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# GROTHENDIECK GROUPS OF ABELIAN GROUP RINGS

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Let $R$ be a noetherian ring, and $G(R)$ the Grothendieck group of finitely generated modules over $R$. For a finite abelian group $\pi$, we describe $G(R\pi)$ as the direct sum of groups $G(R')$. Each $R'$ is the form $R[\zeta_n, 1/n]$, where $n$ is a positive integer and $\zeta_n$ a primitive $n$th root of unity. As an application, we describe the structure of the Grothendieck group of pairs $(H, u)$, where $H$ is an abelian group and $u$ is an automorphism of $H$ of finite order.

## 0. Introduction

The Grothendieck group $G(\mathscr{C})$ of an abelian category $\mathscr{C}$ is defined by generators and relations. There is one generator $[M]$ for each object $M$ of $\mathscr{C}$, and one relation $[M] = [M'] + [M'']$ for every exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{C}$.

Let $R$ be a left noetherian ring with 1. We write $G(R)$ for the Grothendieck group of the category of finitely generated left $R$-modules.

For a group $\pi$, we denote by $R\pi$ the group ring of $\pi$ over $R$. Let $\varrho$ be a finite cyclic group of order $n$, with generator $\tau$, and denote by $\Phi_n$ the $n$th cyclotomic polynomial. As we shall see in Section 2, the two-sided ideal $\Phi_n(\tau)R\varrho$ of $R\varrho$ does not depend on the choice of $\tau$, and we put

$$R(\varrho) = R\varrho/\Phi_n(\tau)R\varrho,$$
$$R\langle\varrho\rangle = R(\varrho)[X]/(nX - 1).$$

These are also left noetherian rings; the zero ring is not excluded.

**0.1. Theorem.** *Let $R$ be a left noetherian ring with 1 and $\pi$ a finite abelian group. Then we have, with the above notations*

$$G(R\pi) \cong \bigoplus_{\pi'} G(R\langle\pi/\pi'\rangle)$$

*where $\pi'$ ranges over all subgroups of $\pi$ for which $\pi/\pi'$ is cyclic.*

For a description of the isomorphism we refer to the proof of the theorem, which is given in Section 4. It is not in any obvious way induced by the natural ring homo-

morphism $R\pi \to \prod_{\pi'} R\langle \pi/\pi' \rangle$. Sections 1, 2 and 3 contain some preparatory material. In Section 5 we describe the behaviour of the isomorphism under change of groups and change of rings.

**0.2. Theorem.** *Let $\pi$ be a finite cyclic group of order $n$. Then*

$$G(\mathbb{Z}\pi) \cong \bigoplus_{d \mid n} (\mathbb{Z} \oplus C(\mathbb{Z}[\zeta_d, 1/d]))$$

*where $\zeta_d$ denotes a primitive $d$th root of unity, $C(\mathbb{Z}[\zeta_d, 1/d])$ denotes the ideal class group of the Dedekind ring $\mathbb{Z}[\zeta_d, 1/d]$, and the direct sum ranges over the divisors $d$ of $n$.*

This theorem is fairly immediate from Theorem 0.1, see Section 7. More generally, for $R$ a Dedekind domain and $\pi$ finite abelian we have

$$G(R\pi) \cong \bigoplus_{\chi} (\mathbb{Z} \oplus C(R\langle \chi \rangle)) \tag{0.3}$$

where $\chi$ ranges over a certain set of characters of $\pi$ and $R\langle \chi \rangle$ is a certain Dedekind domain; see 7.4 for details. The groups $G(R\pi)$ and $\mathbb{Z} \oplus C(R\langle \chi \rangle) \cong G(R\langle \chi \rangle)$ have natural ring structures induced by the tensor product over $R$ and $R\langle \chi \rangle$ respectively, cf. [8, Corollary 1.1]. One might wonder whether (0.3) is a ring isomorphism if multiplication is defined componentwise in the direct sum. Checking the image of the unit element $[R]$ of $G(R\pi)$ one finds that this is only true in the trivial case when the order of $\pi$ is a power of the characteristic of $R$.

For a discussion of the relation between 0.2 and Reiner's description of $G(\mathbb{Z}\pi)$ for $\pi$ cyclic [7], we refer to Section 7.

Sections 8 and 9 are devoted to the group $SSF$ which was investigated by Bass [1] and Grayson [2]. It is defined as follows. Let $\mathcal{S}$ be the category of all pairs $(H, u)$, where $H$ is a finitely generated abelian group and $u$ an automorphism of $H$ for which $u^n - \mathrm{id}_H$ is nilpotent for some positive integer $n$; here $\mathrm{id}_H$ is the identity on $H$. A morphism in $\mathcal{S}$ from $(H, u)$ to $(H', u')$ is defined to be a group homomorphism $f : H \to H'$ for which $f \circ u = u' \circ f$. The pair $(H, u)$ is called a *permutation module* if $H$ admits a $\mathbb{Z}$-basis permuted by $u$. Let $P$ be the subgroup of the Grothendieck group $G(\mathcal{S})$ generated by the classes of all permutation modules. Then $SSF = G(\mathcal{S})/P$.

**0.4. Theorem.** *We have $SSF \cong \bigoplus_{n \geq 1} C(\mathbb{Z}[\zeta_n, 1/n])$.*

This theorem is proved in Section 8. In Section 9 we obtain an almost complete description of $SSF$ as an abelian group, using methods from algebraic number theory.

Rings in this paper are always supposed to have a unit element, and modules are left modules. By $\mathbb{Z}$ and $\mathbb{Q}$ we denote the ring of integers and the field of rational numbers, respectively. Set-theoretic difference is denoted by $-$, and cardinality by $\#$.

## 1. The Grothendieck group of $R[1/n]$

In this section $n$ denotes a positive integer. By $\mathbb{Z}[1/n]$ we denote the subring of $\mathbb{Q}$ generated by $1/n$, and if $M$ is an abelian group we put $M[1/n] = M \otimes_{\mathbb{Z}} \mathbb{Z}[1/n]$. If $R$ is a ring, then $R[1/n]$ is a ring isomorphic to $R[X]/(nX-1)R[X]$, and the element $1 \otimes (1/n)$ of $R[1/n]$ is simply denoted by $1/n$. For any $R$-module $M$ there is a natural $R[1/n]$-module structure on $M[1/n]$, and the functor from the category of $R$-modules to the category of $R[1/n]$-modules mapping $M$ to $M[1/n]$ is exact.

**1.1. Proposition.** *Let $R$ be a left noetherian ring. Then $R[1/n]$ is a left noetherian ring, and $G(R[1/n])$ is isomorphic to $G(R)/H$, where $H$ is the subgroup of $G(R)$ generated by all symbols $[M]$, with $M$ ranging over the finitely generated $R$-modules for which $n \cdot M = 0$.*

**Proof.** The ring $R[1/n]$ is left noetherian because every left ideal of $R[1/n]$ is of the form $\mathfrak{a}[1/n]$, where $\mathfrak{a}$ is a left ideal of $R$.

Since the functor $M \mapsto M[1/n]$ from the category of $R$-modules to the category of $R[1/n]$-modules is exact, and since $M[1/n] = 0$ if $n \cdot M = 0$, there is a group homomorphism

$$\lambda : G(R)/H \to G(R[1/n])$$

mapping the coset of $[M]$ mod $H$ to $[M[1/n]]$.

Let conversely $N$ be a finitely generated $R[1/n]$-module, and let $M$ be a finitely generated $R$-submodule of $N$ which generates $N$ as an $R[1/n]$-module. By a straightforward argument one shows that $([M] \bmod H) \in G(R)/H$ depends only on $N$, and that there is a group homomorphism

$$\mu : G(R[1/n]) \to G(R)/H$$

for which $\mu([N]) = ([M] \bmod H)$ in the situation just described.

To prove 1.1 it now suffices to check that $\lambda$ and $\mu$ are inverse to each other. If $N, M$ are as in the definition of $\mu$, then one easily proves that $N \cong M[1/n]$, so $\lambda\mu$ is the identity on $G(R[1/n])$. Let now $M$ be a finitely generated $R$-module, and let $M_0$ be the image of $M$ under the natural map $M \to M[1/n]$. Then $\mu\lambda([M] \bmod H) = ([M_0] \bmod H)$, so to prove that $\mu\lambda$ is the identity on $G(R)/H$ it suffices to show that $[M] \equiv [M_0] \bmod H$. The kernel $L$ of the natural surjection $M \to M_0$ is given by

$$L = \{x \in M : \exists i \in \mathbb{Z},\ i \geq 0 : n^i \cdot x = 0\}.$$

Since $R$ is noetherian, $L$ is finitely generated, so $n^k \cdot L = 0$ for some $k \in \mathbb{Z}$, $k \geq 0$. Therefore we have $[M] - [M_0] = [L] \in H$, as required. This proves 1.1.

