ERRATUM TO "GENERALISED HEEGNER CYCLES AND THE COMPLEX ABEL–JACOBI MAP"

Abstract. This is an erratum to the article [M. Bertolini, H. Darmon, D. Lilienfeldt, and K. Prasanna, *Generalised Heegner cycles and the complex Abel–Jacobi map*, Math. Z., **298**, 385–418(2021)]. Only Section 10 is concerned by these corrections. The main result (Theorem 2) of Section 10 remains true, but the proof requires some minor fixes.

Overview

In the proof of Lemma 4, the formula of Theorem 1 is applied with respect to the isogenies

$$[pq]\circ\varphi_{p,q,\infty}:\mathbf{C}/\mathcal{O}_K\longrightarrow\mathbf{C}/\Lambda_{p,q,\infty}\longrightarrow\mathbf{C}/\langle 1,\tau_{p,q,\infty}\rangle=\mathbf{C}/\langle 1,pq\tau\rangle,$$
$$[p]\circ\varphi_{p,q,\beta}:\mathbf{C}/\mathcal{O}_K\longrightarrow\mathbf{C}/\Lambda_{p,q,\beta}\longrightarrow\mathbf{C}/\langle 1,\tau_{p,q,\beta}\rangle=\mathbf{C}/\langle 1,(\tau+\beta)p/q\rangle,\qquad 0\le\beta\le q-1.$$

However, these are not isogenies of marked elliptic curves with $\Gamma_1(N)$-level structure between (A,t,ω_A) and $(\mathbf{C}/\langle 1,\tau_{p,q,\beta}\rangle, 1/N+\langle 1,\tau_{p,q,\beta}\rangle, 2\pi i dw)$, as required by the assumptions of Theorem 1. This is the main reason for this erratum. Below is a way to fix this. Along the way, we also indicate other minor fixes and typos.

A. Changes to Section 10.1

A.1. **Explicit isogenies.** When defining $(\varphi_{p,q,\beta},A_{p,q,\beta})\in\mathrm{Isog}^{\mathfrak{N}}(A)$, the point $\tau:=(-d_K+\sqrt{-d_K})/2\in\mathcal{H}$ should be replaced as follows. Recall that $t\in A[\mathfrak{N}]$ is the fixed $\Gamma_1(N)$-level structure. We must have $t=(c\tau+d)/N+\langle 1,\tau\rangle$ for some integers $c,d\in\mathbf{Z}$ with $\gcd(c,d,N)=1$ and $N\nmid c$. The condition $N\nmid c$ is necessary, since otherwise we have $A[\mathfrak{N}]=\langle 1/N+\langle 1,\tau\rangle\rangle$, implying that $A/A[\mathfrak{N}](\mathbf{C})\simeq\mathbf{C}/\langle 1,N\tau\rangle$ has CM by $\mathcal{O}_N$, when it should have CM by $\mathcal{O}_K$. Let $a,b,k\in\mathbf{Z}$ such that $ad-bc-kN=1$ (possible by the gcd condition). Then $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in M_2(\mathbf{Z})$ reduces into $\mathbf{SL}_2(\mathbf{Z}/N\mathbf{Z})$. By modifying $a,b,c,$ and d modulo N if necessary, we may and will assume that $\gamma=\gamma_t:=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathbf{SL}_2(\mathbf{Z})$. Multiplication by $c\tau+d$ then yields and isomorphism of elliptic curves with $\Gamma_1(N)$-level structure

$$[c\tau+d]:(\mathbf{C}/\langle 1,\gamma(\tau)\rangle, 1/N+\langle 1,\gamma(\tau)\rangle)\simeq(A,t).$$

In other words, the point $(A,t)\in X_1(N)(\mathbf{C})=\Gamma_1(N)\setminus\mathcal{H}$ is represented by $\Gamma_1(N)\gamma(\tau)$.

