INTRODUCTION TO ELLIPTIC UNITS

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Abstract. These are the notes of two talks I gave at IMJ–PRG on May 24, 2023. The goal of these lectures is to introduce the audience to a compatible system of units in class fields over imaginary quadratic fields that plays an important role in the study of the arithmetic of CM elliptic curves, notably in the landmark proof of BSD for CM elliptic curves in analytic rank zero by Coates–Wiles [1]. We define elliptic units as values at torsion points on a CM elliptic curve E of certain rational functions on E. We survey some of their salient properties, notably the so-called distribution relation. We then give the complex analytic interpretation of elliptic units in terms of theta functions and present an important link with special values of Hecke L-functions. These expository notes closely follow [2] (and to a lesser degree [1] and [3]) and contain no novel mathematical contributions on my part, except for the mistakes I may have introduced.

1. Motivation: circular units

Consider a compatible collection of roots of unity $\{\zeta_m \mid m \in \mathbb{Z}_{\geq 1}\}$ in the sense that $\zeta_{mn}^n = \zeta_m$ for all $m, n \geq 1$. If m is not a power of a prime, then $\zeta_m - 1$ is a global unit in the cyclotomic field $\mathbb{Q}(\zeta_m)$, i.e., $\zeta_m - 1 \in \mathbb{Z}[\zeta_m]^\times$. This is called a circular or cyclotomic unit. These special units satisfy certain relations among them known as distribution relations or norm relations. Namely, given a prime ℓ and an integer $m \geq 1$ which is not a power of a prime, we have

$$N_{\mathbb{Q}(\zeta_m)}^{\mathbb{Q}(\zeta_{m\ell})}(\zeta_{m\ell} - 1) = \begin{cases} \zeta_m - 1, & \ell \mid m \\ (\zeta_m - 1)^{1-\mathrm{Fr}_\ell^{-1}}, & \ell \nmid m, \end{cases}$$

where $\mathrm{Fr}_\ell \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ denotes the Frobenius automorphism. Note that the cyclotomic fields $\mathbb{Q}(\zeta_m)$ (along with their maximal totally real subfields $\mathbb{Q}(\zeta_m)^+$) are the ray class fields over $\mathbb{Q}$. The so-called Euler system of circular units has many important applications in number theory, notably to special value formulas for Dirichlet L-functions and subsequently to the construction of the Kubota–Leopoldt p-adic L-function. Elliptic units play an analogous role when $\mathbb{Q}$ is replaced by an imaginary quadratic field K.

2. Elliptic units: definitions and properties

Elliptic units are certain special units in ray class fields over an imaginary quadratic field K. The explicit class field theory of imaginary quadratic fields is intimately linked with the theory of CM elliptic curves. We will begin by recalling the main theorem of CM theory in the setting relevant for us. In these notes we will often restrict attention to the case where the class number of K is 1, as in the original [1]. This leaves us with 9 discriminants to choose from. For the more general case, we refer the reader to [3].

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2.1. **Complex multiplication.** Let K be an imaginary quadratic field. Let E be an elliptic curve over a field F with CM by $\mathcal{O}_K$. This means that there is an isomorphism $\mathrm{End}_F(E) \simeq \mathcal{O}_K$ and F contains $K(j(E))$. The main theorem of CM theory asserts that the Hilbert class field H of K is equal to $K(j(E))$. In particular, the isomorphism class of E over $\bar{F}$ contains an elliptic curve, again denoted E by slight abuse, which is defined over H and such that $\mathrm{End}_H(E)$ is isomorphic to $\mathcal{O}_K$. We will choose once and for all this isomorphism sending $\alpha \in \mathcal{O}_K$ to $[\alpha] \in \mathrm{End}_H(E)$ such that the map $[\alpha]^* \colon \Omega^1_{E/H} \longrightarrow \Omega^1_{E/H}$ is multiplication by α. Furthermore, if $\mathfrak{m}$ is an integral ideal of $\mathcal{O}_K$, then $H(E[\mathfrak{m}])/H$ is an abelian extension and the ray class field $K(\mathfrak{m})$ of modulus $\mathfrak{m}$ is obtained by adjoining to H the image of the group of $\mathfrak{m}$-torsion points $E[\mathfrak{m}]$ under the Kummer map $h \colon E \longrightarrow E/\mathrm{Aut}_H(E) = E/\mathcal{O}_K^\times$. In the case when $\mathcal{O}_K^\times = \{\pm 1\}$, and after choosing a plane projective model for E, the map h takes a point P to its x-coordinate. We recall that the Artin map of class field theory induces an isomorphism

$$I_K(\mathfrak{m})/P_{K,1}(\mathfrak{m}) \xrightarrow{\sim} \mathrm{Gal}(K(\mathfrak{m})/K), \qquad [\mathfrak{b}] \mapsto \sigma_{\mathfrak{b}} := [\mathfrak{b}, K(\mathfrak{m})/K]\,, \tag{2.1}$$

where $I_K(\mathfrak{m})$ denotes the group of fractional ideal of K that are prime to $\mathfrak{m}$ and $P_{K,1}(\mathfrak{m})$ consists of those principal ideals that can be written as $\alpha\mathcal{O}_K$ with $\alpha \equiv 1 \pmod{\mathfrak{m}}$. One deduces an isomorphism

$$(\mathcal{O}_K/\mathfrak{m})^\times/\mathcal{O}_K^\times \xrightarrow{\sim} \mathrm{Gal}(K(\mathfrak{m})/H).$$

Finally, there is a unique algebraic Hecke character $\psi := \psi_E$ of H of conductor $\mathfrak{F} = \mathrm{cond}(E)$ (an ideal of H) and valued in K such that

(1) If $\mathfrak{M}$ is any integral ideal of H prime to $\mathfrak{F}$, then $\psi(\mathfrak{M}) \in \mathcal{O}_K$ and $\psi(\mathfrak{M})\mathcal{O}_K = N_K^H\mathfrak{M}$,
(2) If $\mathfrak{M}$ is any integral ideal of H prime to $\mathfrak{F}$ and $\mathfrak{c}$ is any ideal of K prime to $N_K^H\mathfrak{M}$, then
$$[\mathfrak{M}, H(E[\mathfrak{c}])/H](P) = [\psi(\mathfrak{M})](P), \qquad \text{for all } P \in E[\mathfrak{c}].$$
(3) If $\mathfrak{P}$ is a prime of H at which E has good reduction (with reduction $E_{\mathbb{F}_\mathfrak{P}}$), then $[\psi(\mathfrak{P})] \in \mathrm{End}_H(E)$ reduces modulo $\mathfrak{P}$ to the absolute Frobenius endomorphism $\mathrm{Frob}_\mathfrak{P}$ of $E_{\mathbb{F}_\mathfrak{P}}$.