**1.2. Corollary.** *Let $R$ be a left noetherian ring and $\varrho$ a finite cyclic group of order $n$. Then we have, with the notations of the introduction*

$$G(R\langle\varrho\rangle) \cong G(R(\varrho))/H,$$

*where $H$ is the subgroup of $G(R(\varrho))$ generated by all symbols $[M]$, with $M$ ranging over the finitely generated $R(\varrho)$-modules for which $n \cdot M = 0$.*

**Proof.** Immediate from 1.1.

## 2. Filtrations of $R\pi$-modules

Let $\varrho$ be a finite cyclic group of order $n$, with generator $\tau$, and denote by $\Phi_n$ the $n$th cyclotomic polynomial. If $f$ is any ring homomorphism from $\mathbb{Z}\varrho$ to $\mathbb{C}$ which is injective when restricted to $\varrho$, then the kernel of $f$ is generated by $\Phi_n(\tau)$. Hence the ideal $\Phi_n(\tau)\mathbb{Z}\varrho$ of $\mathbb{Z}\varrho$ does not depend on the choice of $\tau$.

Let $R$ be a ring. It follows that the two-sided ideal $\Phi_n(\tau)R\varrho$ is independent of the choice of $\tau$. We define the ring $R(\varrho)$ by

$$R(\varrho) = R\varrho / \Phi_n(\tau)R\varrho.$$

The ring $\mathbb{Z}(\varrho)$ is a domain isomorphic to $\mathbb{Z}[\zeta_n]$, where $\zeta_n$ denotes a primitive $n$th root of unity. Its field of fractions may be identified with $\mathbb{Q}(\varrho)$. The group of units of $\mathbb{Z}(\varrho)$ contains $\varrho$ in a natural way. For arbitrary $R$, we have $R(\varrho) \cong R \otimes_{\mathbb{Z}} \mathbb{Z}(\varrho)$. As an $R$-module, $R(\varrho)$ is free on $\varphi(n)$ generators, where $\varphi$ is the function of Euler. Hence, if $R$ is left noetherian then so is $R(\varrho)$. If $\varrho'$ is a subgroup of $\varrho$, then there is a natural inclusion $R(\varrho') \subset R(\varrho)$.

**2.1. Lemma.** *Let $\varrho$ be a finite cyclic group of order $n$, and suppose that $p^k$ divides $n$, where $p$ is prime and $k \in \mathbb{Z}$, $k \geq 1$. Then in $\mathbb{Z}(\varrho)$ we have $\prod_\sigma (1-\sigma) = p$, where $\sigma$ ranges over the elements of $\varrho$ of order $p^k$.*

**Proof.** The number of such $\sigma$ equals $p^k - p^{k-1}$, and they are zeros of $X^{p^k} - 1$ but not of $X^{p^{k-1}} - 1$. Since $\mathbb{Z}(\varrho)$ is a domain this implies that

$$\prod_\sigma (X - \sigma) = (X^{p^k} - 1)/(X^{p^{k-1}} - 1) = \sum_{i=0}^{p-1} X^{i \cdot p^{k-1}}$$

in $\mathbb{Z}(\varrho)[X]$, and the desired result follows if we substitute 1 for $X$. This proves 2.1.

**2.2. Lemma.** *Let $\varrho$ be a finite cyclic group of order $n$. Denote, for every prime $p$ dividing $n$, by $\varrho_p$ the $p$-primary subgroup of $\varrho$. Let further $R$ be a ring and $M$ an $R(\varrho)$-module for which $n \cdot M = 0$. Then there is a finite chain of $R(\varrho)$-submodules $M = M_0 \supset M_1 \supset \cdots \supset M_t = 0$ of $M$ such that for every $i \in \{1, 2, \ldots, t\}$ there is a prime $p$ dividing $n$ for which $\varrho_p$ acts trivially on $M_{i-1}/M_i$ and $p \cdot (M_{i-1}/M_i) = 0$.*

**Proof.** If $n = p_1 p_2 \cdots p_s$, with $p_i$ prime, then in the chain of $R(\varrho)$-submodules

$$M \supset p_1 M \supset p_1 p_2 M \supset \cdots \supset p_1 p_2 \cdots p_s M = nM = 0$$

every quotient $p_1p_2\cdots p_{i-1}M/p_1p_2\cdots p_iM$ is annihilated by some prime dividing $n$. Hence it suffices to prove the lemma under the added assumption that $p\cdot M=0$, where $p$ is a prime dividing $n$.

Let $p^k = \#\varrho_p$, and let $\sigma_1, \sigma_2, \ldots, \sigma_t$ be the elements of $\varrho$ of order $p^k$; so $t=p^k-p^{k-1}$. Then each $\sigma_i$ is a generator of $\varrho_p$. Put $M_i=(1-\sigma_1)(1-\sigma_2)\cdots(1-\sigma_i)\cdot M$, for $0\le i\le t$. These are $R(\varrho)$-submodules of $M$, and

$$M_0=M,$$
$$M_t=(1-\sigma_1)(1-\sigma_2)\cdots(1-\sigma_t)\cdot M=p\cdot M=0, \text{ by 2.1},$$
$$M_0\supset M_1\supset\cdots\supset M_t.$$

Each module $M_{i-1}/M_i$ is annihilated by $1-\sigma_i$, and since $\sigma_i$ generates $\varrho_p$ this implies that $\varrho_p$ acts trivially on $M_{i-1}/M_i$, for every $i\in\{1,2,\ldots,t\}$. Also $p\cdot(M_{i-1}/M_i)=0$ since $p\cdot M=0$. This proves 2.2.

Now let $\pi$ be a finite abelian group. A *factor group* of $\pi$ is a group of the form $\pi/\pi'$, where $\pi'\subset\pi$ is a subgroup. We stress that two factor groups $\pi/\pi'$ and $\pi/\pi''$ of $\pi$ are only to be considered equal if $\pi'=\pi''$ as subgroups of $\pi$. The set of *cyclic* factor groups of $\pi$ is denoted by $X(\pi)$.

Let $\varrho\in X(\pi)$, and let $R$ be a ring. Then there are natural surjective ring homomorphisms $R\pi\to R\varrho\to R(\varrho)$, and this enables us to identify the $R(\varrho)$-modules with the $R\pi$-modules annihilated by $\ker(R\pi\to R(\varrho))$, as we will do in the sequel.

**2.3. Lemma.** *Let $\pi$ be a finite abelian group, $R$ a ring, and $\varrho', \varrho''\in X(\pi)$, $\varrho'\ne\varrho''$. Suppose that $M$ is an $R\pi$-module which is both an $R(\varrho')$-module and an $R(\varrho'')$-module. Then $p\cdot M=0$ for some prime number $p$ dividing $\#\varrho'$ or $\#\varrho''$.*

**Proof.** Let $\varrho'=\pi/\pi'$ and $\varrho''=\pi/\pi''$. Interchanging $\varrho'$ and $\varrho''$, if necessary, we may assume that $\pi''\not\subset\pi'$. Choose $\sigma\in\pi''-\pi'$. Replacing $\sigma$ by a suitable power we can achieve that $\sigma^p\in\pi'$ for some prime number $p$. The image $\bar\sigma$ of $\sigma$ in $\varrho'$ then has order $p$, so $p$ divides $\#\varrho'$.

Since $M$ is a $\mathbb{Z}(\varrho'')$-module and $\sigma\in\pi''=\ker(\pi\to\varrho'')$, the action of $\sigma$ on $M$ is trivial. Hence $M$ is, as a $\mathbb{Z}(\varrho')$-module, annihilated by $1-\bar\sigma$. Applying 2.1 to $p^k=p$ we find that also $p$ annihilates $M$. This proves 2.3.

For $\varrho\in X(\pi)$, denote by $\mathfrak{m}_\varrho$ the kernel of the ring homomorphism $\mathbb{Q}\pi\to\mathbb{Q}(\varrho)$. Since $\mathbb{Q}(\varrho)$ is a field, $\mathfrak{m}_\varrho$ is a maximal ideal of $\mathbb{Q}\pi$, and from

$$\sigma-1\in\mathfrak{m}_\varrho \Leftrightarrow \sigma\in\ker(\pi\to\varrho)$$

(for $\sigma\in\pi$) we see that $\mathfrak{m}_\varrho\ne\mathfrak{m}_{\varrho'}$ for $\varrho\ne\varrho'$. Hence the Chinese remainder theorem implies that the combined map

$$\mathbb{Q}\pi\to\prod_{\varrho\in X(\pi)}\mathbb{Q}(\varrho) \tag{2.4}$$

is a surjective ring homomorphism. It is also $\mathbb{Q}$-linear, and to prove that it is an isomorphism it suffices to show that the $\mathbb{Q}$-dimensions $\#\pi$ and $\sum_{\varrho \in X(\pi)} \varphi(\#\varrho)$ are the same.

