Experiments with Ceresa classes of Fermat quotients

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THE HEBREW UNIVERSITY OF JERUSALEM

The Ceresa cycle

- C = smooth projective curve over a field $K \subset \mathbb{C}$
- g = genus of $C = \dim H^0(C, \Omega^1_C) \geq 2$
- $P \in C(K)$
- $J = \mathrm{Pic}^0_{C/K}$ = Jacobian (abelian) variety of dimension g

Embeddings: $C \hookrightarrow J$

$$\iota_P(Q) = [(Q) - (P)] \qquad \iota_P^-(Q) = [(P) - (Q)]$$

The Ceresa cycle:

$$\kappa_P(C) := [\iota_P(C)] - [\iota_P^-(C)] \in \mathrm{CH}_1(J) := F_{\mathrm{ab}}(\text{curves} \subset J)/\sim_{\mathrm{rat}}$$

Observation: $\kappa_P(C)$ is null-homologous

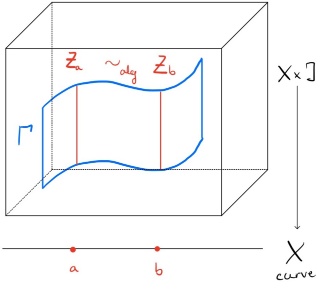
$$\kappa_P(C) \in \mathrm{CH}_1(J)_0 := \ker(\mathrm{CH}_1(J) \xrightarrow{\mathrm{cl}_{\mathbb{C}}} H^{2g-2}(J(\mathbb{C}), \mathbb{Z})).$$

Proof: $\iota_P^- = [-1] \circ \iota_P$ and $[-1]$ acts as $+1$ on $H^{2g-2}(J) = \wedge^{2g-2} H^1(J)$.

The Ceresa class

Algebraic equivalence:

$$\mathrm{CH}_1(J)_{\mathrm{alg}} := \sum_{\substack{X/K \text{ sm. proj. curve} \\ \Gamma \subset X \times J \text{ dim}=2}} \Gamma_*(\mathrm{Pic}^0(X))$$



The Ceresa class: $\kappa(C) := [\kappa_P(C)]_{\mathrm{alg}} \in \mathrm{CH}_1(J)_0/\,\mathrm{CH}_1(J)_{\mathrm{alg}} =: \mathrm{Gr}_1(J)$

Question: Is $\kappa(C) = 0$ in $\mathrm{Gr}_1(J)$?

A brief history of the Ceresa cycle

Observation: C hyperelliptic $\implies \kappa(C) = 0$

Proof: If $\sigma =$ hyper. inv., then $\iota_P^- = \iota_{\sigma(P)} \circ \sigma$, so $\iota_P^-(C) = \iota_{\sigma(P)}(C)$. □

Infinite order Ceresa class:

Very general curves $/\mathbb{C}$	$g \geq 3$	Ceresa/Top	1983/89
$F^4 \colon X^4 + Y^4 + Z^4 = 0$	$g = 3$	Harris/Bloch	1983/84
$XY^3 + YZ^3 + ZX^3 = 0$	$g = 3$	Kimura/Tadokoro	2000/08
$F^m \colon X^m + Y^m + Z^m = 0$ if $\exists p > 7$ such that $p \mid m$	$g(m)$	Eskandari–Murty	2021

Question: (Clemens, $g = 3$) C hyperelliptic $\iff \kappa(C) = 0$?

Theorem (Beauville–Schoen, 2021)

For $C \colon y^3 = x^4 + x$, $\kappa(C) = 0 \in \mathrm{Gr}_1(J) \otimes \mathbb{Q}$.