2.2. **Elliptic units.** From now on we let K be an imaginary quadratic field with class number one. We then have $H = K$ and the theory of the previous section simplifies considerably. Fix E an elliptic curve over $\mathbb{C}$ with CM by $\mathcal{O}_K$. Pick a Weierstrass model

$$E \colon y^2 = 4x^3 - Ax - B, \qquad \Delta(E) = A^3 - 27B^2 \neq 0, \qquad A, B \in \mathbb{C}. \tag{2.2}$$

Note that we can always choose an isomorphic model with coefficients in K by the discussion of the previous section.

Definition 2.1. Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ prime to 6 with generator $\alpha \in \mathcal{O}_K$. Define a rational function on E by

$$\Theta_{E,\mathfrak{a}} := \alpha^{-12}\Delta(E)^{N(\mathfrak{a})-1} \prod_{P \in E[\mathfrak{a}]\setminus 0} \frac{1}{(x - x(P))^6}.$$

The rational function $\Theta_{E,\mathfrak{a}}$ is independent of the choice of α (since $\#\mathcal{O}_K^\times \mid 12$) and of the choice of Weierstrass model (the model is unique up to $(A, B) \mapsto (u^4A, u^6B)$ with $u \in \mathbb{C}^\times$). In other words, we have

$$\Theta_{E,\mathfrak{a}} = \Theta_{E',\mathfrak{a}} \circ \phi, \qquad \text{for all } \phi \in \mathrm{Isom}(E, E'). \tag{2.3}$$

Finally, it is clear from this that if E is defined over F, then $\Theta_{E,\mathfrak{a}} \in F(E)$ (since a Weierstrass equation with coefficients in F may be chosen, so that $\alpha \in F$ and $\Delta(E) \in F$).

Proposition 2.2. *Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ prime to 6 and let $\mathfrak{b}$ be an ideal of $\mathcal{O}_K$ prime to $\mathfrak{a}$. Let $Q \in E[\mathfrak{b}]$ be a proper $\mathfrak{b}$-section (a generator of $E[\mathfrak{b}]$ as a free rank one $\mathcal{O}_K/\mathfrak{b}$-module). Then $\Theta_{E,\mathfrak{a}}(Q) \in K(\mathfrak{b})$ and if $\mathfrak{c} = (c)$ is any ideal of $\mathcal{O}_K$ prime to $\mathfrak{b}$ and $\sigma_\mathfrak{c} := [\mathfrak{c}, K(\mathfrak{b})/K] \in \mathrm{Gal}(K(\mathfrak{b})/K)$, then $\sigma_\mathfrak{c}(\Theta_{E,\mathfrak{a}}(Q)) = \Theta_{E,\mathfrak{a}}([c]Q)$.*

Proof. This follows from CM theory. The fact that $\Theta_{E,\mathfrak{a}}(Q) \in K(E[\mathfrak{b}])$ is clear given that $\Theta_{E,\mathfrak{a}} \in K(E)$. The fact that $\Theta_{E,\mathfrak{a}}(Q) \in K(\mathfrak{b}) = K(h(E[\mathfrak{b}]))$, where $h\colon E \longrightarrow E/\mathrm{Aut}(E)$ is the Kummer map, then follows from the invariance property (2.3). Let ψ be the Hecke character of K valued in $K^\times$ attached to E by CM theory. We then have

$$\sigma_\mathfrak{c}(\Theta_{E,\mathfrak{a}}(Q)) = \Theta_{E,\mathfrak{a}}(\sigma_\mathfrak{c}(Q)) = \Theta_{E,\mathfrak{a}}([\psi(\mathfrak{c})](Q)).$$

The first equality follows from $\Theta_{E,\mathfrak{a}}$ being a K-rational function, while the second equality is property (3) of §2.1. By (1) of §2.1, we have $\psi(\mathfrak{c})\mathcal{O}_K = \mathfrak{c} = c\mathcal{O}_K$, whence $\psi(\mathfrak{c}) = cu$ for some $u \in \mathcal{O}_K^\times = \mathrm{Aut}(E)$. The result then follows by (2.3). □

Theorem 2.3. *Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ prime to 6 and let $\mathfrak{b}$ be an ideal of $\mathcal{O}_K$ prime to $\mathfrak{a}$. Let $Q \in E[\mathfrak{b}]$ be a proper $\mathfrak{b}$-section. If $\mathfrak{b}$ is not a power of a prime ideal, then $\Theta_{E,\mathfrak{a}}(Q) \in \mathcal{O}_{K(\mathfrak{b})}^\times$. If $\mathfrak{b} = \mathfrak{p}^n$ for some prime ideal $\mathfrak{p}$ and $n > 0$, then $\Theta_{E,\mathfrak{a}}(Q) \in \mathcal{O}_{K(\mathfrak{b})}[1/\mathfrak{p}]^\times$.*

Proof. We give a sketch of the proof. Let $\mathfrak{p}$ be a prime ideal and let $n = \mathrm{ord}_\mathfrak{p}(\mathfrak{a})$. Fix an extension of the valuation $\mathrm{ord}_\mathfrak{p}$ to $\bar{K}$ such that $\mathrm{ord}_\mathfrak{p}(\pi) = 1$ for $(\pi) = \mathfrak{p}$. We may and will assume that E has good reduction modulo $\mathfrak{p}$. Indeed, there exists an elliptic curve E' over K that is isomorphic to E over $\bar{K}$ and has good reduction at $\mathfrak{p}$ [2, Corollary 5.22] and we can apply (2.3). In particular, we have $\mathrm{ord}_\mathfrak{p}(\Delta(E)) = 0$ and

$$\mathrm{ord}_\mathfrak{p}(\Theta_{E,\mathfrak{a}}(Q)) = -12n - 6 \sum_{P \in E[\mathfrak{a}] \setminus 0} \mathrm{ord}_\mathfrak{p}(x(Q) - x(P)). \tag{2.4}$$

Let $K_\mathfrak{p}$ denote the $\mathfrak{p}$-adic completion of K and let $E_1(\bar{K}_\mathfrak{p})$ denote the residue disc modulo $\mathfrak{p}$ of the origin (the kernel of reduction modulo $\mathfrak{p}$ of $E_{\bar{K}_\mathfrak{p}}$). Note that $Q \in E(\bar{K})$ belongs to $E_1(\bar{K}_\mathfrak{p})$ if and only if $\mathrm{ord}_\mathfrak{p}(x(Q)) < 0$. We will need the following classical fact [2, Theorem 3.15]: if $\mathfrak{m}$ is an ideal prime to $\mathfrak{p}$, then $E[\mathfrak{m}]$ injects into $E_{\mathbb{F}_\mathfrak{p}}[\mathfrak{m}]$, where $E_{\mathbb{F}_\mathfrak{p}}$ denotes the reduction of E modulo $\mathfrak{p}$.