Let $\hat{\pi}$ be the set of group homomorphisms from $\pi$ to the unit circle. Then $\pi/\ker(\chi) \in X(\pi)$ for all $\chi \in \hat{\pi}$. Conversely, for each $\varrho \in X(\pi)$ there are precisely $\varphi(\#\varrho)$ distinct $\chi \in \hat{\pi}$ for which $\varrho = \pi/\ker(\chi)$. Hence $\sum_{\varrho \in X(\pi)} \varphi(\#\varrho) = \#\hat{\pi} = \#\pi$, and (2.4) is an isomorphism.

Since $\mathbb{Z}\pi$ is contained in $\mathbb{Q}\pi$, it follows that the map $\mathbb{Z}\pi \to \prod_{\varrho \in X(\pi)} \mathbb{Z}(\varrho)$ is injective, so if we put $\mathfrak{p}_\varrho = \ker(\mathbb{Z}\pi \to \mathbb{Z}(\varrho)) = \mathfrak{m}_\varrho \cap \mathbb{Z}\pi$ then we have $\bigcap_{\varrho \in X(\pi)} \mathfrak{p}_\varrho = 0$.

**2.5. Lemma.** *Let $R$ be a ring, $\pi$ a finite abelian group, and $M$ an $R\pi$-module. Then there is a finite chain of $R\pi$-submodules $M = M_0 \supset M_1 \supset \cdots \supset M_t = 0$ of $M$ such that for each $i \in \{1, 2, \ldots, t\}$ there exists $\varrho \in X(\pi)$ for which $M_{i-1}/M_i$ is an $R(\varrho)$-module.*

**Proof.** Write $X(\pi) = \{\varrho_1, \varrho_2, \ldots, \varrho_t\}$ and choose $M_i = \mathfrak{p}_{\varrho_1} \cdots \mathfrak{p}_{\varrho_i} M$ for $0 \le i \le t$. Here the $\mathfrak{p}_\varrho$ are as above, and $M$ is considered as a $\mathbb{Z}\pi$-module via the obvious map $\mathbb{Z}\pi \to R\pi$. Since the actions of $R$ and $\mathbb{Z}\pi$ on $M$ commute, the $M_i$ are $R\pi$-submodules of $M$. Further $M_0 = M$, and $M_t = 0$ since $\mathfrak{p}_{\varrho_1} \cdots \mathfrak{p}_{\varrho_t} \subset \bigcap_{\varrho \in X(\pi)} \mathfrak{p}_\varrho = 0$. Finally, each $M_{i-1}/M_i$ is annihilated by $\mathfrak{p}_{\varrho_i}$ and is therefore a module over $R \otimes \mathbb{Z}(\varrho_i) = R(\varrho_i)$, for $1 \le i \le t$. This proves 2.5.

## 3. Notation for the proof of Theorem 0.1

In this section we establish the notation used in Section 4. By $\pi$ we denote a finite abelian group and by $R$ a left noetherian ring. Instead of "finitely generated module" we simply write "module". The class of an $R\pi$-module $M$ in $G(R\pi)$ is denoted by $[M, \pi]$.

As in the previous section, we denote by $X(\pi)$ the set of cyclic factor groups of $\pi$, and for $\varrho \in X(\pi)$ we identify the $R(\varrho)$-modules with the $R\pi$-modules annihilated by the kernel of the natural surjective ring homomorphism $R\pi \to R(\varrho)$. Using 1.2, we will view the group $G(R\langle\varrho\rangle)$ as being defined by generators and relations; one generator $[M, \langle\varrho\rangle]$ for each $R(\varrho)$-module $M$, one relation $[M, \langle\varrho\rangle] = [M', \langle\varrho\rangle] + [M'', \langle\varrho\rangle]$ for each exact sequence $0 \to M' \to M \to M'' \to 0$ of $R(\varrho)$-modules, and one relation $[M, \langle\varrho\rangle] = 0$ for each $R(\varrho)$-module $M$ with $(\#\varrho) \cdot M = 0$. For $\varrho' \in X(\pi)$, we consider $G(R\langle\varrho'\rangle)$ as being embedded in $\bigoplus_{\varrho \in X(\pi)} G(R\langle\varrho\rangle)$ in the obvious way. This allows us to add symbols $[M, \langle\varrho\rangle]$ with distinct $\varrho$'s.

By $P(\pi)$ we denote the set of prime numbers dividing $\#\pi$. If $p$ is a prime number, then $\pi_p$ is the $p$-primary part of $\pi$. There is a canonical isomorphism $\pi \cong \bigoplus_{p \text{ prime}} \pi_p$, and $\pi_p$ is non-trivial if and only if $p \in P(\pi)$. Let $S$ be a set of prime numbers, and let $\pi_S$ denote the subgroup of $\pi$ generated by $\pi_p$ for $p \in S$. Then $\pi \cong \pi_S \oplus \pi_{P(\pi)-S}$, and the composite of the canonical maps $\pi \to \pi_S \to \pi$ induces a ring homomorphism $R\pi \to R\pi$. The functor from the category of $R\pi$-modules to itself induced by this ring

homomorphism is denoted by $N_S$. Thus, if $M$ is an $R\pi$-module, then $N_SM$ is equal to $M$ as an $R$-module, and the actions of $\pi_S$ on $N_SM$ and $M$ coincide, but $\pi_{P(\pi)-S}$ acts trivially on $N_SM$. Hence $N_SM \cong M$ over $R\pi$ if and only if $\pi_{P(\pi)-S}$ acts trivially on $M$. Further we have $N_SN_TM \cong N_{S\cap T}M$ for any two sets of prime numbers $S$ and $T$ and any $R\pi$-module $M$.

Let $\varrho \in X(\pi)$, and let $S$ again be a set of prime numbers. Since there are canonical surjections $\pi \to \varrho \to \varrho_S$ we may consider $\varrho_S$ as an element of $X(\pi)$. Also, $\varrho_S$ is, as a subgroup of $\varrho$, equal to the image of $\pi_S$ under the canonical map $\pi \to \varrho$; so $\varrho_S \in X(\pi_S)$. Since the diagram of natural maps and inclusions

$$\begin{array}{ccccc} R\pi & \to & R\pi_S & \subset & R\pi \\ & \searrow & \downarrow & & \downarrow \\ & & R(\varrho_S) & \subset & R(\varrho) \end{array}$$

is commutative we see that for every $R(\varrho)$-module $M$ the $R\pi$-module $N_SM$ is actually an $R(\varrho_S)$-module. This remark will play an essential role in Section 4.

## 4. Proof of Theorem 0.1

In this section we establish the isomorphism

$$\bigoplus_{\varrho \in X(\pi)} G(R\langle\varrho\rangle) \cong G(R\pi).$$

Let $\varrho \in X(\pi)$. We claim that there is a group homomorphism

$$\varphi_\varrho : G(R\langle\varrho\rangle) \to G(R\pi)$$

for which

$$\varphi_\varrho([M, \langle\varrho\rangle]) = \sum_{S \subset P(\varrho)} (-1)^{\#(P(\varrho)-S)} \cdot [N_SM, \pi].$$

To prove this, we have to show that this assignment respects the relations defining $G(R\langle\varrho\rangle)$, cf. Section 3. This is certainly true for the relations arising from short exact sequences of $R(\varrho)$-modules, since $N_S$ is obviously an exact functor. So it suffices to check that if $(\#\varrho)\cdot M = 0$, then

$$\sum_{S \subset P(\varrho)} (-1)^{\#(P(\varrho)-S)} \cdot [N_SM, \pi] = 0.$$

By Lemma 2.2 we may assume that $\varrho_p$ acts trivially on $M$ and that $p \cdot M = 0$, for some $p \in P(\varrho)$. Then $N_{P(\varrho)-\{p\}}M \cong M$, so for every $S \subset P(\varrho)$ we have

$$N_SM \cong N_SN_{P(\varrho)-\{p\}}M \cong N_{S-\{p\}}M \quad \text{over } R\pi$$

and

$$[N_SM, \pi] = [N_{S-\{p\}}M, \pi].$$

Thus we find

$$\sum_{S \subset P(\varrho)} (-1)^{\#(P(\varrho)-S)} \cdot [N_S M, \pi]$$

$$= \sum_{S \subset P(\varrho), p \in S} (-1)^{\#(P(\varrho)-S)} \cdot ([N_S M, \pi] - [N_{S-\{p\}} M, \pi]) = 0,$$

as required. This proves that $\varphi_\varrho$ is well defined.

Combining the maps $\varphi_\varrho$ we obtain a group homomorphism

$$\varphi : \bigoplus_{\varrho \in X(\pi)} G(R\langle\varrho\rangle) \to G(R\pi).$$

Before defining a map in the other direction we prove a lemma.