A criterion of Beauville

The complex Abel–Jacobi map: $\mathrm{AJ}_{\mathbb{C}} \colon \mathrm{CH}_1(J)_0 \longrightarrow \mathcal{J}_1(J_{\mathbb{C}})$

$$\mathcal{J}_1(J_{\mathbb{C}}) := \mathrm{Fil}^{g-1} \backslash H^{2g-3}(J(\mathbb{C}), \mathbb{C}) / H^{2g-3}(J(\mathbb{C}), \mathbb{Z})$$

Idea: If $\sigma \in \mathrm{Aut}(C)$ and $\sigma(P) = P$, then

- $\rightsquigarrow$ σ fixes $\kappa_P(C)$ in $\mathrm{CH}_1(J)_0$
- $\rightsquigarrow$ σ fixes $\mathrm{AJ}_{\mathbb{C}}(\kappa_P(C))$ in $\mathcal{J}_1(J_{\mathbb{C}})$
- $\rightsquigarrow$ If $\mathrm{Fix}(\sigma) \subset \mathcal{J}_1(J_{\mathbb{C}})$ is finite, then $\mathrm{AJ}_{\mathbb{C}}(\kappa_P(C))$ is torsion

Let $V = H^0(C_{\mathbb{C}}, \Omega^1_{C_{\mathbb{C}}})$. The finiteness of $\mathrm{Fix}(\sigma)$ is equivalent to:

Criterion ($\star$): 1 is not an eigenvalue for σ acting on

$$T_0(\mathcal{J}_1(J_{\mathbb{C}})) = \overline{\mathrm{Fil}^{g-1}} = (\wedge^{g-3} V \otimes \wedge^g V^*) \oplus (\wedge^{g-2} V \otimes \wedge^{g-1} V^*).$$

Theorem (Beauville, 2021)

For $C \colon y^3 = x^4 + x$, $\mathrm{AJ}_{\mathbb{C}}(\kappa_{(0,0)}(C))$ is torsion in $\mathcal{J}_1(J_{\mathbb{C}})$.

Proof: $V = \{dx/y^2, xdx/y^2, dx/y\}$ and $\sigma(x, y) = (\zeta_9^{-3} x, \zeta_9^{-1} y)$. $\square$

Cyclic Fermat quotients

Let $m \in \mathbb{N}$, $0 < a, b < m$, $\gcd(m, a, b, a+b) = 1$.

$$\begin{array}{lccc} f^m_{a,b}\colon & F^m\colon X^m + Y^m + Z^m = 0 & \xrightarrow{\mu_m} & C^m_{a,b}\colon v^m = (-1)^{a+b} u^a (1-u)^b \\ & (x:y:1) & \mapsto & (-x^m, x^a y^b) \end{array}$$

Definitions/Properties

- $g^m_{a,b} = (m - (\gcd(m,a) + \gcd(m,b) + \gcd(m,a+b)) + 2)/2$
- $\mu_m \subset \mathrm{Aut}(C^m_{a,b})$ generated by $\sigma_m\colon (u,v) \mapsto (u, \zeta_m v)$
- $I^m_{a,b} := \{(r,s) \in (\mathbb{Z}/m\mathbb{Z})^{\oplus 2} \mid r, s, r+s \neq 0, br \equiv as \pmod{m}\}$
- $(f^m_{a,b})^* V^m_{a,b} = \{x^{\langle r \rangle - m} y^{\langle s \rangle - 1} dy \mid (r,s) \in I^m_{a,b}, \langle r \rangle + \langle s \rangle < m\}$ basis of σ_m-eigenvectors.

If $\{a, b, m-a-b\} \equiv \{ta', tb', t(m-a'-b')\} \pmod{m}$ for some $t \in (\mathbb{Z}/m\mathbb{Z})^\times$ (written $(a,b) \sim_m (a',b')$), then $C^m_{a,b} \simeq C^m_{a',b'}$.

Observation: $C^m_{a,b}$ is hyperelliptic if and only if $(a,b) \sim_m (1,1)$ or $m = 2n$ and $(a,b) \sim_m (1,n)$.

Testing Beauville's criterion

Recall: $I^m_{a,b} = \{(r,s) \in (\mathbb{Z}/m\mathbb{Z})^{\oplus 2} \mid r, s, r+s \neq 0, br \equiv as \pmod{m}\}$.