First, suppose that $n > 0$ and write

$$\mathrm{ord}_\mathfrak{p}(\Theta_{E,\mathfrak{a}}(Q)) = -12n - 6 \sum_{P \in E[\mathfrak{a}] \setminus E[\mathfrak{p}^n]} \mathrm{ord}_\mathfrak{p}(x(Q) - x(P)) - 6 \sum_{P \in E[\mathfrak{p}^n] \setminus 0} \mathrm{ord}_\mathfrak{p}(x(Q) - x(P)). \tag{2.5}$$

Note that $\mathfrak{p} \nmid \mathfrak{b}$ since $(\mathfrak{a}, \mathfrak{b}) = 1$. In particular, we have $E[\mathfrak{b}] \hookrightarrow E_{\mathbb{F}_\mathfrak{p}}[\mathfrak{b}]$. Since $Q \neq 0$, we see that $Q \not\equiv 0 \pmod{\mathfrak{p}}$. Equivalently, $Q \notin E_1(\bar{K}_\mathfrak{p})$, or $\mathrm{ord}_\mathfrak{p}(x(Q)) \geq 0$. Let us write $\mathfrak{a} = \mathfrak{a}'\mathfrak{p}^n$, and $\mathfrak{p} = (\pi)$. If $P \in E[\mathfrak{a}] \setminus E[\mathfrak{p}^n]$, then $0 \neq \pi^n P \in E[\mathfrak{a}'] \hookrightarrow E_{\mathbb{F}_\mathfrak{p}}[\mathfrak{a}']$. Thus, $\pi^n P \notin E_1(\bar{K}_\mathfrak{p})$, which implies that $P \notin E_1(\bar{K}_\mathfrak{p})$, i.e., $\mathrm{ord}_\mathfrak{p}(x(P)) \geq 0$. We deduce that $\mathrm{ord}_\mathfrak{p}(x(Q) - x(P)) \geq 0$. Observe that

$$\mathrm{ord}_\mathfrak{p}(x(Q) - x(P)) > 0 \iff x(Q) \equiv x(P) \pmod{\mathfrak{p}} \iff Q \equiv \pm P \pmod{\mathfrak{p}}.$$

Since $(\mathfrak{a}, \mathfrak{b}) = 1$, we have $0 \neq \pi^n(P \pm Q) \in E[\mathfrak{a}'\mathfrak{b}] \hookrightarrow E_{\mathbb{F}_\mathfrak{p}}[\mathfrak{a}'\mathfrak{b}]$. In particular, $\pi^n(Q \pm P) \not\equiv 0 \pmod{\mathfrak{p}}$, whence $Q \not\equiv \pm P \pmod{\mathfrak{p}}$. We conclude that $\mathrm{ord}_\mathfrak{p}(x(Q) - x(P)) = 0$ and

$$\mathrm{ord}_\mathfrak{p}(\Theta_{E,\mathfrak{a}}(Q)) = -12n - 6 \sum_{m=1}^{n} \sum_{P \in E[\mathfrak{p}^m] \setminus E[\mathfrak{p}^{m-1}]} \mathrm{ord}_\mathfrak{p}(x(Q) - x(P)). \tag{2.6}$$

We now use the fact [2, Lemma 7.2] that if $P \in E[\mathfrak{p}^m] \setminus E[\mathfrak{p}^{m-1}]$, then

$$\mathrm{ord}_\mathfrak{p}(x(P)) = -\frac{2}{N(\mathfrak{p})^m - N(\mathfrak{p})^{m-1}}. \tag{2.7}$$

Since $\mathrm{ord}_{\mathfrak{p}}(x(Q)) \geq 0$, we obtain

$$\mathrm{ord}_{\mathfrak{p}}(\Theta_{E,\mathfrak{a}}(Q)) = -12n + \frac{12}{N(\mathfrak{p})^m - N(\mathfrak{p})^{m-1}} \sum_{m=1}^{n} \sum_{P \in E[\mathfrak{p}^m] \setminus E[\mathfrak{p}^{m-1}]} 1 = -12n + 12 \sum_{m=1}^{n} 1 = 0.$$

Next, suppose that $n = 0$ and $\mathfrak{p} \nmid \mathfrak{b}$. Then $\mathfrak{p} \nmid \mathfrak{a}\mathfrak{b}$. Because $Q \neq 0$ and $E[\mathfrak{b}] \hookrightarrow E_{\mathbb{F}_\mathfrak{p}}[\mathfrak{b}]$, we see that $Q \not\equiv 0 \pmod{\mathfrak{p}}$, i.e., $Q \notin E_1(\bar{K}_\mathfrak{p})$, which is equivalent to $\mathrm{ord}_{\mathfrak{p}}(x(Q)) \geq 0$. The same is true for P, i.e., $\mathrm{ord}_{\mathfrak{p}}(x(P)) \geq 0$. In particular, we have $\mathrm{ord}_{\mathfrak{p}}(x(Q) - x(P)) \geq 0$. Observe that

$$\mathrm{ord}_{\mathfrak{p}}(x(Q) - x(P)) > 0 \iff x(Q) \equiv x(P) \pmod{\mathfrak{p}} \iff Q = \pm P \pmod{\mathfrak{p}}.$$

But $E[\mathfrak{a}\mathfrak{b}]$ injects into $E_{\mathbb{F}_\mathfrak{p}}[\mathfrak{a}\mathfrak{b}]$. Since $(\mathfrak{a}, \mathfrak{b}) = 1$, we have $0 \neq P \pm Q \in E[\mathfrak{a}\mathfrak{b}]$, and thus $P \not\equiv \pm Q \pmod{\mathfrak{p}}$. We conclude that $\mathrm{ord}_{\mathfrak{p}}(x(Q) - x(P)) = 0$, and by (2.4) we deduce that $\mathrm{ord}_{\mathfrak{p}}(\Theta_{E,\mathfrak{a}}(Q)) = 0$.