**4.1. Lemma.** *Let $\varrho', \varrho'' \in X(\pi)$, and suppose that $M$ is an $R\pi$-module which is both an $R(\varrho')$-module and an $R(\varrho'')$-module. Then we have*

$$\sum_{S \subset P(\varrho')} [N_S M, \langle \varrho'_S \rangle] = \sum_{S \subset P(\varrho'')} [N_S M, \langle \varrho''_S \rangle]$$

*in the group $\bigoplus_{\varrho \in X(\pi)} G(R\langle\varrho\rangle)$.*

**Proof.** Let $S$ be any subset of $P(\pi)$. We prove that $S$ yields the same contribution on both sides. This is certainly true if, on each side, $S$ gives the zero contribution or no contribution at all. So suppose it gives a non-zero contribution on the left hand side, i.e.:

$$S \subset P(\varrho'), \qquad [N_S M, \langle \varrho'_S \rangle] \neq 0.$$

If now $\varrho''_S = \varrho'_S$, then $S = P(\varrho'_S) = P(\varrho''_S) \subset P(\varrho'')$, so $S$ gives on both sides the contribution $[N_S M, \langle \varrho'_S \rangle]$, as required. If $\varrho''_S \neq \varrho'_S$ then applying Lemma 2.3 to $N_S M$, $\varrho'_S$, $\varrho''_S$ we find that $p \cdot N_S M = 0$ for some prime $p$ dividing $\#\varrho'_S$ or $\#\varrho''_S$. Then $p \in S = P(\varrho'_S)$, so $p \cdot N_S M = 0$ implies that $(\#\varrho'_S) \cdot N_S M = 0$, contradicting our assumption that $[N_S M, \langle \varrho'_S \rangle] \neq 0$. This proves 4.1.

If $M$ is an $R\pi$-module which for some $\varrho' \in X(\pi)$ is an $R(\varrho')$-module we put

$$\psi(M) = \sum_{S \subset P(\varrho')} [N_S M, \langle \varrho'_S \rangle] \in \bigoplus_{\varrho \in X(\pi)} G(R\langle\varrho\rangle).$$

By Lemma 4.1 this only depends on $M$, not on the choice of $\varrho'$. If $M$ is an $R(\varrho')$-module and $M' \subset M$ is an $R\pi$-submodule, then $M'$ and $M/M'$ are $R(\varrho')$-modules, and

$$\psi(M) = \psi(M') + \psi(M/M'). \tag{4.2}$$

Now let $M$ be any $R\pi$-module. By Lemma 2.5 there exists a finite chain of submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_t = 0 \tag{4.3}$$

such that each $M_{i-1}/M_i$ is an $R(\varrho_i)$-module for some $\varrho_i \in X(\pi)$, $i = 1, 2, \ldots, t$. We consider the expression

$$\sum_{i=1}^{t} \psi(M_{i-1}/M_i). \tag{4.4}$$

By (4.2), this expression does not change if the chain (4.3) is replaced by a refinement. It is also clear that the expression does not change if (4.3) is replaced by an equivalent chain, i.e. a chain $M = M'_0 \supset M'_1 \supset \cdots \supset M'_t = 0$ for which $M_{i-1}/M_i \cong M'_{\sigma(i)-1}/M'_{\sigma(i)}$ for some permutation $\sigma$ of $\{1, 2, \ldots, t\}$ and all $i \in \{1, 2, \ldots, t\}$. Since by Schreier's theorem any two chains have equivalent refinements, we conclude that the expression (4.4) only depends on $M$; let us denote it by $\psi(M)$.

If $M' \subset M$ is a submodule then combining a chain for $M'$ and a chain for $M/M'$ into one for $M$ we see that $\psi(M) = \psi(M') + \psi(M/M')$. Hence $\psi$ is additive for short exact sequences and therefore induces a group homomorphism

$$\psi : G(R\pi) \to \bigoplus_{\varrho \in X(\pi)} G(R\langle\varrho\rangle)$$

denoted by the same letter, for which

$$\psi([M, \pi]) = \sum_{i=1}^{t} \sum_{S \subset P(\varrho_i)} [N_S M_{i-1}/M_i, \langle \varrho_{i,S} \rangle]$$

if $M_i$, $\varrho_i$ are as above.

To conclude the proof of Theorem 0.1 it now suffices to check that $\varphi$ and $\psi$ are inverse to each other.

First we consider $\psi\varphi$. Let $\varrho \in X(\pi)$, and let $M$ be an $R(\varrho)$-module. Then

$$\begin{aligned}\psi\varphi([M, \langle\varrho\rangle]) &= \psi\Big(\sum_{S \subset P(\varrho)} (-1)^{\#(P(\varrho)-S)} \cdot [N_S M, \pi]\Big) \\ &= \sum_{S \subset P(\varrho)} (-1)^{\#(P(\varrho)-S)} \cdot \psi([N_S M, \pi]).\end{aligned}$$

Each $N_S M$ is an $R(\varrho_S)$-module, so

$$\begin{aligned}\psi([N_S M, \pi]) &= \sum_{T \subset P(\varrho_S)} [N_T N_S M, \langle (\varrho_S)_T \rangle] \\ &= \sum_{T \subset S} [N_T M, \langle \varrho_T \rangle].\end{aligned}$$

Hence we find

$$\begin{aligned}\psi\varphi([M, \langle\varrho\rangle]) &= \sum_{S \subset P(\varrho)} (-1)^{\#(P(\varrho)-S)} \cdot \sum_{T \subset S} [N_T M, \langle \varrho_T \rangle] \\ &= \sum_{T \subset P(\varrho)} \Big( \sum_{S,\, T \subset S \subset P(\varrho)} (-1)^{\#(P(\varrho)-S)} \Big) \cdot [N_T M, \langle \varrho_T \rangle] \\ &= [N_{P(\varrho)} M, \langle \varrho_{P(\varrho)} \rangle] = [M, \langle \varrho \rangle],\end{aligned}$$

where we have used the trivial identity

$$\sum_{S,T\subset S\subset P(\varrho)}(-1)^{\#(P(\varrho)-S)}=\begin{cases}1 & \text{if } T=P(\varrho),\\ 0 & \text{if } T\neq P(\varrho).\end{cases}$$

This proves that $\psi\varphi$ is the identity on $\bigoplus_{\varrho\in X(\pi)}G(R\langle\varrho\rangle)$.

Finally we check that $\varphi\psi$ is the identity on $G(R\pi)$. It suffices to prove that $\varphi\psi([M,\pi])=[M,\pi]$ for every $R\pi$-module $M$ which is an $R(\varrho)$-module for some $\varrho\in X(\pi)$, since by Lemma 2.5 the group $G(R\pi)$ is generated by the classes of these modules. For such $M$ and $\varrho$ we have, using the same identity as before:

$$\begin{aligned}\varphi\psi([M,\pi]) &= \varphi\Big(\sum_{S\subset P(\varrho)}[N_SM,\langle\varrho_S\rangle]\Big)\\ &= \sum_{S\subset P(\varrho)}\sum_{T\subset S}(-1)^{\#(S-T)}\cdot[N_TN_SM,\pi]\\ &= \sum_{T\subset P(\varrho)}\Big((-1)^{\#(P(\varrho)-T)}\sum_{S,T\subset S\subset P(\varrho)}(-1)^{\#(P(\varrho)-S)}\Big)\cdot[N_TM,\pi]\\ &= [N_{P(\varrho)}M,\pi]=[M,\pi],\end{aligned}$$

as required.

This completes the proof of Theorem 0.1.

## 5. Change of groups and change of rings

In this section we investigate the behaviour of the isomorphism of Theorem 0.1 under change of groups and change of rings. The notation introduced in Section 3 remains in force.

First, let $R$ be a left noetherian ring, $\pi$ and $\pi'$ finite abelian groups, and $\pi\to\pi'$ a group homomorphism. This homomorphism induces a functor, which we denote by $F$, from the category of $R\pi'$-modules to the category of $R\pi$-modules. Since $F$ is exact, it gives rise to a group homomorphism

$$G(R\pi')\to G(R\pi),\qquad [M,\pi']\mapsto[FM,\pi].$$

Let $\varrho'\in X(\pi')$, and let $\varrho$ be the image of the composed map $\pi\to\pi'\to\varrho'$. Then $\varrho$ may be considered as an element of $X(\pi)$. Since $R(\varrho)\subset R(\varrho')$, there is a natural forgetful functor, which again may be denoted by $F$, from the category of $R(\varrho')$-modules to the category of $R(\varrho)$-modules. If in addition $P(\varrho')=P(\varrho)$, then a map

$$G(R\langle\varrho'\rangle)\to G(R\langle\varrho\rangle),\qquad [M,\langle\varrho'\rangle]\mapsto[FM,\langle\varrho\rangle]$$

is induced. If $P(\varrho')\neq P(\varrho)$, let

$$G(R\langle\varrho'\rangle)\to G(R\langle\varrho\rangle)$$

be the zero map.

**5.1. Proposition.** *The following diagram is commutative:*

$$\begin{array}{ccc} G(R\pi') & \cong & \bigoplus_{\varrho' \in X(\pi')} G(R\langle\varrho'\rangle) \\ \downarrow & & \downarrow \\ G(R\pi) & \cong & \bigoplus_{\varrho \in X(\pi)} G(R\langle\varrho\rangle). \end{array}$$

*Here the vertical maps are induced by the maps defined above, and the horizontal isomorphisms come from* 0.1.

