we have

(2.4) $\phi(d) < \phi(b).$

We distinguish three cases:

(i) $d > |b|$

(ii) $d < -|b|$

(iii) $d \in \{r, -s\}$.

In case (iii), (2.2) follows by (2.4). In each of the cases (i) and (ii) we derive a contradiction.

Case (i). The triple $(r',s',b') = (d-|b|,|b|,d)$ has the properties corresponding to (2.1). By (2.4) we may apply the induction hypothesis, and we find

$$\phi(d-|b|) < \phi(d) \text{ or } \phi(-|b|) < \phi(d).$$

But the first alternative is excluded by the minimality assumption on $\phi(d)$, and the second one by (2.3) and (2.4).

Case (ii). Applying the induction hypothesis to the triple $(r',s',b') =$

$= (|b|, -d-|b|, d)$ we get

$$\phi(|b|) < \phi(d) \quad \text{or} \quad \phi(d+|b|) < \phi(d).$$

The first possibility contradicts (2.3) and (2.4), the second one our choice of d.
This finishes the proof of theorem 2.

3. Proof of theorem 3.

Let $\phi : \mathbb{N} \to \mathbb{R}_+$ satisfy

$$\phi(ab) = \phi(a) \,.\, \phi(b) \tag{3.1}$$

$$\phi(a+b) \geq \min\{\phi(a), \phi(b)\}, \tag{3.2}$$

for all a, $b \in \mathbb{N}$. From (3.1) it follows that $\phi(1) = 1$, and using (3.2) inductively we find $\phi(a) \geq 1$ for all $a \in \mathbb{N}$. Define

$$\psi(a) = \frac{\log \phi(a)}{\log a} \qquad \text{for } a \in \mathbb{N},\ a \geq 2.$$

Then $\psi(a) \geq 0$, and $\phi(a) = a^{\psi(a)}$, for $a \geq 2$.

We first construct a natural number $k \geq 2$ such that

$$\psi(a) \geq \psi(k) \qquad \text{for all } a \geq 2. \tag{3.3}$$

Let p be any prime number, $\alpha = \psi(p)$. If $\psi(q) \geq \alpha$ for all primes q, then $k = p$ works. So choose a prime q such that $\beta = \psi(q) < \alpha$. If $\psi(r) \geq \beta$ for all $r \geq 2$ we can take $k = q$. So let $r \geq 2$ be a natural number such that $\gamma = \psi(r) < \beta$. Then $\beta > 0$, and replacing $\phi(a)$ by $\phi(a)^{1/\beta}$ for all a we may suppose

$$\phi(q) = q, \quad \beta = 1, \quad 0 \leq \gamma < 1 < \alpha.$$

Now choose a natural number M such that

(3.4) $\qquad M \geq r$

(3.5) $\qquad \dfrac{M^{1-\gamma}}{pq} > \dfrac{1}{\sqrt{(1-p^{\gamma-\alpha})}}$.

Let $k \in \mathbb{N}$, $2 \leq k \leq M$ be chosen such that

$$\delta = \psi(k) = \min \{\psi(a) \mid 2 \leq a \leq M\}.$$

By (3.4) we have $\delta \leq \gamma < 1$.
We assert that k has property (3.3). Otherwise, let $a \in \mathbb{N}$ be minimal such that $\psi(a) < \delta = \psi(k)$. We derive a contradiction. By definition of δ, we have $a > M$, so (3.5) implies $a^{1-\gamma} / (pq) > 1$, i.e. $q.a^{\gamma} < \frac{1}{p}.a$. Let q^n be the highest power of q which is smaller than $q.a^{\gamma}$. Then

$$a^{\gamma} \leq q^n < q.a^{\gamma} < \frac{1}{p}a.$$

Choose $\quad c \in \{1, 2, \ldots, p\}$ such that $a + c.q^n \equiv 0 \bmod p$. Then