Finally, assume that $m := \mathrm{ord}_{\mathfrak{p}}(\mathfrak{p}) > 0$. If $\mathfrak{b}$ is not a power of $\mathfrak{p}$, then $Q \in E[\mathfrak{b}] \setminus E[\mathfrak{p}^m]$ and as above we obtain $\mathrm{ord}_{\mathfrak{p}}(x(Q) - x(P)) = 0$, whence $\mathrm{ord}_{\mathfrak{p}}(\Theta_{E,\mathfrak{a}}(Q)) = 0$ (since $n = 0$). If $\mathfrak{b} = \mathfrak{p}^m$, then since $\mathrm{ord}_{\mathfrak{p}}(x(P)) \geq 0$, we use (2.7) to obtain

$$\mathrm{ord}_{\mathfrak{p}}(\Theta_{E,\mathfrak{a}}(Q)) = \frac{2(N(\mathfrak{a}) - 1)}{N(\mathfrak{p})^m - N(\mathfrak{p})^{m-1}} > 0.$$

Hence $\Theta_{E,\mathfrak{a}}(Q)$ is only a $\mathfrak{p}$-unit in this case. □

2.3. **The distribution relation.**

Theorem 2.4. *Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ prime to 6, and let $\mathfrak{b} = \beta\mathcal{O}_K$ be an ideal prime to $\mathfrak{a}$. For all $Q \in E(\bar{K})$, we have*

$$\prod_{R \in E[\mathfrak{b}]} \Theta_{E,\mathfrak{a}}(Q + R) = \Theta_{E,\mathfrak{a}}([\beta](Q)).$$

Proof. Let $F := \prod_{R \in E[\mathfrak{b}]} \Theta_{E,\mathfrak{a}} \circ t_R$ and $G := \Theta_{E,\mathfrak{a}} \circ [\beta]$. We begin by observing that

$$\mathrm{div}(\Theta_{E,\mathfrak{a}}) = -6 \sum_{P \in E[\mathfrak{a}] \setminus 0} ([P] + [-P] - 2[O]) = 12N(\mathfrak{a})[O] - 12 \sum_{P \in E[\mathfrak{a}]} [P].$$

Using this, we deduce that

$$\mathrm{div}(F) = \sum_{R \in E[\mathfrak{b}]} (12N(\mathfrak{a})[R] - 12 \sum_{P \in E[\mathfrak{a}]} [P + R]) = 12N(\mathfrak{a}) \sum_{R \in E[\mathfrak{b}]} [R] - 12 \sum_{S \in E[\mathfrak{a}\mathfrak{b}]} [S].$$

On the other hand, note that for $P \in E[\mathfrak{a}] \setminus 0$, the function $x([\beta]Q) - x(P)$ in the variable Q has poles at points Q for which $[\beta]Q = 0$, i.e., at $Q \in E[\beta]$. The zeros are at points Q such that $[\beta]Q = P \in E[\mathfrak{a}] \setminus 0$. In particular, $Q \in E[\mathfrak{a}\mathfrak{b}] \setminus E[\mathfrak{b}]$. As P ranges over $E[\mathfrak{a}] \setminus 0$, Q ranges over $E[\mathfrak{a}\mathfrak{b}] \setminus E[\mathfrak{b}]$. We deduce that

$$\mathrm{div}(G) = 12(N(\mathfrak{a}) - 1) \sum_{R \in E[\mathfrak{b}]} [R] - 12 \sum_{S \in E[\mathfrak{a}\mathfrak{b}] \setminus E[\mathfrak{b}]} [S] = \mathrm{div}(F).$$

As a consequence, F/G is a constant λ. We compare Laurent expansions at $s = 0$, or equivalently we take the limit of $F(Q)/G(Q)$ as $Q \to O$. We have

$$\frac{F(Q)}{G(Q)} = \alpha^{-12(N(\mathfrak{b})-1)} \Delta(E)^{(N(\mathfrak{a})-1)(N(\mathfrak{b})-1)} \prod_{P \in E[\mathfrak{a}] \setminus 0} \left(\frac{x(Q) - x(P)}{x([\beta]Q) - x(P)} \right)^{-6} \prod_{R \in E[\mathfrak{b}] \setminus 0} (x(Q+R) - x(P))^{-6}.$$

We have

$$\lim_{Q \to O} \frac{x(Q) - x(P)}{x([\beta]Q) - x(P)} = \beta^2,$$

as can be seen using the complex uniformization via the Weierstrass $\wp$-function (see §3.1). We deduce that

$$\lambda = \frac{\Delta(E)^{(N(\mathfrak{a})-1)(N(\mathfrak{b})-1)}}{\alpha^{12(N(\mathfrak{b})-1)}\beta^{12(N(\mathfrak{a})-1)}} \prod_{\substack{P\in E[\mathfrak{a}]\setminus 0\\ R\in E[\mathfrak{b}]\setminus 0}} (x(R)-x(P))^{-6}.$$

Note that $x(P)=x(-P)$. Since $2\nmid\mathfrak{a}$, we have $P\neq -P$. We thus obtain

$$\lambda = \frac{\Delta(E)^{(N(\mathfrak{a})-1)(N(\mathfrak{b})-1)}}{\alpha^{12(N(\mathfrak{b})-1)}\beta^{12(N(\mathfrak{a})-1)}} \prod_{\substack{P\in (E[\mathfrak{a}]\setminus 0)/\pm 1\\ R\in E[\mathfrak{b}]\setminus 0}} (x(R)-x(P))^{-12}.$$

Let $u_K := \#\mathcal{O}_K$. Then $u_K \mid 12$ and $u_K \mid N(\mathfrak{a})-1$. To see the latter, assume first that $u_K=2$. Since $2\nmid N(\mathfrak{a})$, $N(\mathfrak{a})-1$ is even. If $u_K=4$, then we are in the case $K=\mathbb{Q}(i)$ and $N(\mathfrak{a})=a^2+b^2$ with $\mathfrak{a}=(a+bi)$. Thus, $N(\mathfrak{a})\equiv 0,1,2 \pmod 4$. Since $2\nmid N(\mathfrak{a})$, we must have $N(\mathfrak{a})\equiv 1 \pmod 4$, whence $u_K\mid N(\mathfrak{a})-1$. If $u_K=6$, then we are in the case $K=\mathbb{Q}(\sqrt{-3})$ and $\mathcal{O}_K=\mathbb{Z}[(1+\sqrt{-3})/2]$. If $\mathfrak{a}=(a+b(1+\sqrt{-3})/2)$, then $N(\mathfrak{a})=(a+b/2)^2+3b^2/4=a^2+ab+b^2\equiv 0,1 \pmod 3$. But $3\nmid N(\mathfrak{a})$ so $3\mid N(\mathfrak{a})-1$. Since $2\nmid N(\mathfrak{a})$, we obtain $u_K\mid N(\mathfrak{a})-1$. In any case, we may write

$$\lambda = \left(\frac{\Delta(E)^{(N(\mathfrak{a})-1)(N(\mathfrak{b})-1)/u_K}}{\alpha^{12(N(\mathfrak{b})-1)/u_K}\beta^{12(N(\mathfrak{a})-1)/u_K}} \prod_{\substack{P\in (E[\mathfrak{a}]\setminus 0)/\pm 1\\ R\in E[\mathfrak{b}]\setminus 0}} (x(R)-x(P))^{-12/u_K} \right)^{u_K}.$$