**Proof.** It suffices to check that, for an $R(\varrho')$-module $M$, the two images of $[M, \pi'] \in G(R\pi')$ in $\bigoplus_{\varrho \in X(\pi)} G(R\langle\varrho\rangle)$ coincide. This is routine, and left to the reader.

**5.2. Corollary.** *If the map $\pi \to \pi'$ is surjective, then $X(\pi')$ may be considered as a subset of $X(\pi)$, and the map $G(R\pi') \to G(R\pi)$ defined above corresponds to the natural inclusion*

$$\bigoplus_{\varrho \in X(\pi')} G(R\langle\varrho\rangle) \subset \bigoplus_{\varrho \in X(\pi)} G(R\langle\varrho\rangle)$$

*under the isomorphisms of* 0.1.

**Proof.** Left to the reader.

In particular, we see that $G(R\pi') \to G(R\pi)$ is *injective* if $\pi \to \pi'$ is surjective, a fact which seems not to be obvious otherwise.

Next let $R$ and $R'$ be left noetherian rings, and let an exact functor $F$ from the category of $R$-modules to the category of $R'$-modules be given. Let $\pi$ be an abelian group. Then $F$ can be used to transport $\pi$-actions, and therefore gives rise to an exact functor from the category of $R\pi$-modules to the category of $R'\pi$-modules, thus inducing a map $G(R\pi) \to G(R'\pi)$. It is not difficult to see that in a similar way a map $G(R\langle\varrho\rangle) \to G(R'\langle\varrho\rangle)$ is induced, for any finite cyclic group $\varrho$.

**5.3. Proposition.** *The following diagram is commutative:*

$$\begin{array}{ccc} G(R\pi) & \cong & \bigoplus_{\varrho \in X(\pi)} G(R\langle\varrho\rangle) \\ \downarrow & & \downarrow \\ G(R'\pi) & \cong & \bigoplus_{\varrho \in X(\pi)} G(R'\langle\varrho\rangle). \end{array}$$

*Here the vertical maps are those just defined, and the horizontal isomorphisms come from* 0.1.

**Proof.** Immediate, and left to the reader.

## 6. Abelian categories

All results obtained in Sections 2 to 5 can be formulated and proved more generally, by replacing the category of finitely generated left modules over a left noetherian ring $R$ by an arbitrary abelian category $A$. The category $A\pi$ then has as its objects pairs consisting of an object of $A$ and an action of $\pi$ on this object. Morphisms of $A\pi$ are morphisms of $A$ respecting the $\pi$-actions. If $\varrho$ is cyclic of order $n$, with generator $\tau$, then $A(\varrho)$ is defined to be the full subcategory of $A\varrho$ consisting of those pairs for which the action of $\Phi_n(\tau)$ on the object is the zero action. We define $G(A\langle\varrho\rangle)$ in the way suggested by 1.2:

$$G(A\langle\varrho\rangle) = G(A(\varrho))/H,$$

where $H$ is the subgroup of $G(A(\varrho))$ generated by the classes of those objects $M$ of $A(\varrho)$ for which $\#\varrho \cdot M = 0$. The Grothendieck group of $A\pi$ can now be described as in Theorem 0.1:

$$G(A\pi) \cong \bigoplus_{\varrho \in X(\pi)} G(A\langle\varrho\rangle).$$

Up to terminology, the proof of this result is identical to the proof of 0.1. The results of Section 5 generalize in a similar way.

## 7. Dedekind rings

In this section $R$ denotes a noetherian *domain*, i.e. a commutative noetherian ring without zero-divisors and with $1 \neq 0$. The field of fractions of $R$ is denoted by $K$ and its characteristic by char $(K)$. If $n$ is divisible by char $(K)$ then $R[1/n]$ is the zero ring. If $n$ is not divisible by char $(K)$, then $R[1/n]$ may be identified with the subring of $K$ generated by $R$ and the inverse of $n \cdot 1$. For these $n$, we denote by $\zeta_n$ a primitive $n$th root of unity in a fixed algebraic closure $\bar{K}$ of $K$. We call $R$ a *Dedekind ring* if it is integrally closed inside $K$ and every non-zero prime ideal of $R$ is maximal. The class group of a Dedekind ring $R$ is denoted by $C(R)$. The following lemma is well known.

**7.1. Lemma.** *Let $R$ be a Dedekind ring. Then $G(R) \cong \mathbb{Z} \oplus C(R)$.*

**Proof.** From [7, § 6, (4)] (with $\Lambda = R$) or [8, Proposition 1.1] (with $\pi = 1$) we know that $G(R) \cong K_0(R)$, where $K_0(R)$ denotes the Grothendieck group of the category of finitely generated projective $R$-modules. By [6, Corollary 1.11] we have $K_0(R) \cong \mathbb{Z} \oplus C(R)$. This proves 7.1.

For later use we remark that $[R]$ is mapped to $1 \in \mathbb{Z}$ under the projection $G(R) \cong \mathbb{Z} \oplus C(R) \to \mathbb{Z}$.

**7.2. Theorem.** *Let $\pi$ be a finite abelian group. Then*

$$G(\mathbb{Z}\pi) \cong \bigoplus_{\varrho \in X(\pi)} (\mathbb{Z} \oplus C(\mathbb{Z}[\zeta_{\#\varrho}, 1/\#\varrho])),$$

*with $X(\pi)$ as defined in Section 2.*

**Proof.** Immediate from 0.1, 7.1 and the observation that $\mathbb{Z}\langle\varrho\rangle \cong \mathbb{Z}[\zeta_{\#\varrho}, 1/\#\varrho]$ is a Dedekind domain. This proves 7.2.

If $\pi$ is cyclic of order $n$, then $\pi$ has exactly one cyclic factor group of order $d$ for every divisor $d$ of $n$. Hence Theorem 0.2 is a special case of 7.2. It can also be derived from the description of $G(\mathbb{Z}\pi)$ given by Reiner [7, § 6, (27)]:

$$G(\mathbb{Z}\pi) \cong (\bigoplus_{d|n} \mathbb{Z}) \oplus ((\bigoplus_{d|n} C(\mathbb{Z}[\zeta_d]))/W)$$

where $W$ is a certain subgroup of $\bigoplus_{d|n} C(\mathbb{Z}[\zeta_d])$ defined by Reiner. Let $N$ be the automorphism of $\bigoplus_{d|n} C(\mathbb{Z}[\zeta_d])$ induced by all norm maps

$$C(\mathbb{Z}[\zeta_e]) \to C(\mathbb{Z}[\zeta_d]) \quad \text{for } d|e|n, \gcd(d, e/d) = 1.$$

Then it is not difficult to check that $W$ is "diagonalized" by $N$:

$$N[W] = \bigoplus_{d|n} W_d$$

where $W_d \subset C(\mathbb{Z}[\zeta_d])$ is the subgroup generated by the classes of the prime ideals dividing $d$. Since it is well known that $C(\mathbb{Z}[\zeta_d])/W_d \cong C(\mathbb{Z}[\zeta_d, 1/d])$, this yields a new proof of 0.2. The two isomorphisms obtained in this way differ only by an automorphism of $\bigoplus_{d|n} \mathbb{Z}$, if the inclusions $\mathbb{Z}[\zeta_d] \subset \mathbb{Z}[\zeta_e]$ used for the above norm maps are well chosen.

Before generalizing 7.2 to Dedekind domains we introduce some terminology. By a *character* of a finite abelian group $\pi$ we mean a group homomorphism from $\pi$ to the multiplicative group of $\bar{K}$. Two characters $\chi, \chi'$ of $\pi$ are called *conjugate* over $K$ if $\chi = \sigma \circ \chi'$ for some $K$-automorphism $\sigma$ of $\bar{K}$. If $\chi$ is a character of $\pi$ then $\pi/\ker(\chi)$ belongs to $X(\pi)$, and has order not divisible by char $(K)$. Conversely, if $\varrho \in X(\pi)$ has order $n$, with $n$ not divisible by char $(K)$, then the set of $K$-conjugacy classes of characters $\chi$ for which $\varrho = \pi/\ker(\chi)$ is in bijective correspondence with the set of monic irreducible factors of $\Phi_n$ in $K[X]$. For a character $\chi$, let $R\langle\chi\rangle$ be the subring of $\bar{K}$ generated by $R$, the image of $\chi$, and the inverse of $\#\chi[\pi] \cdot 1$. The *exponent* of $\pi$ is the l.c.m. of the orders of the elements of $\pi$.