$$c.q^n < p.q.a^{\gamma} < a.$$

Therefore (3.5) yields

$$(1 - \frac{c^2q^{2n}}{a^2})^{\delta} > 1 - \frac{c^2q^{2n}}{a^2} \geq 1 - (pqa^{\gamma-1})^2$$

$$> 1 - \sqrt{(1-p^{\gamma-\alpha})}^2 = p^{\gamma-\alpha}.$$

Also

$$0 < a - c.q^n < a, \qquad\qquad 0 < \frac{a+c.q^n}{p} < a$$

so the minimality condition on a implies

$$\phi(a-c.q^n) \geq (a-c.q^n)^{\delta}, \quad \phi(\frac{a+c.q^n}{p}) \geq (\frac{a+c.q^n}{p})^{\delta}.$$

Hence

$$\begin{aligned}\phi(a^2-c^2.q^{2n}) &= \phi(p).\phi(a-c.q^n).\phi(\frac{a+c.q^n}{p}) \\ &\geqq p^{\alpha}.(a-c.q^n)^{\delta}.(\frac{a+c.q^n}{p})^{\delta} \\ &= p^{\alpha-\delta}.(1-\frac{c^2q^{2n}}{a^2})^{\delta}.a^{2\delta} \\ &> p^{\alpha-\delta}.p^{\gamma-\alpha}.a^{2\delta} \\ &\geq a^{2\delta}.\end{aligned}$$

Also

$$\phi(c^2.q^{2n}) \geq \phi(q^{2n}) = q^{2n} \geq a^{2\gamma} \geq a^{2\delta}.$$

We conclude

$$\phi(a^2) \geq \min\{\phi(a^2-c^2.q^{2n}), \phi(c^2.q^{2n})\} \geq a^{2\delta},$$

$$\phi(a) \geq a^{\delta}, \qquad \psi(a) \geq \delta,$$

contradicting our choice of a. This finishes the construction of k.

Now fix k such that (3.3) holds. Putting $A = \psi(k)$ we have

(3.6) $$\psi(a) \geq A = \psi(k), \qquad \phi(a) \geq a^A \qquad \text{for all } a \geq 2.$$

If $\psi(p) = A$ for all primes p, theorem 3 follows by taking $B = 0$, p = any prime. So suppose

$$\psi(p) = A + B > A, \qquad B > 0,$$

for some prime p. We remark

(3.7) $$p|a \Rightarrow \phi(a) = \phi(\frac{a}{p}).\phi(p) \geq$$

$$\geq (\frac{a}{p})^A.p^{A+B} = a^A.p^B.$$

Since $\phi(k) = k^A$ it follows that $p \nmid k$.

To prove theorem 3 it is clearly sufficient to show that $\psi(s) = A$ for all primes $s \neq p$. So let s be a prime $\neq p$. Suppose $n, m \in \mathbb{N}$ satisfy

$$k^n > s^m.$$

If $N \in \mathbb{N}$ is divisible by $p - 1$ we have

$$p | k^{n.N} - s^{m.N}$$

and taking N sufficiently large we find by (3.7):

$$\begin{aligned}\phi(k^{n.N} - s^{m.N}) &\geq (k^{n.N} - s^{m.N})^A \ . \ p^B \\ &= k^{n.N.A} \ . \ (1 - \frac{s^{mN}}{k^{nN}})^A \ . \ p^B \\ &> k^{n.N.A} = \phi(k^{n.N}).\end{aligned}$$

Using (3.2) with $a = k^{n.N} - s^{m.N}$ and $b = s^{m.N}$ we get

$$\phi(k^{n.N}) \geq \phi(s^{m.N})$$

$$\phi(k)^n \geq \phi(s)^m.$$

If $\phi(k) = 1$, $\psi(k) = 0$ we conclude $\phi(s) = 1$, $\psi(s) = 0 = A$, as desired. If $\phi(k) > 1$, the preceding discussion shows:

$$\frac{n}{m} > \frac{\log s}{\log k} \implies \frac{n}{m} \geq \frac{\log \phi(s)}{\log \phi(k)} .$$

Since the rational numbers are dense in the reals this implies

$$\frac{\log s}{\log k} \geq \frac{\log \phi(s)}{\log \phi(k)}$$

$$A = \psi(k) = \frac{\log \phi(k)}{\log k} \geq \frac{\log \phi(s)}{\log s} = \psi(s).$$

By (3.6) we conclude $\psi(s) = A$, as desired.
This completes the proof of theorem 3.