Using the same method as in the proof of Theorem 2.3, one can show that the quantity in brackets is a unit in $\mathcal{O}_K$. Hence $\lambda=1$. □

Corollary 2.5. *Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ prime to 6, and let $\mathfrak{b}$ be an ideal prime to $\mathfrak{a}$. Let $\mathfrak{p}=(\pi)$ be a prime ideal dividing $\mathfrak{b}$ and write $\mathfrak{b}=\mathfrak{p}\mathfrak{b}'$. Assume that the map $\mathcal{O}_K^\times \longrightarrow (\mathcal{O}_K/\mathfrak{b}')^\times$ is injective. Given a proper $\mathfrak{b}$-section $Q\in E[\mathfrak{b}]$, we have*

$$N^{K(\mathfrak{b})}_{K(\mathfrak{b}')}\Theta_{E,\mathfrak{a}}(Q) = \begin{cases} \Theta_{E,\mathfrak{a}}(\pi Q), & \mathfrak{p}\mid\mathfrak{b}' \\ \Theta_{E,\mathfrak{a}}(\pi Q)^{1-\mathrm{Fr}_\mathfrak{p}^{-1}}, & \mathfrak{p}\nmid\mathfrak{b}'. \end{cases}$$

Proof. We write $\mathfrak{b}=(\beta)$, $\mathfrak{b}'=(\beta')$, and $\mathfrak{p}=(\pi)$. By (2.1), the Artin map of class field theory yields an isomorphism

$$I_K(\mathfrak{b})/P_{K,1}(\mathfrak{b}) \simeq \mathrm{Gal}(K(\mathfrak{b})/K), \qquad \mathfrak{c}\mapsto\sigma_\mathfrak{c}.$$

Using the fact that $h_K=1$, we see that the left hand side is isomorphic to $(\mathcal{O}_K/\mathfrak{b})^\times/\mathcal{O}_K^\times$. The same is true with $\mathfrak{b}$ replaced by $\mathfrak{b}'$. The Galois group $\mathrm{Gal}(K(\mathfrak{b})/K(\mathfrak{b}'))$ is thus seen to be isomorphic to the kernel of the map

$$(\mathcal{O}_K/\mathfrak{b})^\times/\mathcal{O}_K^\times \longrightarrow (\mathcal{O}_K/\mathfrak{b}')^\times/\mathcal{O}_K^\times. \tag{2.8}$$

The kernel of (2.8) is thus given by $C := 1+\mathfrak{b}'(\mathcal{O}_K/\mathfrak{b}) \hookrightarrow (\mathcal{O}_K/\mathfrak{b})^\times/\mathcal{O}_K^\times$ (the latter map is an injection since its kernel is $(1+\mathfrak{b}'(\mathcal{O}_K/\mathfrak{b}))\cap\mathcal{O}_K^\times=\{1\}$ since the image of $1+\mathfrak{b}'(\mathcal{O}_K/\mathfrak{b})$ in $(\mathcal{O}_K/\mathfrak{b}')^\times$ is 1 and $\mathcal{O}_K^\times\hookrightarrow(\mathcal{O}_K/\mathfrak{b}')^\times$ by assumption). If $c\in C$, we let $\mathfrak{c}=(c)$. Using Proposition 2.2, we see that

$$N^{K(\mathfrak{b})}_{K(\mathfrak{b}')}\Theta_{E,\mathfrak{a}}(Q) = \prod_{\sigma\in\mathrm{Gal}(K(\mathfrak{b})/K(\mathfrak{b}'))} \sigma(\Theta_{E,\mathfrak{a}}(Q)) = \prod_{c\in C}\sigma_\mathfrak{c}(\Theta_{E,\mathfrak{a}}(Q)) = \prod_{c\in C}\Theta_{E,\mathfrak{a}}([c]Q).$$

Observe that $[c]Q\notin E[\mathfrak{b}']$. Indeed, otherwise $\Theta_{E,\mathfrak{a}}(Q)=\sigma_\mathfrak{c}^{-1}(\Theta_{E,\mathfrak{a}}([c]Q))=\Theta_{E,\mathfrak{a}}([c]Q)$ is defined over $K(\mathfrak{b}')$, contradicting CM theory. If $c\in C$, we write $c=1+\beta'x$ with $x\in\mathcal{O}_K/\mathfrak{b}$. Then $[c]Q=Q+[\beta']xQ$. Observe that $[\beta']xQ\in E[\mathfrak{p}]$. If $\mathfrak{p}\mid\mathfrak{b}'$, then $[\beta']xQ\in E[\mathfrak{b}']$ and $[\beta']([c]Q)=[\beta']Q\neq 0$. Hence, the condition $[c]Q\notin E[\mathfrak{b}']$ is automatically satisfied. However, when $\mathfrak{p}\nmid\mathfrak{b}'$, we have $[\beta']xQ \notin E[\mathfrak{b}']$. Let $S \in E[\mathfrak{b}']$ be a proper $\mathfrak{b}'$-section. Observe that $[\pi]S$ is again a proper $\mathfrak{b}'$-section and that $[\pi]Q \in E[\mathfrak{b}']$. Thus, there exists a unique $x_0 \in \mathcal{O}_K/\mathfrak{b}'$ such that $[\pi]x_0S = [\pi]Q$. But then $S_0 := x_0S \in E[\mathfrak{b}']$ such that $[\pi]S_0 = [\pi]Q$. Let $R_0 := S_0 - Q$. Then $Q + R_0 = S_0 \in E[\mathfrak{b}']$ and $R_0 \in E[\mathfrak{p}]$ by construction. In conclusion, in the case $\mathfrak{p} \nmid \mathfrak{b}'$, the condition $[c]Q \notin E[\mathfrak{b}']$ is not automatically satisfied and we must exclude the case $c = 1 + \beta'x$ with $x = x_0$. We deduce that

$$N^{K(\mathfrak{b})}_{K(\mathfrak{b}')}\Theta_{E,\mathfrak{a}}(Q) = \begin{cases} \prod_{x\in\mathcal{O}_K/\mathfrak{b}} \Theta_{E,\mathfrak{a}}(Q + [\beta']xQ) = \prod_{R\in E[\mathfrak{p}]} \Theta_{E,\mathfrak{a}}(Q + R), & \mathfrak{p} \mid \mathfrak{b}' \\ \prod_{\substack{x\in\mathcal{O}_K/\mathfrak{b} \\ x\neq x_0}} \Theta_{E,\mathfrak{a}}(Q + [\beta']xQ) = \prod_{\substack{x\in R \\ R\neq R_0}} \Theta_{E,\mathfrak{a}}(Q + R), & \mathfrak{p} \nmid \mathfrak{b}'. \end{cases}$$

