JOURNAL OF ALGEBRA **180**, 513–516 (1996)
ARTICLE NO. 0079

# Abelian Subvarieties

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*Communicated by Wilberd Van der Kallen*

Received May 9, 1995

Let $K$ be a field and let $X$ be an abelian variety over $K$. The group Aut $X$ of $K$-automorphisms of $X$ acts in a natural way on the set of abelian subvarieties of $X$ that are defined over $K$. In this paper it is proved that the number of orbits is finite. The proof makes use of a finiteness result about semisimple algebras.


Let $K$ be a field. All abelian varieties and morphisms between them occurring below are assumed to be defined over $K$. We shall prove the following result.

THEOREM. *Let $K$ be a field, let $X$ be an abelian variety defined over $K$, and denote by* Aut $X$ *the group of automorphisms of $X$. Then the number of*

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0021-8693/96 $18.00

*orbits of the set of abelian subvarieties of X under the natural action of* Aut $X$ *is finite*.

It clearly follows that for each abelian variety $X$ there are, up to isomorphism, only finitely many abelian varieties that admit an embedding in $X$ as an abelian subvariety. In the case that $K$ is algebraically closed and of characteristic zero, this assertion is due to Bertrand [1, Section 2, Lemma 2], and if $K$ is finite it follows from a finiteness theorem for abelian varieties of given dimension over $K$; see [7, 4.1]. Our theorem generalizes several other finiteness results for abelian subvarieties that can be found in the literature (see [2, Section 1.3, Proposition 1(ii); 5, Theorem 18.7; 4, Proposition 2]).

The proof of the theorem is given below. It depends on a consequence of the Jordan–Zassenhaus theorem.

By "ring" we mean "ring with 1". A ring is called *semisimple* if it is the sum of its minimal nonzero right ideals. We denote by $\mathbf{Z}$ the ring of integers and by $\mathbf{Q}$ the field of rational numbers. If $L$ is an abelian group, then we write $L_{\mathbf{Q}} = L \otimes_{\mathbf{Z}} \mathbf{Q}$. By a *lattice* in a finite dimensional $\mathbf{Q}$-vector space $V$ we mean a finitely generated subgroup $L$ of the additive group of $V$ for which the natural map $L_{\mathbf{Q}} \to V$ is an isomorphism. A $\mathbf{Q}$-*algebra* is a ring $F$ provided with a ring homomorphism from $\mathbf{Q}$ to the center of $F$. If $F$ is a $\mathbf{Q}$-algebra, then we denote the dimension of $F$ as a vector space over $\mathbf{Q}$ by $[F: \mathbf{Q}]$.

PROPOSITION. *Let $F$ be a semisimple* $\mathbf{Q}$-*algebra with* $[F: \mathbf{Q}] < \infty$, *let $M$ be a finitely generated right $F$-module, and let $L$ be a lattice in $M$. Denote by $G$ the group of those automorphisms $\sigma$ of the $F$-module $M$ for which $\sigma L = L$. Then the number of orbits of the set of $F$-submodules of $M$ under the natural action of $G$ is finite*.

*Proof.* We recall that an *order* in $F$ is a lattice in $F$ that is a subring of $F$. Let $A$ be an order in $F$ that is maximal in the sense that it is not contained in a larger order; the existence of such an order is proved in [6, Section 10].

Let $L'$ be the $A$-submodule of $M$ generated by $L$. This is a lattice in $M$ that contains $L$ as a sublattice of finite index; denote this index by $n$. Write $G'$ for the group of all automorphisms $\sigma$ of the $F$-module $M$ for which $\sigma L' = L'$. Clearly, $G$ is a subgroup of $G'$. One easily checks that $G$ contains the kernel of the natural map $G' \to \mathrm{Aut}(L'/nL')$, and since the latter group is finite this implies that $G$ is of finite index in $G'$. Hence it will suffice to prove the proposition with $G$ replaced by $G'$, that is, with $L$ replaced by $L'$. In other words, we may assume without loss of generality that $L$ is an $A$-submodule of $M$.