4. Proof of theorem 1.

Let $\phi : \mathbb{Z} - \{0\} \to \mathbb{R}_+$ be a multiplicative division algorithm. Then $\phi(-1)^2 = \phi(1)^2 = \phi(1)$ so $\phi(-1) = 1$. Therefore $\phi(-a) = (a)$ for all a. From theorem 2 we get

$$\phi(a+b) > \min\{\phi(a),\phi(b)\}$$

for all $a > 0$, $b > 0$. Using theorem 3 we find a prime p and reals $A \geq 0$, $B \geq 0$ such that $\phi(a) = |a|^A \cdot a_p^B$ for all $a \in \mathbb{Z}$, $a \neq 0$. Since

$$(p+1)^A = \phi(p+1) > \min\{\phi(p),\phi(1)\} = 1$$

we have $A > 0$. This proves the first part of theorem 1.

That, conversely, the function ϕ defined by $\phi(a) = |a|^A \cdot a_p^B$ is a multiplicative division algorithm for any prime p and all $A > 0$, $B \geq 0$, is easy to check.

If A and p^{A+B} are positive integers, it is clear that ϕ assumes only integral values. To prove the converse, we recall a simple fact from analysis.

For a function $f : \mathbb{R}_+ \to \mathbb{R}$ we define $\Delta f : \mathbb{R}_+ \to \mathbb{R}$ by $\Delta f(x) = f(x+1) - f(x)$, and inductively $\Delta^1 f = \Delta f$, $\Delta^n f = \Delta.\Delta^{n-1} f$, $n \in \mathbb{N}$, $n \geq 2$.

Lemma

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be n times differentiable, $n \in \mathbb{N}$. Then for all $y \in \mathbb{R}_+$ there exists a $v \in [y, y+n]$ such that

$$f^{(n)}(v) = \Delta^n f(y).$$

Proof. Let $h(x) = \sum_{i=0}^{n} h_i x^i$ be the unique polynomial of degree $\leq n$ for which $g(x) = f(x) - h(x)$ has zeros in $x = y, y+1, \ldots, y+n$. Using Rolle's theorem repeatedly we find $v \in [y, y+n]$ with

$$g^{(n)}(v) = 0.$$

Furthermore, it is clear that

$$\Delta^n g(y) = 0, \qquad \Delta^n h(x) = h^{(n)}(x) = n!h_n \qquad \text{for } x \in \mathbb{R}_+$$

so

$$\Delta^n f(y) = \Delta^n g(y) + \Delta^n h(y) = 0 + n!h_n = g^{(n)}(v) + h^{(n)}(v) = f^{(n)}(v).$$

This proves the lemma.

We apply this lemma with $f(x) = (p.x+1)^A$. Then $\phi[\mathbb{Z} - \{0\}] \subset \mathbb{Z}$ implies $f[\mathbb{N}] \subset \mathbb{Z}$, hence by induction on n we get

$$\Delta^n f(y) \in \mathbb{Z}, \qquad \text{for all } n,\ y \in \mathbb{N}.$$

Choose $n > A$ fixed. Then for y sufficiently large the lemma yields

$$|\Delta^n f(y)| \leq \max_{v \in [y,y+n]} |f^{(n)}(v)| = |A.(A-1)\ldots(A-n+1).p^n(py+1)^{A-n}| < 1.$$

Hence $\Delta^n f(y) = 0$ for $y \in \mathbb{N}$ sufficiently large. So there exists a polynomial f_1 of degree $\leq n - 1$ such that $f(y) = f_1(y)$ for all $y \in \mathbb{N}$ sufficiently large. Then

$$\lim_{y \in \mathbb{N}, y \to \infty} \frac{f_1(y)}{f(y)} = 1$$

so A = degree f_1 is an integer, which we knew already to be positive.
Also $\phi(p) = p^{A+B}$ is a positive integer.
This concludes the proof of theorem 1.