Since $\mathfrak{p} \mid \mathfrak{b}$ and $(\mathfrak{a}, \mathfrak{b}) = 1$, we can apply Theorem 2.4 to obtain

$$\prod_{R\in E[\mathfrak{p}]} \Theta_{E,\mathfrak{a}}(Q + R) = \Theta_{E,\mathfrak{a}}([\pi]Q).$$

This concludes the proof in the case $\mathfrak{p} \mid \mathfrak{b}'$. When $\mathfrak{p} \nmid \mathfrak{b}'$, we get

$$\Theta_{E,\mathfrak{a}}(Q + R_0)N^{K(\mathfrak{b})}_{K(\mathfrak{b}')}\Theta_{E,\mathfrak{a}}(Q) = \Theta_{E,\mathfrak{a}}([\pi]Q).$$

To finish the proof, we observe using Proposition 2.2 (since $Q + R_0 \in E[\mathfrak{b}']$ and $(\mathfrak{b}', \mathfrak{p}) = 1$) that

$$\Theta_{E,\mathfrak{a}}(Q + R_0)^{\mathrm{Fr}_{\mathfrak{p}}} = \sigma_{\mathfrak{p}}(\Theta_{E,\mathfrak{a}}(Q + R_0)) = \Theta_{E,\mathfrak{a}}([\pi](Q + R_0)) = \Theta_{E,\mathfrak{a}}([\pi]Q).$$

□

3. Analytic theory of elliptic units

We use the complex uniformization of elliptic curves provided by the Weierstrass $\wp$-function to give a description of elliptic units using theta functions. We then link these analytic elliptic units to special values of Hecke L-functions, giving a proof of Damerell's algebraicity result.

3.1. **Complex uniformization.** Let E be an elliptic curve given by a Weierstrass equation

$$E\colon y^2 = 4x^3 - Ax - B, \qquad \Delta(E) = A^3 - 27B^2 \neq 0, \qquad A, B \in \mathbb{C}. \tag{3.1}$$

The standard differential on E is $\omega_E := dx/y$. The period lattice

$$L_E := \left\{ \int_\gamma \omega_E \;\middle|\; [\gamma] \in H_1(E(\mathbb{C}), \mathbb{Z}) \right\}$$

is then a lattice in $\mathbb{C}$ and there is an isomorphism

$$E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/L_E, \qquad P \longrightarrow \int_O^P \omega_E.$$

The inverse of this map is given by

$$\xi\colon \mathbb{C}/L_E \xrightarrow{\sim} E(\mathbb{C}), \qquad z \longrightarrow (\wp(z; L_E), \wp'(z; L_E)),$$

where for any lattice $L \subset \mathbb{C}$

$$\wp(z; L) := \frac{1}{z^2} + \sum_{0\neq\lambda\in L} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

In fact, the Weierstrass $\wp$-function satisfies the equation

$$\wp'(z; L)^2 = 4\wp(z; L)^3 - 60G_4(L)\wp(z; L) - 140G_6(L), \tag{3.2}$$

where $G_k(L) := \sum_{0\neq\lambda\in L} \lambda^{-k}$ for $k \geq 4$ even. The fact that ξ is the inverse map then translates to $A = 60G_4(L_E)$ and $B = 140G_6(L_E)$.

3.2. **Special functions.** Let $L \subset \mathbb{C}$ be a lattice. Define the (non-periodic) Weierstrass σ-function by

$$\sigma(z;L) := z \prod_{0\neq\lambda\in L} \left(1-\frac{z}{\lambda}\right) e^{\frac{z}{\lambda}+\frac{1}{2}\left(\frac{z}{\lambda}\right)^2}.$$

Applying the logarithmic derivative yields the Weierstrass zeta function

$$\zeta(z;L) := \frac{d}{dz}\log(\sigma(z;L)) = \frac{1}{z} + \sum_{0\neq\lambda\in L}\left(\frac{1}{z-\lambda}+\frac{1}{\lambda}+\frac{z}{\lambda^2}\right).$$

It is non-periodic and has a single simple pole and no zeros. The derivative of the zeta function is

$$\zeta'(z;L) = -\wp(z;L),$$

which is periodic. Choose an oriented basis $L = w_1\mathbb{Z}\oplus w_2\mathbb{Z}$ such that $\tau = w_1/w_2 \in \mathcal{H}$. The discriminant of (3.2) is

$$\Delta(L) = (60G_4(L))^3 - 27(140G_6(L))^2 = \left(\frac{2\pi i}{w_2}\right)^{12} q\prod_{n\geq 1}(1-q^n)^{24}, \qquad q = e^{2\pi i\tau}.$$

We also define

$$\begin{cases} G_2(L) := \lim\limits_{s\to 0^+}\sum_{0\neq\lambda\in L}\frac{1}{\lambda^2|\lambda|^{2s}}, \\ A(L) := \mathrm{Vol}(\mathbb{C}/L)/\pi, \\ \eta(z;L) := G_2(L)z - A(L)^{-1}\bar{z}. \end{cases}$$

We point out the transformation law

$$\zeta(z+\lambda;L) = \zeta(z;L) + \eta(\lambda;L), \qquad \forall z\in\mathbb{C}, \lambda\in L.$$

Finally, we define the fundamental theta function by

$$\theta(z;L) := \Delta(L)e^{-6\eta(z;L)z}\sigma(z;L)^{12}.$$

Note that it is a non-holomorphic function.

3.3. **Analytic elliptic units.** Let K be an imaginary quadratic field with class number 1 and let E be an elliptic curve over $\mathbb{C}$ with CM by $\mathcal{O}_K$. Let $L = L_E$ denote the period lattice of E and recall the complex uniformization $\xi\colon \mathbb{C}/L \simeq E(\mathbb{C})$. Because E has CM by $\mathcal{O}_K$, the lattice L is homothetic to a K-fractional ideal, and since the class number is 1, there exists $\Omega \in \mathbb{C}^\times$ such that $L = \Omega\mathcal{O}_K$. Given an ideal $\mathfrak{a} = (\alpha)$ of $\mathcal{O}_K$ prime to 6, we define

$$\Theta_{L,\mathfrak{a}}(z) := \Theta_{E,\mathfrak{a}}(\xi(z)) = \alpha^{-12}\Delta(L)^{N(\mathfrak{a})-1}\prod_{w\in\mathfrak{a}^{-1}L/L\setminus 0}\left(\frac{1}{\wp(z;L)-\wp(w;L)}\right)^{-6}.$$