Proposition: Criterion $(\star)$ for $C^m_{a,b}$ and $\sigma = \sigma_m$ is equivalent to:

For all pairwise distinct $(r_n, s_n)_{n=1}^3 \in I^m_{a,b}$, $\sum_{n=1}^3 (r_n, s_n) \neq (0,0) \qquad (\star\star)$.

Theorem (L.–Shnidman)

Up to $\sim_m$-equivalence, the only non-hyperelliptic cyclic Fermat quotients $C^m_{a,b}$ with $m \leq 100$ that satisfy Criterion $(\star\star)$ are:

$$\begin{array}{llll} C^9_{1,2} \simeq & y^3 = x^4 + x & g = 3 & (\text{Beauville}) \\ C^{12}_{1,3} \simeq & y^3 = x^4 + 1 & g = 3 & \\ C^{15}_{1,5} \simeq & y^3 = x^5 + 1 & g = 4. & \end{array}$$

In particular, $\mathrm{AJ}_{\mathbb{C}}(\kappa_{(0,0)}(C))$ is torsion for these three curves.

Upper bounds: $\#\,\mathrm{Fix}(\sigma_9) = 81$, $\#\,\mathrm{Fix}(\sigma_{12}) = 36$, $\#\,\mathrm{Fix}(\sigma_{15}) = 2025$.

Algebraic cycles and L-functions

C = smooth projective curve over a **number field** K, $g \geq 2$.

Conjecture (Beilinson–Bloch, 1984)

$$\operatorname{rank} \mathrm{CH}_1(J_K)_0 = \operatorname{ord}_{s=g-1} L(H^{2g-3}(J_K), s).$$

The L-function is attached to the ℓ-adic $\mathrm{Gal}(\bar{K}/K)$-reps $H^{2g-3}_{\mathrm{et}}(J_{\bar{K}}, \mathbb{Q}_\ell)$

- ⇝ Euler product for $\Re(s) > g - 1/2$
- ⇝ Conjectural functional equation centered at $s = g - 1$.

Conjecture (Beilinson–Bloch, 1984)

If $H^{2g-3}(J_K) = M_K \oplus I_K$ of pure type $(g-3, g) + (g, g-3)$ and $(g-2, g-1) + (g-1, g-2)$ respectively, then

$$\operatorname{rank} \mathrm{Gr}_1(J_K) = \operatorname{ord}_{s=g-1} L(M_K, s)$$

L-function calculations

The Jacobian $J^m_{a,b}$ of $C^m_{a,b}$ has CM by $\mathbb{Q}(\zeta_m)$

$$\implies H^{2g-3}((J^m_{a,b})_{\mathbb{Q}(\zeta_m)^+}) = (M^m_{a,b})_{\mathbb{Q}(\zeta_m)^+} \oplus (I^m_{a,b})_{\mathbb{Q}(\zeta_m)^+}.$$

Theorem (L.–Shnidman)

$$\begin{aligned} \operatorname{ord}_{s=2} L((M^9_{1,2})_{\mathbb{Q}(\zeta_9)^+}, s) &= 1 \\ \operatorname{ord}_{s=2} L((M^{12}_{1,3})_{\mathbb{Q}(\zeta_{12})^+}, s) &= 0 \\ \operatorname{ord}_{s=3} L((M^{15}_{1,5})_{\mathbb{Q}(\zeta_{15})^+}, s) &= 2 \end{aligned}$$