5. Valuations of the natural numbers.

Let R be a commutative domain and $F : R \to \mathbb{R}_+ \cup \{0\}$ a function.
Suppose there exists a constant $C \in \mathbb{R}_+$ such that

$$F(a) = 0 \iff a = 0$$

(5.1) $$F(ab) = F(a)F(b)$$

(5.2) $$F(a+b) \le C \,.\, \max \{F(a), F(b)\}$$

for all $a, b \in R$. Then F is called a valuation of R.

By analogy, let us call a function $F : \mathbb{N} \to \mathbb{R}_+$ a valuation of $\mathbb{N}$ if there is a constant $C \in \mathbb{R}_+$ such that (5.1) and (5.2) hold for all $a, b \in \mathbb{N}$.

The following lemma is frequently used to determine all valuations of $\mathbb{Z}$, cf. [2], ch.I, §3, lemma 3.

Lemma

Let F be a valuation of $\mathbb{N}$. Then either $F(a) \le 1$ for all $a \in \mathbb{N}$, or there is a $\lambda \in \mathbb{R}_+$ such that $F(a) = a^\lambda$ for all $a \in \mathbb{N}$.

For the proof of this lemma we refer to [2].
Using theorem 3, we can complete the conclusion of the lemma in the following way.

Theorem 4.

Let $F : \mathbb{N} \to \mathbb{R}_+$ be a function. Then F is a valuation of $\mathbb{N}$ if and only if there exist a prime p and real numbers λ, μ such that $\mu \le 0$, $\lambda\mu \ge 0$, $F(a) = a^\lambda \,.\, a_p^{\ \mu}$ for all $a \in \mathbb{N}$.

Proof of theorem 4, cf. [2], ch. I, §3, lemma 4. First assume F is a valuation of $\mathbb{N}$.
If $F(a) = a^\lambda$ for some $\lambda \in \mathbb{R}_+$ and all $a \in \mathbb{N}$ we can put $\mu = 0$, $p=$ any prime number. So by the lemma we may assume $F(a) \le 1$ for all a. Let $n \in \mathbb{N}$, $N = 2^n - 1$. By induction on n, we get from (5.2)

$$F\Big(\sum_{i=0}^{N} a_i\Big) \le C^n \,.\, \max \{F(a_i) \mid 0 \le i \le N\}, \qquad \text{for } a_i \in \mathbb{N}.$$

Applying this to

$$(a+b)^N = \sum_{i=0}^{N} \binom{N}{i} a^i b^{N-i}$$

and using

$$F(\binom{N}{i} a^i b^{N-i}) \leq F(a)^i \cdot F(b)^{N-i} \leq \max \{F(a), F(b)\}^N$$

we find

$$F((a+b)^N) \leq C^n \cdot \max \{F(a), F(b)\}^N.$$

Taking N-th roots and letting n go to infinity we conclude

$$F(a+b) \leq \max \{F(a), F(b)\}.$$

Define $\phi(a) = F(a)^{-1}$; then theorem 3 applies to ϕ, so there is a prime p and there are reals $A \geq 0$, $B \geq 0$ such that

$$\phi(a) = a^A \cdot a_p{}^B$$

for all $a \in \mathbb{N}$. Putting $\lambda = -A$ and $\mu = -B$ proves the "only if"part. The "if"-part may be left to the reader.

References.

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2. A. Weil, Basic number theory, Springer, Berlin etc., 1967.