**7.3. Proposition.** *Let $R$ be a noetherian domain and $\pi$ a finite abelian group. Suppose that for every $n$ dividing the exponent of $\pi$ but not divisible by* char $(K)$, *at least one of the irreducible factors of $\Phi_n$ in $K[X]$ has coefficients in $R[1/n]$ and leading coefficient* 1. *Then*

$$G(R\pi) \cong \bigoplus_{\chi \in Y} G(R\langle\chi\rangle)$$

*where $Y$ is a set of representatives for the $K$-conjugacy classes of characters of $\pi$.*

**Proof.** Let $\varrho \in X(\pi)$ have order $n$, with $n$ not divisible by char $(K)$, and let $f$ be an irreducible factor of $\Phi_n$ which has coefficients in $R[1/n]$ and leading coefficient 1. Let $\zeta_n \in \bar{K}$ be a zero of $f$. This is a primitive $n$th root of unity, and $f = \prod_{d \in H} (X - \zeta_n^d)$ for a certain finite set $H$ of integers containing 1. Every monic irreducible factor of $\Phi_n$ in $K[X]$ is of the form $f_a = \prod_{d \in H} (X - \zeta_n^{ad})$ for some $a \in \mathbb{Z}$, gcd $(a, n) = 1$. By the main theorem on symmetric functions all these $f_a$ have coefficients in $R[1/n]$. The resultant of any two distinct $f_a, f_b$ divides the discriminant of $X^n - 1$, which equals $\pm n^n$, and is therefore a unit in $R[1/n]$. Thus any two distinct $f_a, f_b$ generate the unit ideal in $R[1/n][X]$. The Chinese remainder theorem now yields

$$R[1/n][X]/\Phi_n R[1/n][X] \cong \prod R[1/n][X]/f_a R[1/n][X],$$

the product ranging over the distinct irreducible factors $f_a$ of $\Phi_n$ in $K[X]$.

The left hand ring is isomorphic to $R\langle\varrho\rangle$. If $\chi \in Y$ satisfies $\varrho = \pi/\ker(\chi)$ and corresponds to the irreducible factor $f_a$ of $\Phi_n$, then

$$R\langle\chi\rangle \cong R[1/n][X]/f_a R[1/n][X].$$

Thus we have proved that

$$R\langle\varrho\rangle \cong \prod_{\chi \in Y,\, \varrho = \pi/\ker(\chi)} R\langle\chi\rangle.$$

This formula is also valid if char$(K)$ does divide $\#\varrho$, since in that case both sides are the zero ring. Hence

$$\prod_{\varrho \in X(\pi)} R\langle\varrho\rangle \cong \prod_{\chi \in Y} R\langle\chi\rangle.$$

Taking the Grothendieck groups of both sides and applying 0.1 we obtain 7.3.

We observe that the condition on $\Phi_n$ in 7.3 is satisfied if $R$ is integrally closed inside $K$, and in particular if $R$ is a Dedekind ring.

**7.4. Theorem.** *Let $R$ be a Dedekind ring, $\pi$ a finite abelian group and $Y$ as in* 7.3. *Then for each $\chi \in Y$ the ring $R\langle\chi\rangle$, defined before* 7.3, *is a Dedekind ring, and*

$$G(R\pi) \cong \bigoplus_{\chi \in Y} (\mathbb{Z} \oplus C(R\langle\chi\rangle)).$$

**Proof.** Let $\chi \in Y$ and $n = \#\chi[\pi]$. Then $R\langle\chi\rangle \cong R[1/n][\zeta_n]$, where $f(\zeta_n) = 0$ for some irreducible factor $f$ of $\Phi_n$ in $R[X]$. The discriminant of $f$ divides the discriminant of $X^n - 1$, which equals $\pm n^n$, and is therefore a unit in $R[1/n]$. Hence $R[1/n][\zeta_n]$ is the integral closure of $R[1/n]$ in $K(\zeta_n)$. Since $R$ is a Dedekind ring, so is $R[1/n]$, and the theorem of Krull–Akizuki now implies that the same is true for $R[1/n][\zeta_n]$. Thus we have proved that $R\langle\chi\rangle$ is a Dedekind ring, and the rest of 7.4 follows from 7.3 and 7.1.

For the calculation of the class groups occurring in 7.2 and 7.4 it may further be remarked that

$$C(R[1/n, \zeta_n]) \cong C(R_n)/W_n \tag{7.5}$$

for $R$ a Dedekind ring and $n$ not divisible by char($K$); here $R_n$ denotes the integral closure of $R$ in $K(\zeta_n)$, and $W_n$ the subgroup of $C(R_n)$ generated by the classes of the $R_n$-ideals dividing $nR_n$.

## 8. The group *SSF*

In this section we prove Theorem 0.4. For the definitions of $\mathscr{S}$, "permutation module", $P$ and $SSF$ we refer to the paragraph of the introduction preceding 0.4.

Let $\mathscr{S}_0$ be the full subcategory of $\mathscr{S}$ consisting of all $(H, u)$ for which $u$ has finite order. Every object of $\mathscr{S}$ admits a finite filtration with successive quotients in $\mathscr{S}_0$, so the Schreier refinement theorem implies that the natural map $G(\mathscr{S}_0) \to G(\mathscr{S})$ is an isomorphism. The permutation modules belong to $\mathscr{S}_0$, and we conclude that

$$SSF \cong G(\mathscr{S}_0)/P$$

where $P$ is now considered as a subgroup of $G(\mathscr{S}_0)$.

For a positive integer $n$, let $\mathscr{S}_n$ be the full subcategory of $\mathscr{S}_0$ consisting of all $(H, u)$ with $u^n = \mathrm{id}_H$, and let $P_n$ be the subgroup of $G(\mathscr{S}_n)$ generated by the classes of the permutation modules belonging to $\mathscr{S}_n$. There is an obvious isomorphism

$$G(\mathscr{S}_0) \cong \varinjlim_n G(\mathscr{S}_n),$$

the limit ranging over the positive integers, ordered by divisibility. It follows that

$$SSF \cong \varinjlim_n G(\mathscr{S}_n)/P_n. \tag{8.1}$$

For a positive integer $n$, let $\pi_n$ be a cyclic group of order $n$ with a fixed generator $\tau_n$. Letting the action of $u$ correspond to multiplication by $\tau_n$, we have an isomorphism

$$G(\mathscr{S}_n) \cong G(\mathbb{Z}\pi_n).$$

Under this isomorphism, $P_n$ corresponds to the subgroup of $G(\mathbb{Z}\pi_n)$ generated by the elements $[\mathbb{Z}\pi_e]$ of $G(\mathbb{Z}\pi_n)$, for $e \mid n$; here $\mathbb{Z}\pi_n$ acts on $\mathbb{Z}\pi_e$ via the map $\pi_n \to \pi_e$ sending $\tau_n$ to $\tau_e$.

By Theorem 0.1 there is an isomorphism

$$\psi : G(\mathbb{Z}\pi_n) \to \bigoplus_{d \mid n} G(\mathbb{Z}\langle \pi_d \rangle).$$

Checking the definition of $\psi$ in Section 4 one finds that

$$\psi([\mathbb{Z}\pi_e]) = ([M_{de}])_{d \mid n} \in \bigoplus_{d \mid n} G(\mathbb{Z}\langle \pi_d \rangle)$$

for $e \mid n$, where

$$M_{de} = 0 \quad \text{if } d \text{ does not divide } e,$$
$$M_{ee} = \mathbb{Z}\langle \pi_e \rangle$$

(it is easy to give a formula for $M_{de}$ if $d \mid e$, $d \neq e$, but it will not be used in the sequel). Composing $\psi$ with the isomorphism of Lemma 7.1 and the canonical projections $\mathbb{Z} \oplus C(\mathbb{Z}\langle \pi_d \rangle) \to \mathbb{Z}$ we obtain a map

$$\lambda : G(\mathbb{Z}\pi_n) \xrightarrow{\sim} \bigoplus_{d \mid n} G(\mathbb{Z}\langle \pi_d \rangle) \xrightarrow{\sim} \bigoplus_{d \mid n} (\mathbb{Z} \oplus C(\mathbb{Z}\langle \pi_d \rangle)) \to \bigoplus_{d \mid n} \mathbb{Z}.$$

Using the remark following 7.1 we see that

$$\lambda([\mathbb{Z}\pi_e]) = (a_{de})_{d \mid n} \in \bigoplus_{d \mid n} \mathbb{Z},$$

for $e \mid n$, where

$$a_{de} = 0 \quad \text{if } d \text{ does not divide } e,$$
$$a_{ee} = 1.$$

Letting $e$ range over the divisors of $n$ we obtain a triangular matrix $(a_{de})_{d \mid n, e \mid n}$ of determinant 1, and therefore $\lambda$ induces an isomorphism

$$P_n \xrightarrow{\sim} \bigoplus_{d \mid n} \mathbb{Z}.$$

We conclude that there is an isomorphism

$$G(\mathcal{S}_n)/P_n \xrightarrow{\sim} \bigoplus_{d \mid n} C(\mathbb{Z}\langle \pi_d \rangle).$$

Further, if $n$ divides $m$, then a straightforward verification depending on 5.2 shows that the diagram

$$\begin{array}{ccc} G(\mathcal{S}_n)/P_n & \xrightarrow{\sim} & \bigoplus_{d \mid n} C(\mathbb{Z}\langle \pi_d \rangle) \\ \downarrow & & \downarrow \\ G(\mathcal{S}_m)/P_m & \xrightarrow{\sim} & \bigoplus_{d \mid m} C(\mathbb{Z}\langle \pi_d \rangle) \end{array}$$

is commutative, the left hand vertical arrow being induced by the inclusion $\mathcal{S}_n \subset \mathcal{S}_m$ and the right hand vertical map being the obvious inclusion. Thus (8.1) becomes

$$SSF \cong \varinjlim_n \bigoplus_{d \mid n} C(\mathbb{Z}\langle \pi_d \rangle) \cong \bigoplus_{d \geq 1} C(\mathbb{Z}\langle \pi_d \rangle).$$

Since $\mathbb{Z}\langle \pi_d \rangle \cong \mathbb{Z}[\zeta_d, 1/d]$, this proves Theorem 0.4.