Proposition 3.1. *We have*

$$\Theta_{L,\mathfrak{a}}(z) = \frac{\theta(z;L)^{N(\mathfrak{a})}}{\theta(z;\mathfrak{a}^{-1}L)}.$$

Proof. Using properties of the fundamental theta function, the function $f(z) := \theta(z;L)^{N(\mathfrak{a})}/\theta(z;\mathfrak{a}^{-1}L)$ is holomorphic and elliptic. Its divisor is

$$\mathrm{div}(f) = 12N(\mathfrak{a})[0] - 12\sum_{w\in\mathfrak{a}^{-1}L/L}[w] = \mathrm{div}(\Theta_{L,\mathfrak{a}}).$$

Hence, the two functions differ by a constant, which can be determined by analyzing the Taylor series expansions of both sides at $z = 0$. These are both given by

$$\alpha^{-12}\Delta(L)^{N(\mathfrak{a})-1}z^{12(N(\mathfrak{a})-1)}(1+O(z)).$$

□

Definition 3.2. For $k \geq 1$, define the Eisenstein series

$$E_k(z;L) := \lim_{s\to 0} \sum_{\lambda\in L} \frac{1}{(z-\lambda)^k |z-\lambda|^{2s}}.$$

Proposition 3.3. *We have*

$$\begin{aligned} E_1(z;L) &= \zeta(z;L) - \eta(z;L) \\ E_2(z;L) &= -E_1(z;L)' = \wp(z;L) + G_2(L) \\ E_k(z;L) &= \frac{-E_{k-1}(z;L)'}{k-1} = \frac{(-1)^k}{(k-1)!}\left(\frac{d}{dz}\right)^{k-2} \wp(z;L), \quad k \geq 3. \end{aligned}$$

Proposition 3.4. *For all $k \geq 1$, we have*

$$\left(\frac{d}{dz}\right)^k \log(\Theta_{L,\mathfrak{a}}(z)) = 12(-1)^{k-1}(k-1)!(N(\mathfrak{a})E_k(z;L) - E_k(z;\mathfrak{a}^{-1}L)).$$

Proof. Using Proposition 3.1, we see that

$$\log(\Theta_{L,\mathfrak{a}}(z)) = N(\mathfrak{a})\log(\theta(z;L)) - \log(\theta(z;\mathfrak{a}^{-1}L)).$$

Recall that $\theta(z;L) := \Delta(L)e^{-6\eta(z;L)z}\sigma(z;L)^{12}$ and thus

$$\log(\theta(z;L)) = \log(\Delta(L)) - 6\eta(z;L)z + 12\log(\sigma(z;L)).$$

Taking the derivative yields

$$\log(\theta(z;L))' = -6G_2(L)z - 6\eta(z;L) + 12\zeta(z;L) = 12E_1(z;L) + 6A(L)^{-1}\bar{z}.$$

We deduce that

$$\begin{aligned} \log(\Theta_{L,\mathfrak{a}}(z)) &= 12(N(\mathfrak{a})E_1(z;L) - E_1(z;\mathfrak{a}^{-1}L)) + 6(N(\mathfrak{a})A(L)^{-1} - A(\mathfrak{a}^{-1}L)^{-1})\bar{z} \\ &= 12(N(\mathfrak{a})E_1(z;L) - E_1(z;\mathfrak{a}^{-1}L)), \end{aligned}$$

since $A(\mathfrak{a}^{-1}L) = N(\mathfrak{a})^{-1}A(L)$. The result for $k \geq 2$ follows from Proposition 3.3. □

3.4. **Hecke L-functions.** Let ψ be the Hecke character of K with values in $K^\times$ attached to E by CM theory. Let $\mathfrak{f} = \mathrm{cond}(\psi) = \mathrm{cond}(E)$. The Hecke L-function associated to powers of $\bar{\psi}$ is

$$L(\bar{\psi}^k, s) := \sum_{\mathfrak{b}} \frac{\bar{\psi}^k(\mathfrak{b})}{N(\mathfrak{b})^s},$$

where the sum is taken over integral ideals of $\mathcal{O}_K$ prime to the conductor of $\bar{\psi}^k$. If $\mathfrak{m}$ is an ideal divisible by $\mathfrak{f}$ and $\mathfrak{c}$ is prime to $\mathfrak{m}$, we also define

$$L_{\mathfrak{m}}(\bar{\psi}^k, s, \mathfrak{c}) := \sum_{\substack{(\mathfrak{b},\mathfrak{m})=1 \\ \sigma_{\mathfrak{b}}=\sigma_{\mathfrak{c}}}} \frac{\bar{\psi}^k(\mathfrak{b})}{N(\mathfrak{b})^s},$$

where the equality of automorphisms takes place in $\mathrm{Gal}(K(\mathfrak{m})/K)$.

Proposition 3.5. *Let $v \in KL/L$ be a point of order $\mathfrak{m}$, where $\mathfrak{m}$ is an ideal divisible by $\mathfrak{f}$. Recall that $L = \Omega\mathcal{O}_K$ and let $\mathfrak{c} := \Omega^{-1}v\mathfrak{m}$. Then for all $k \geq 1$, we have*

$$E_k(v;L) = v^{-k}\psi(\mathfrak{c})^k L_{\mathfrak{m}}(\bar{\psi}^k, k, \mathfrak{c}).$$

Proof. Let $\mathfrak{m} = (\mu)$ and write $v = c\Omega/\mu + L$ for some $c \in K$ with $((c), \mathfrak{m}) = 1$. Note that $c = \Omega^{-1}v\mu$, whence $(c) = \mathfrak{c}$. Writing

$$E_k(z; L) = \lim_{s\to k} \sum_{\lambda\in L} \frac{1}{(z-\lambda)^k |z-\lambda|^{2s-2k}} = \lim_{s\to k} \sum_{\lambda\in L} \frac{(\bar{z}-\bar{\lambda})^k}{|z-\lambda|^{2s}},$$

we see that

$$E_k(v; L) = \lim_{s\to k} \frac{N\mu^s}{\bar{\mu}^k} \frac{\bar{\Omega}^k}{|\Omega|^{2s}} \sum_{\substack{\beta\in\mathcal{O}_K \\ \beta\equiv c \pmod{\mathfrak{m}}}} \frac{\bar{\beta}^k}{|\beta|^{2s}}.$$