Assumption 1. *Throughout Section* 10*, the extra condition on p and q that they do not divide $c|c\tau+d|^2$ should be imposed.*

In the definition of $\Lambda_{p,q,\beta}$ in Section 10.1, replace τ by $\gamma(\tau)$:

$$\Lambda_{p,q,\infty}:=\mathbf{Z}\frac{1}{pq}\oplus\mathbf{Z}\gamma(\tau),\qquad \Lambda_{p,q,\beta}:=\mathbf{Z}\frac{1}{p}\oplus\mathbf{Z}\frac{\gamma(\tau)+\beta}{q},\ \text{for } 0\le\beta\le q-1.$$

Define, for $\beta\in\mathbf{P}_1(\mathbb{F}_q)$, the isogeny

$$\varphi_{p,q,\beta}:\mathbf{C}/\mathcal{O}_K\overset{[(c\tau+d)^{-1}]}{\longrightarrow}\mathbf{C}/\langle 1,\gamma(\tau)\rangle\overset{\text{quot}}{\longrightarrow}\mathbf{C}/\Lambda_{p,q,\beta},$$

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where the last isogeny is the natural quotient isogeny arising from the inclusion of lattices $\langle 1, \gamma(\tau)\rangle \subset \Lambda_{p,q,\beta}$. Redefine $\Delta_{p,q,\beta} := \Delta_{\varphi_{p,q,\beta}} \in \mathrm{CH}^{r+1}(X_r)_0(\bar{\mathbf{Q}})$ with respect to these modified isogenies.

A.2. **Proof of Proposition 7.** In the proof of Proposition 7, it is stated that the elliptic curves $A_{p,q,\beta}$ with $\beta \in \mathbf{P}_1(\mathbb{F}_q)$ have CM by $\mathcal{O}_{pq}$. This is indeed the case if the prime q is inert in K. If q splits in K, then there are exactly two choices of $\beta \in \mathbb{F}_q \subset \mathbf{P}_1(\mathbb{F}_q)$ for which the elliptic curves $A_{q,\beta}$ (defined on page 412 line 7) have CM by $\mathcal{O}_K$. These two choices correspond to the canonical isogenies $A \longrightarrow A/A[\mathfrak{q}]$ and $A \longrightarrow A/A[\bar{\mathfrak{q}}]$, where $q\mathcal{O}_K = \mathfrak{q}\bar{\mathfrak{q}}$ for some prime ideal $\mathfrak{q}$ of $\mathcal{O}_K$. It follows that for these two choices of β, the elliptic curves $A_{p,q,\beta}$ have CM by $\mathcal{O}_p$. In any case, it is true that $\Delta_{p,q,\beta}$ is rational over F_{pq} no matter the splitting behaviour of q. In later sections, q is always assumed to be inert.

Remark 1. Proposition 7 remains true with the modified elliptic curves and isogenies of Section A.1. The condition that q is inert, $q \nmid c$, and $p \nmid c|c\tau + d|^2$ should be added. A detailed proof is given in Section 7.2 of [D. T.-B. G. Lilienfeldt, *Heegner cycles in Griffiths groups of Kuga-Sato varieties*, Preprint, arXiv:2107.06731v3].

B. Changes to Section 10.2

Retain the new notations introduced in Section A.1. The main theorem of Section 10.2 (Theorem 3) remains true for the redefined cycles. The assumption that $p, q \nmid |c\tau+d|^2$ should be added to the statement.

B.1. **Proof of Lemma 4.** In the statement of Lemma 4, the assumptions that $p, q \nmid |c\tau + d|^2$ should be added, and the quantity $\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta})(\omega_f \wedge \omega_A^j \eta_A^{r-j})$ should be replaced by $\mathrm{AJ}_{\mathbf{C}}(\delta_\beta \Delta_{p,q,\beta})(\omega_f \wedge \omega_A^j \eta_A^{r-j})$, where

$$\delta_\beta := \begin{cases} (pq)^{2j-r} & \beta = \infty \\ p^{2j-r} & \beta \neq \infty. \end{cases}$$

We will now explain why. Let $\omega_A \in H^{1,0}(A/\mathbf{C})$ be the differential form such that

$$\xi^*(\omega_A) = (c\tau + d)^{-1} 2\pi i dw \in H^{1,0}(\mathbf{C}/\mathcal{O}_K),$$

where we recall that $\xi : \mathbf{C}/\mathcal{O}_K = A(\mathbf{C})$ is the fixed analytic identification. Recall that $\eta_A \in H^{0,1}(A/\mathbf{C})$ is the corresponding differential form determined by ω_A.