## 9. The structure of *SSF*

In this section we investigate the structure of *SSF* as an abelian group. If $n$ is a prime power $>1$, then every prime ideal of $\mathbb{Z}[\zeta_n]$ dividing $n$ is generated by $1-\zeta_n$ and therefore principal, so (7.5) yields

$$C(\mathbb{Z}[\zeta_n, 1/n]) \cong C(\mathbb{Z}[\zeta_n]).$$

Since $C(\mathbb{Z}[\zeta_n])=0$ for only finitely many positive integers $n$, cf. [5], it follows that infinitely many of the groups $C(\mathbb{Z}[\zeta_n, 1/n])$ are non-zero. Hence *SSF* is an infinite abelian group.

Since each group $C(\mathbb{Z}[\zeta_n, 1/n])$ is finite, *SSF* can be written as the direct sum of a collection of finite cyclic groups. It is reasonable to conjecture that

$$SSF \cong \bigoplus_{n\geq 1} (\mathbb{Z}/n\mathbb{Z}) \qquad \text{(non-canonically)}$$

which can be more sensibly written as

$$SSF \cong \bigoplus_{p \text{ prime}} \bigoplus_{m=1}^{\infty} (\mathbb{Z}/p^m\mathbb{Z})^{(\infty)}$$

where $M^{(\infty)}$ denotes the direct sum of a countably infinite collection of copies of $M$.

The following theorem shows that only the 2-primary part of this conjecture escapes me.

**9.1. Theorem.** *There is a strictly increasing sequence $(m_i)_{i=1}^{\infty}$ of positive integers such that*

$$SSF \cong [\bigoplus_{i\geq 1} (\mathbb{Z}/2^{m_i}\mathbb{Z})^{(\infty)}] \oplus [\bigoplus_{\substack{p \text{ prime} \\ p\geq 3}} \bigoplus_{m\geq 1} (\mathbb{Z}/p^m\mathbb{Z})^{(\infty)}].$$

It is not asserted that the isomorphism in 9.1 is canonical. It would be of interest to study the structure of *SSF* as a module over the Galois group of $\bigcup_{n\geq 1}\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$.

We begin the proof of 9.1 with a series of lemmas. For background in algebraic number theory we refer to [4]. Notation: $R_K$ is the ring of integers of $K$; if $K\subset L$ is a Galois extension, then $\mathrm{Gal}(L/K)$ is its Galois group; $[\mathfrak{a}]$ is the ideal class of $\mathfrak{a}$; and the exponent $\exp(A)$ of a finite group $A$ is the l.c.m. of the orders of its elements.

**9.2 Lemma.** *For any positive integer $m$ there exists a quadratic number field $K$ of discriminant relatively prime to $m$, such that $\exp(C(R_K))$ is divisible by $m$.*

**Proof.** This follows from [9, Theorem 1 or 2].

**9.3. Lemma.** *Let $K$ be a quadratic number field of discriminant $\Delta$. Then $K\subset\mathbb{Q}(\zeta_{|\Delta|})$, and the cokernel of the norm map $C(\mathbb{Z}[\zeta_{|\Delta|}])\to C(R_K)$ has exponent $\leq 2$.*

**Proof.** There is a unique decomposition $\Delta = \prod_{i=1}^{t} \Delta_i$, where each $\Delta_i$ belongs to the set $\{-4, -8, 8\} \cup \{p: p \text{ is prime, } p \equiv 1 \bmod 4\} \cup \{-q: q \text{ is prime, } q \equiv 3 \bmod 4\}$. If $|\Delta_i|$ is prime, then $\mathbb{Q}(\zeta_{|\Delta_i|})$ has a unique quadratic subfield, and checking the ramification behaviour one finds that this must be the field $\mathbb{Q}(\sqrt{\Delta_i})$ of discriminant $\Delta_i$. Hence $\mathbb{Q}(\sqrt{\Delta_i}) \subset \mathbb{Q}(\zeta_{\Delta_i})$, an inclusion which can be verified directly for $\Delta_i \in \{-4, -8, 8\}$. So $K = \mathbb{Q}(\sqrt{\Delta}) \subset L = \mathbb{Q}(\sqrt{\Delta_1}, \ldots, \sqrt{\Delta_t}) \subset \mathbb{Q}(\zeta_{|\Delta_1|}, \ldots, \zeta_{|\Delta_t|}) = \mathbb{Q}(\zeta_{|\Delta|})$. Again checking the ramification behaviour, one finds that $\mathrm{Gal}(\mathbb{Q}(\zeta_{|\Delta|})/L)$ is generated by the inertia groups of the primes dividing $\Delta$. Therefore the only subfield $M \subset \mathbb{Q}(\zeta_{|\Delta|})$ which is a totally unramified extension of $L$ is $M = L$ itself.

Now let $F$ be the maximal totally unramified extension of $K$ inside $\mathbb{Q}(\zeta_{|\Delta|})$. Applying the above to $M = L \cdot F$ we find that $F \subset L$; in fact, one can check that $F$ equals $L$ if $\Delta < 0$ and the maximal real subfield of $L$ if $\Delta > 0$. Hence $\mathrm{Gal}(F/K)$ is a quotient of $\mathrm{Gal}(L/K)$ and therefore of exponent $\leq 2$.

Let $H_K$ denote the Hilbert class field of $K$. Then $\mathrm{Gal}(H_K/K) \cong C(R_K)$, and the subfield corresponding to the image of the norm $C(\mathbb{Z}[\zeta_{|\Delta|}]) \to C(R_K)$ equals $H_K \cap \mathbb{Q}(\zeta_{|\Delta|}) = F$. Therefore the cokernel of $C(\mathbb{Z}[\zeta_{|\Delta|}]) \to C(R_K)$ is isomorphic to $\mathrm{Gal}(F/K)$, and we have just seen that this group has exponent $\leq 2$. This proves 9.3.

**9.4. Lemma.** *For any positive integer $q$ there exists a positive integer $n$ relatively prime to $q$ such that* $\exp(C(\mathbb{Z}[\zeta_n, 1/n]))$ *is divisible by $q$.*

**Proof.** Let $K$ be a quadratic number field of discriminant $\Delta$ relatively prime to $q$, such that $\exp(C(R_K))$ is divisible by $4q$. The existence of such a $K$ is guaranteed by 9.2, with $m = 4q$. Put $n = |\Delta|$. By 9.3 the cokernel of the norm map $C(\mathbb{Z}[\zeta_{|\Delta|}]) \to C(R_K)$ has exponent $\leq 2$. Factoring out the subgroup generated by the classes of the prime ideals dividing $n$, we see that also the cokernel of the induced map $C(\mathbb{Z}[\zeta_n, 1/n]) \to C(R_K[1/n])$ has exponent $\leq 2$. Hence to prove that $C(\mathbb{Z}[\zeta_n, 1/n])$ has exponent divisible by $q$ it suffices to prove that $C(R_K[1/n])$ has exponent divisible by $2q$.

If $\mathfrak{p}$ is a prime of $R_K$ dividing $n = |\Delta|$, then $\mathfrak{p}^2$ is generated by a rational prime number and is therefore principal. This shows that the kernel of the surjective map $C(R_K) \to C(R_K[1/n])$ has exponent $\leq 2$. Since $\exp(C(R_K))$ is divisible by $4q$ it follows that $\exp(C(R_K[1/n]))$ is divisible by $2q$, as required. This proves 9.4.