The Jordan–Zassenhaus theorem (see [6, Theorem (26.4)]) asserts that for any non-negative integer $t$ there are, up to isomorphism, only finitely many right $A$-modules whose additive group is isomorphic to $\mathbf{Z}^t$. Hence the set $S$ of pairs of isomorphism types of right $A$-modules $L_1, L_2$ for which $L_1 \oplus L_2 \cong_A L$ is finite.

Let $N$ be an $F$-submodule of $M$, and put $L_1 = N \cap L$ and $L_2 = L/L_1$. Since $A$ is a maximal order, $L_2$ is projective as an $A$-module (see [6, Corollary (21.5)]), so that there is an isomorphism $L \cong_A L_1 \oplus L_2$ that is the identity on $L_1$. This shows that there is a well-defined map from the set of $F$-submodules of $M$ to $S$ that sends $N$ to the pair $L_1, L_2$ (up to isomorphism). To prove the proposition it will suffice to show that if two $F$-submodules $N, N'$ of $M$ have the same image in $S$, then there exists $\sigma \in G$ such that $\sigma N = N'$. Thus, assume that there are isomorphisms $N \cap L \to N' \cap L$ and $L/(N \cap L) \to L/(N' \cap L)$ of $A$-modules. Taking the direct sum we obtain an $A$-isomorphism $L \to L$ that extends the isomorphism $N \cap L \to N' \cap L$. Tensoring with $\mathbf{Q}$ we find an $F$-automorphism $\sigma$ of $M$ with $\sigma L = L$ and $\sigma N = N'$, as required. This proves the proposition.

An alternative proof can be derived from a result of Borel and Harish-Chandra [3, Theorem 6.9].

Let $K$ and $X$ be as in the theorem. We write $R$ for the ring of endomorphisms of $X$, and $R^*$ for the group of units of $R$; so we have $R^* = \operatorname{Aut} X$. It is known that the $\mathbf{Q}$-algebra $R_{\mathbf{Q}}$ is semisimple, that $[R_{\mathbf{Q}}: \mathbf{Q}] < \infty$, and that $R$ may be viewed as a lattice in $R_{\mathbf{Q}}$ (see [5, Section 12]).

For an abelian subvariety $Y \subset X$ we write $I(Y) = \{r \in R: rX \subset Y\}$. This is a right ideal of $R$.

LEMMA. *Suppose that $Y, Y'$ are abelian subvarieties of $X$, and that $u \in R^* = \operatorname{Aut} X$ is such that $uI(Y)_{\mathbf{Q}} = I(Y')_{\mathbf{Q}}$. Then we have $uY = Y'$.*

*Proof.* Since we clearly have $uI(Y) = I(uY)$, the proof immediately reduces to the case $u = 1$, which we now assume.

By the theorem of Poincaré–Weil [5, Proposition 12.1] there is an abelian subvariety $Z$ of $X$ such that the natural map $Y \times Z \to X$ is an isogeny. Hence there is a surjective morphism $X \to Y$, so $rX = Y$ for some $r \in I(Y)$. The hypothesis $I(Y)_{\mathbf{Q}} = I(Y')_{\mathbf{Q}}$ implies that $nr \in I(Y')$ for some positive integer $n$. Since multiplication by $n$ is a surjective morphism $Y \to Y$, we now obtain $Y = nY = nrX \subset Y'$. By symmetry we have $Y' \subset Y$. This proves the lemma.

We now prove the theorem. By the lemma, the orbit space of the set of abelian subvarieties of $X$ under the action of $\operatorname{Aut} X$ maps injectively to the orbit space of the set of right ideals of $R_{\mathbf{Q}}$ under the action of $R^*$

(acting by left multiplication). To see that the latter orbit space is finite, it suffices to apply the proposition to $F = R_{\mathbf{Q}}$, $M = F$ (viewed as a right $F$-module), and $L = R$; note that the group $G$ appearing in the proposition can then be identified with $R^*$. This completes the proof of the theorem.

## ACKNOWLEDGMENTS

We thank Mikhail Borovoi and Ke-Zheng Li for clarifying discussions. The third author is grateful for the hospitality enjoyed at the Mathematisch Instituut of the Universiteit Utrecht while work on this paper was being done. The first author was supported by NSF Grant DMS 92-24205. The first and third authors were supported by NWO, the Nederlandse Organisatie voor Wetenschappelijk Onderzoek.

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