Note that $(\varphi_{p,q,\beta}, \mathbf{C}/\Lambda_{p,q,\beta}) = (\psi_{p,q,\beta}, \mathbf{C}/\langle 1, \tau_{p,q,\beta}\rangle)$ in $\mathrm{Isog}^{\mathfrak{N}}(A)$, where

$$\tau_{p,q,\infty} := pq\gamma(\tau), \qquad \tau_{p,q,\beta} := \frac{p}{q}(\gamma(\tau) + \beta), \qquad 0 \leq \beta \leq q-1,$$

and

$$\psi_{p,q,\infty} := [pq] \circ \varphi_{p,q,\infty}, \qquad \psi_{p,q,\beta} := [p] \circ \varphi_{p,q,\beta}, \qquad 0 \leq \beta \leq q-1.$$

Given p and q two distinct odd primes both congruent to 1 modulo N, and $\beta \in \mathbf{P}_1(\mathbb{F}_q)$,

$$\psi_{p,q,\infty} : (A, t, pq\omega_A) \longrightarrow (\mathbf{C}/\langle 1, \tau_{p,q,\infty}\rangle, 1/N + \langle 1, \tau_{p,q,\infty}\rangle, 2\pi i dw)$$
$$\psi_{p,q,\beta} : (A, t, p\omega_A) \longrightarrow (\mathbf{C}/\langle 1, \tau_{p,q,\beta}\rangle, 1/N + \langle 1, \tau_{p,q,\beta}\rangle, 2\pi i dw), \qquad 0 \leq \beta \leq q-1$$

are isogenies of marked elliptic curves with $\Gamma_1(N)$-level structure. The upshot is that the isogenies $\psi_{p,q,\beta}$ satisfy all assumptions necessary to apply Theorem 1.

Remark 2. There is a typo in the definition of κ_β (page 406, line 27). Instead, it should be

$$\kappa_\beta := \begin{cases} (pq)^{2j-r} & \beta = \infty \\ p^{2j-r}q^r & \beta \neq \infty. \end{cases}$$

Redefine the quantity γ_β (page 406, line 27) by

$$\gamma_\beta := (-1)^{j+1} i^{r+1} (2\pi i)^{j+1} \kappa_\beta (|c\tau + d|^2)^{r-j} (\tau - \bar{\tau})^{j-r}.$$

With the above definitions in hand, and with the above notations, Equation (64) should read

$$\mathrm{AJ}_{\mathbf{C}}(\delta_\beta \Delta_{p,q,\beta})(\omega_f \wedge \omega_A^j \eta_A^{r-j}) = \gamma_\beta \int_{Y_\beta}^{\infty} (y - Y_\beta)^j (y + Y_\beta)^{r-j} f(X_\beta + iy) dy,$$

where $\tau_{p,q,\beta} = X_\beta + iY_\beta$.

Remark 3. The appearance of δ_β is due to the fact that $\psi^*_{p,q,\beta}(2\pi i dw)$ is $pq\omega_A$ or $p\omega_A$ depending on whether $\beta = \infty$ or not, coupled with the fact that scaling ω_A by a constant $\lambda \in \mathbf{C}^\times$ results in a scaling of η_A by λ^{-1}.

The rest of the proof of Lemma 4 holds with the new definitions introduced here, but in Equation (65), $\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta})(\omega_f \wedge \omega_A^j \eta_A^{r-j})$ should be replaced by $\mathrm{AJ}_{\mathbf{C}}(\delta_\beta \Delta_{p,q,\beta})(\omega_f \wedge \omega_A^j \eta_A^{r-j})$. In conclusion, $\mathrm{AJ}_{\mathbf{C}}(\delta_\beta \Delta_{p,q,\beta})(\omega_f \wedge \omega_A^j \eta_A^{r-j})$ is asymptotically equivalent, as a function of p and q, to

$$B_\beta := \gamma_\beta e^{2\pi i X_\beta} A_\beta = \gamma_\beta e^{2\pi i X_\beta} \int_{Y_\beta}^{\infty} (y - Y_\beta)^j (y + Y_\beta)^{r-j} e^{-2\pi y} dy,$$

as p/q tends to infinity.