**9.5. Lemma.** *Let $n$ and $r$ be positive integers, and let $p$ be a prime number not dividing $\phi(nr)/\phi(n)$; here $\phi$ denotes the Euler function. Then the $p$-primary subgroup of $C(\mathbb{Z}[\zeta_n, 1/nr])$ is isomorphic to a direct summand of $C(\mathbb{Z}[\zeta_{nr}, 1/nr])$.*

**Proof.** The ring $\mathbb{Z}[\zeta_{nr}, 1/nr]$ is the integral closure of $\mathbb{Z}[\zeta_n, 1/nr]$ in $\mathbb{Q}(\zeta_{nr})$, so there are natural group homomorphisms

$$i : C(\mathbb{Z}[\zeta_n, 1/nr]) \to C(\mathbb{Z}[\zeta_{nr}, 1/nr])$$
$$N : C(\mathbb{Z}[\zeta_{nr}, 1/nr]) \to C(\mathbb{Z}[\zeta_n, 1/nr])$$

such that $N \circ i(c) = c^m$ for all $c \in C(\mathbb{Z}[\zeta_n, 1/nr])$, where $m = [\mathbb{Q}(\zeta_{nr}) : \mathbb{Q}(\zeta_n)] = \phi(nr)/\phi(n)$. Since $\gcd(p, \phi(nr)/\phi(n)) = 1$, the restriction of $N \circ i$ to the $p$-primary part of $C(\mathbb{Z}[\zeta_n, 1/nr])$ is an automorphism, and 9.5 follows.

**9.6. Lemma.** *For every positive integer $n$ there exist infinitely many positive integers $r$ such that the 2-primary part of $C(\mathbb{Z}[\zeta_n, 1/n])$ is isomorphic to a direct summand of $C(\mathbb{Z}[\zeta_{nr}, 1/nr])$.*

**Proof.** If $n$ is a power of 2, then according to a theorem of Weber (cf. [3]) the class number of $\mathbb{Q}(\zeta_n)$ is odd. In that case the 2-primary part of $C(\mathbb{Z}[\zeta_n, 1/n])$ is trivial, and the lemma is obvious. Assume, therefore, that $n$ is not a power of 2, and let $q$ be an odd prime number dividing $n$. Then for $r = q^k$, with $k \in \mathbb{Z}_{\geq 0}$, we have $\gcd(2, \phi(nr)/\phi(n)) = \gcd(2, q^k) = 1$, and $\mathbb{Z}[\zeta_n, 1/n] = \mathbb{Z}[\zeta_n, 1/nr]$. Hence 9.5 shows that for each of the infinitely many choices of $k$ the 2-primary part of $C(\mathbb{Z}[\zeta_n, 1/n])$ is a direct summand of $C(\mathbb{Z}[\zeta_{nr}, 1/nr])$. This proves 9.6.

**9.7. Lemma.** *Let $p$ be an odd prime number, $n$ a positive integer which is not divisible by $p$, and $c \in C(\mathbb{Z}[\zeta_n, 1/n])$. Then there exist infinitely many prime numbers $r$ such that*

(i) $r \equiv 1 \bmod n$, $r \not\equiv 1 \bmod p$;

(ii) *there is a prime ideal $\mathfrak{r}$ of $\mathbb{Z}[\zeta_n, 1/n]$ with $r \in \mathfrak{r}$ whose ideal class is $c$.*

**Proof.** Let $F$ be the maximal abelian totally unramified extension of $\mathbb{Q}(\zeta_n)$ in which all primes dividing $n$ split completely. By class field theory there is an isomorphism

$$\lambda : C(\mathbb{Z}[\zeta_n, 1/n]) \xrightarrow{\sim} \mathrm{Gal}(F/\mathbb{Q}(\zeta_n))$$

such that

$$\lambda([\mathfrak{p}]) = (\mathfrak{p}, F/\mathbb{Q}(\zeta_n)) \qquad \text{(the Artin symbol)}$$

for every non-zero prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_n, 1/n]$.

The field $F$ is a Galois extension of $\mathbb{Q}$ and it is unramified at $p$ since $p$ does not divide $n$. Therefore $F \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$, and $\mathrm{Gal}(F(\zeta_p)/\mathbb{Q}) \cong \mathrm{Gal}(F/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. Let $\sigma \in \mathrm{Gal}(F(\zeta_p)/\mathbb{Q})$ be such that

$$\sigma | F = \lambda(c), \qquad \sigma | \mathbb{Q}(\zeta_p) \neq \text{identity}$$

($|$ denotes restriction); here we use that $p \neq 2$. By the theorem of Tchebotarev, there are infinitely many prime numbers $r$ not dividing $np$ for which

$$[\mathfrak{r}', F(\zeta_p)/\mathbb{Q}] = \sigma \qquad \text{(the Frobenius symbol)}$$

for some prime $\mathfrak{r}'$ of $F(\zeta_p)$ lying over $r$.

Let $r$ be such a prime number. We claim that (i) and (ii) are satisfied. From

$$\sigma | \mathbb{Q}(\zeta_n) = \lambda(c) | \mathbb{Q}(\zeta_n) = \text{identity},$$
$$\sigma | \mathbb{Q}(\zeta_p) \neq \text{identity}$$

it follows that $r$ splits completely in $\mathbb{Q}(\zeta_n)$ but not in $\mathbb{Q}(\zeta_p)$. This implies (i), since $r$ does not divide $np$. As for (ii), let $\mathfrak{r}'$ be as above and let $\mathfrak{r}$ be its restriction to $\mathbb{Q}(\zeta_n)$. Then $\mathfrak{r}$ has degree one over $r$, and therefore the Artin symbol $(\mathfrak{r}, F/\mathbb{Q}(\zeta_n))$ is equal to the first power of $\sigma|F$. This yields

$$\lambda([\mathfrak{r}]) = (\mathfrak{r}, F/\mathbb{Q}(\zeta_n)) = \sigma|F = \lambda(c)$$

and therefore $[\mathfrak{r}] = c$, as required. This proves 9.7.

**9.8. Lemma.** *Let $q$ be an odd prime power and $n$ a positive integer relatively prime to $q$ for which* exp $(C(\mathbb{Z}[\zeta_n, 1/n]))$ *is divisible by $q$. Then there are infinitely many positive integers $r$ such that $C(\mathbb{Z}[\zeta_{nr}, 1/nr])$ has a cyclic direct summand of order $q$.*

**Proof.** Let $G = \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, and choose $c_1, c_2, \ldots, c_t \in C(\mathbb{Z}[\zeta_n, 1/n])$ such that they generate $C(\mathbb{Z}[\zeta_n, 1/n])^q$ as a module over $\mathbb{Z}G$. Let $p$ be the prime number dividing $q$ (assuming that $q \neq 1$). Applying 9.7 to $c = c_i$ we find prime ideals $\mathfrak{r}_1, \ldots, \mathfrak{r}_t$ of $\mathbb{Z}[\zeta_n, 1/n]$ and prime numbers $r_i \in \mathfrak{r}_i$ such that $r_i \neq r_j$ for $i \neq j$ and

$$[\mathfrak{r}_i] = c_i, \quad r_i \equiv 1 \bmod n, \quad r_i \not\equiv 1 \bmod p.$$

Let $r = \prod_{i=1}^{t} r_i$. Then the prime ideals of $\mathbb{Z}[\zeta_n, 1/n]$ dividing $r$ are precisely the conjugates under $G$ of $\mathfrak{r}_1, \ldots, \mathfrak{r}_t$, and the classes of these prime ideals generate $C(\mathbb{Z}[\zeta_n, 1/n])^q$. Therefore

$$C(\mathbb{Z}[\zeta_n, 1/nr]) \cong C(\mathbb{Z}[\zeta_n, 1/n])/C(\mathbb{Z}[\zeta_n, 1/n])^q.$$

Since $q$ divides the exponent of $C(\mathbb{Z}[\zeta_n, 1/n])$, it follows that $C(\mathbb{Z}[\zeta_n, 1/nr])$ has a cyclic direct summand of order $q$.

From $r_i \not\equiv 1 \bmod p$ we see that $p$ does not divide $\phi(nr)/\phi(n) = \prod_{i=1}^{t}(r_i - 1)$. Hence we can apply 9.5, and we find that $C(\mathbb{Z}[\zeta_{nr}, 1/nr])$ also has a cyclic direct summand of order $q$. Since there are infinitely many choices for $t, r_1, \ldots, r_t$ this finishes the proof of 9.8.

**Proof of 9.1.** Applying 0.4 and writing each group $C(\mathbb{Z}[\zeta_n, 1/n])$ as a direct sum of finitely many cyclic groups of prime power order we see that *SSF* can be written as the direct sum of countably many cyclic groups of prime power order. The question is, which prime powers occur, and how often. From 9.4 it follows that arbitrarily high powers of any fixed prime occur. Further, every power of 2 which does occur, occurs infinitely often, by 9.6. Finally, every odd prime power occurs infinitely often, by 9.4 and 9.8. This proves 9.1.

**Remark.** From the above proof it is clear that to prove the conjecture stated before 9.1, it suffices to show that every power of 2 occurs as the order of a cyclic direct summand of $C(\mathbb{Z}[\zeta_n, 1/n])$ for some $n$.

By techniques similar to those used above it can be shown that $m_{i+1} - m_i \leq 3$ for all $i \geq 1$ in Theorem 9.1.

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