Since $\psi(\mathfrak{b})\mathcal{O}_K = (\beta)$, we see that $\epsilon(\beta) := \psi((\beta))/\beta \in \mathcal{O}_K^\times$. This defines a homomorphism $\epsilon\colon (\mathcal{O}_K/\mathfrak{f})^\times \longrightarrow \mathcal{O}_K^\times$. Thus, if $\beta \equiv c \pmod{\mathfrak{m}}$, then in particular $\beta \equiv c \pmod{\mathfrak{f}}$ and thus $\epsilon(\beta) = \epsilon(c)$, which implies that

$$\bar{\beta} = \frac{\psi(\bar{\beta}\mathfrak{c})}{c}.$$

It follows that

$$E_k(v; L) = \lim_{s\to k} \frac{N\mu^s}{\bar{\mu}^k} \frac{\bar{\Omega}^k}{|\Omega|^{2s}} \frac{\psi(\mathfrak{c})^k}{c^k} \sum_{\substack{(\mathfrak{b},\mathfrak{m})=1 \\ \sigma_{\mathfrak{b}}=\sigma_{\mathfrak{c}}}} \frac{\bar{\psi}^k(\mathfrak{b})}{N(\mathfrak{b})^s} = \lim_{s\to k} \left(\frac{N(\mu)}{N(\Omega)}\right)^{s-k} v^{-k}\psi(\mathfrak{c})^k L_{\mathfrak{m}}(\bar{\psi}^k, s, \mathfrak{c}).$$

The result follows by taking the limit. □

Definition 3.6. Let $\mathfrak{f} = (f)$ and fix a set B of representatives of $I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f}) = (\mathcal{O}_K/\mathfrak{f})^\times/\mathcal{O}_K^\times$ so that $\mathrm{Gal}(K(\mathfrak{f})/K) = \{\sigma_{\mathfrak{b}} \mid \mathfrak{b} \in B\}$. Let $\mathfrak{a}$ be an ideal prime to $6\mathfrak{f}$. Let $u = \Omega/f + L \in \mathfrak{f}^{-1}L/L$ (a point of order $\mathfrak{f}$) and define

$$\Lambda_{L,\mathfrak{a},f}(z) := \prod_{\mathfrak{b}\in B} \Theta_{L,\mathfrak{a}}(\psi(\mathfrak{b})u + z).$$

Theorem 3.7. *For all $k \geq 1$, we have*

$$\left(\frac{d}{dz}\right)^k \log(\Lambda_{L,\mathfrak{a},f}(z))\bigg|_{z=0} = 12(-1)^{k-1}(k-1)!f^k(N(\mathfrak{a}) - \psi(\mathfrak{a})^k)\Omega^{-k}L_{\mathfrak{f}}(\bar{\psi}^k, k).$$

Proof. We have

$$\begin{aligned}\left(\frac{d}{dz}\right)^k \log(\Lambda_{L,\mathfrak{a},f}(z))\bigg|_{z=0} &= \sum_{\mathfrak{b}\in B} \left(\frac{d}{dz}\right)^k \log(\Theta_{L,\mathfrak{a}}(z))\bigg|_{z=\psi(\mathfrak{b})u} \\ &= 12(-1)^{k-1}(k-1)! \sum_{\mathfrak{b}\in B} (N(\mathfrak{a})E_k(\psi(\mathfrak{b})u; L) - E_k(\psi(\mathfrak{b})u; \mathfrak{a}^{-1}L)).\end{aligned}$$

Recall that $u \in KL/L$ is a point of order $\mathfrak{f}$. Since $(\mathfrak{b}, \mathfrak{f}) = 1$, the point $\psi(\mathfrak{b})u$ is again a point of order $\mathfrak{f}$. Note that $\Omega^{-1}\psi(\mathfrak{b})u\mathfrak{f} = \mathfrak{b}$. We apply Proposition 3.5 to obtain

$$E_k(\psi(\mathfrak{b})u; L) = (\psi(\mathfrak{b})u)^{-k}\psi(\mathfrak{b})^k L_{\mathfrak{m}}(\bar{\psi}^k, k, \mathfrak{b}) = u^{-k}L_{\mathfrak{f}}(\bar{\psi}^k, k, \mathfrak{b}),$$

and

$$\sum_{\mathfrak{b}\in B} E_k(\psi(\mathfrak{b})u; L) = u^{-k}L_{\mathfrak{f}}(\bar{\psi}^k, k).$$

Observe that

$$E_k(z; \mathfrak{a}^{-1}L) = \lim_{s\to 0} \sum_{\lambda\in L} \frac{1}{(z-\psi(\mathfrak{a})^{-1}\lambda)^k |z-\psi(\mathfrak{a})^{-1}\lambda|^{2s}} = \psi(\mathfrak{a})^k E_k(\psi(\mathfrak{a})z; L).$$

Hence,

$$E_k(\psi(\mathfrak{b})u; \mathfrak{a}^{-1}L) = \psi(\mathfrak{a})^k E_k(\psi(\mathfrak{a}\mathfrak{b})u; L) = \psi(\mathfrak{a})^k u^{-k} L_{\mathfrak{f}}(\bar{\psi}^k, k, \mathfrak{a}\mathfrak{b}),$$

and

$$\sum_{\mathfrak{b}\in B} E_k(\psi(\mathfrak{b})u;\mathfrak{a}^{-1}L) = \psi(\mathfrak{a})^k u^{-k} L_{\mathfrak{f}}(\bar{\psi}^k, k).$$

The result follows by noting that $u^{-k} = f^k\Omega^{-k}$. □

Corollary 3.8 (Damerell). *For every $k \geq 1$,*

$$\frac{L(\bar{\psi}^k, k)}{\Omega} \in K.$$

Proof. We know that $\Lambda_{L,\mathfrak{a},f}(z) \in K(\wp(z;L), \wp(z;L)')$. Using the Weierstrass equation (3.2) satisfied by $\wp(z;L)$ and $\wp(z;L)'$ with coefficients in K, we see that $\left(\frac{d}{dz}\right)^k \wp(z;L) \in K(\wp(z;L), \wp(z;L)')$ for all $k \geq 2$. The result then follows from Theorem 3.7. □

Remark 3.9. By a classical result of Deuring, we have $L(E/K,s) = L(\psi,s)L(\bar{\psi},s)$. In particular, $L(E/K,1) \neq 0$ implies $L(\bar{\psi},1) \neq 0$. Theorem 3.7 for $k=1$ then translates this information into a statement of non-triviality for elliptic units. More precisely, under this non-vanishing assumption, the so-called Euler system of elliptic units has non-trivial bottom class. The machinery of Euler systems then allows one to bound the torsion in class groups of abelian extensions of K. The latter are related to Selmer groups of E by explicit descent, resulting ultimately in bounds on ranks of Mordell–Weil groups. This is the strategy employed by Coates and Wiles [1] to prove BSD for CM elliptic curves in analytic rank zero (in the class number one case), namely they prove that

$$L(E/K,1) \neq 0 \implies \#E(K) < 0.$$

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