Proof: $A^m_{a,b} = (J^m_{a,b})^{\text{new}}$ has dimension $\varphi(m)/2$ and CM by $\mathbb{Q}(\zeta_m)$

$$\implies L(A^m_{a,b}/\mathbb{Q}, s) = L(\tau^m_{a,b}/\mathbb{Q}(\zeta_m), s)$$

for a Hecke character $\tau^m_{a,b} \colon I_{\mathbb{Q}(\zeta_m)}(m) \longrightarrow \mathbb{C}^\times$ given by Jacobi sum (Weil)

$$\tau^m_{a,b}(\mathfrak{p}) := -\sum_{z \in \mathbb{F}_p \setminus \{0,1\}} \chi^a_{\mathfrak{p}}(z)\chi^b_{\mathfrak{p}}(1-z), \qquad \chi_{\mathfrak{p}}(z) := z^{\frac{N(\mathfrak{p})-1}{m}} \pmod{\mathfrak{p}}.$$

E.g. $L((M^9_{1,2})_{\mathbb{Q}(\zeta_9)^+}, s) = L(\tau^9_{1,2}\tau^9_{2,4}\tau^9_{5,1}/\mathbb{Q}(\zeta_9), s)$, sign $= -1$, $L'(2) \neq 0$.

Torsion in the Griffiths group

So far:

- $\mathrm{AJ}_{\mathbb{C}}(\kappa_{(0,0)}(C^{12}_{1,3}))$ torsion $\overset{\text{conj}}{\Longrightarrow}$ $\kappa_{(0,0)}(C^{12}_{1,3})$ torsion in $\mathrm{CH}_1(J^{12}_{1,3})_0$
- $L((M^{12}_{1,3})_{\mathbb{Q}(\zeta_{12})^+}, 2) \neq 0 \overset{\text{conj}}{\Longrightarrow} \# \, \mathrm{Gr}_1((J^{12}_{1,3})_{\mathbb{Q}(\zeta_{12})^+}) < \infty$
- $C^{12}_{1,3} \simeq y^3 = x^4 + 1$.

Theorem (L.–Shnidman)

$$\kappa(C^{12}_{1,3}) = 0 \in \mathrm{Gr}_1(J^{12}_{1,3}) \otimes \mathbb{Q}$$

Proof: Adaptation of Beauville–Schoen's proof that $\kappa(C^{9}_{1,2})$ is torsion.
Key input: $J^{12}_{1,3}/\langle \sigma_{12} \rangle$ is uniruled. □

All three $J^{9}_{1,2}/\langle\sigma_9\rangle$, $J^{12}_{1,3}/\langle\sigma_{12}\rangle$, $J^{15}_{1,5}/\langle\sigma_{15}\rangle$ are uniruled and currently constitute the only known (to my knowledge) non-hyperelliptic examples.

There can be no such examples for $g = 5$ or $g \geq 21$ (Beauville, 2022).

Recent related developments

- Bisogno–Li–Litt–Srinivasan (2020): $\mathrm{AJ}_{\mathrm{et}}(\kappa_P(C_{\mathrm{FM}}))$ torsion for C_{FM} = the Fricke–Macbeath curve (the $g = 7$ Hurwitz curve) ($\overset{\text{conj}}{\Longrightarrow}$ $\kappa_P(C_{\mathrm{FM}})$ torsion in $\mathrm{CH}_1(J_{\mathrm{FM}})_0$).

- Gross (2021): $L(C_{\mathrm{FM}}^3, 2) \neq 0$ ($\overset{\text{conj}}{\Longrightarrow}$ $\kappa(C)$ torsion in $\mathrm{Gr}_1(J_{\mathrm{FM}})$).

- Qiu–Zhang (2022): If $\exists G \subset \mathrm{Aut}(C)$ such that $H^0(G, H^1(C)^{\otimes 3}) = 0$, then $\kappa_\xi(C)$ is torsion in $\mathrm{CH}_1(J)_0$

  $\rightsquigarrow$ applies to C_{FM}, $C_{1,3}^{12}$, C_{Bring}, and 1-dimensional family in $g = 4, 5$.

- Ellenberg–Logan–Srinivasan–Venkatesh (2023?): $\sim$ 200.000 smooth plane quartics over $\mathbb{Q}$ have $\kappa_P(C) \neq 0 \in \mathrm{CH}_1(J)_0 \otimes \mathbb{Q}$.

Thank you for your attention!

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