The positive real number A_β tends to zero exponentially as p/q tends to ∞. In fact, we have

$$A_\beta = e^{-2\pi Y_\beta} \sum_{i=0}^{j} \sum_{k=0}^{r-j} \sum_{l=0}^{k+i} (-1)^{j-i} \binom{j}{i} \binom{r-j}{k} \frac{(k+i)!}{(k+i-l)!} \frac{Y_\beta^{r-l}}{(2\pi)^{l+1}}, \tag{1}$$

and

$$Y_\beta = \begin{cases} pq|c\tau + d|^{-2}\sqrt{d_K}/2 & \beta = \infty \\ (p/q)|c\tau + d|^{-2}\sqrt{d_K}/2 & \beta \neq \infty \end{cases}$$

B.2. **Proof of Theorem 3.** With the above modifications, Lemma 4 implies that for all $p, q \nmid |c\tau + d|^2$ distinct odd primes satisfying $p, q \equiv 1 \pmod{N}$, the order of $\mathrm{AJ}_{\mathbf{C}}(\delta_\beta \Delta_{p,q,\beta})$ tends to ∞, which in particular implies that the order of $\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta})$ tends to ∞.

C. Changes to Section 10.3

The results remain unchanged with the new definitions in Section A.1, under the additional Assumption 1.

C.1. **Proof of Corollary 2.** When $\beta = \infty$ and $\gamma \neq \infty$, the upper bound for $|B_\infty/B_\gamma|$ (page 408, line 15) is slightly incorrect as stated. This does not affect the result of Corollary 2.

Instead, with the new definitions of Section A.1, define the polynomial

$$P(X) := \sum_{i=0}^{j}\sum_{k=0}^{r-j}\sum_{l=0}^{k+i}(-1)^{j-i}\binom{j}{i}\binom{r-j}{k}\frac{(k+i)!}{(k+i-l)!}\frac{(|c\tau+d|^{-2}\sqrt{d_K}/2)^{r-l}}{(2\pi)^{l+1}}X^{r-l} \in \mathbf{R}[X].$$

Then, by (1), we have

$$A_\infty = e^{-pq\pi|c\tau+d|^{-2}\sqrt{d_K}}P(pq) \qquad \text{and} \qquad A_\gamma = e^{-\frac{p}{q}\pi|c\tau+d|^{-2}\sqrt{d_K}}P\left(p/q\right).$$

It follows that

$$\left|\frac{B_\infty}{B_\gamma}\right| = q^{2(j-r)}e^{-\frac{p}{q}(q^2-1)\pi|c\tau+d|^{-2}\sqrt{d_K}}\frac{P(pq)}{P(p/q)}.$$

In particular, the ratio tends to 0 as p/q tends to ∞. The rest of the proof is unchanged.

D. Changes to Section 10.4

Retain the notations of Section A.1. Given a prime q coprime to cN, define

$$\Lambda^t_{q,\infty} := \mathbf{Z}\frac{1}{q} \oplus \mathbf{Z}\gamma(\tau), \qquad \Lambda^t_{q,\beta} := \mathbf{Z} \oplus \mathbf{Z}\frac{\gamma(\tau)+\beta}{q}, \qquad 0 \leq \beta \leq q-1,$$

along with the isogenies

$$\psi_{q,\beta} : \mathbf{C}/\mathcal{O}_K \overset{[(c\tau+d)^{-1}]}{\longrightarrow} \mathbf{C}/\langle 1, \gamma(\tau)\rangle \overset{\text{quot}}{\longrightarrow} \mathbf{C}/\Lambda^t_{q,\beta}.$$

Observe that

$$\ker(\psi_{q,\beta}) = \begin{cases} \langle (\tau + c^{-1}d)/q + \langle 1,\tau\rangle\rangle, & \beta = \infty \\ \langle 1/q + \langle 1,\tau\rangle\rangle, & a + c\beta \equiv 0 \pmod{q} \\ \langle (\tau + (a+c\beta)^{-1}(b+d\beta))/q + \langle 1,\tau\rangle\rangle, & \beta \neq \infty, a + c\beta \not\equiv 0 \pmod{q}, \end{cases}$$

and thus

$$(\psi_{q,\beta}, \mathbf{C}/\Lambda^t_{q,\beta}) = \begin{cases} (\varphi_{q,c^{-1}d}, A_{q,c^{-1}d}), & \beta = \infty \\ (\varphi_{q,\infty}, A_{q,\infty}), & a + c\beta \equiv 0 \pmod{q} \\ (\varphi_{q,(a+c\beta)^{-1}(b+d\beta)}, A_{q,(a+c\beta)^{-1}(b+d\beta)}), & \beta \neq \infty, a + c\beta \not\equiv 0 \pmod{q}, \end{cases}$$

as elements of $\mathrm{Isog}^{\mathfrak{N}}(A)$, in the notations of Section 10.4. In particular, there is an equality of subsets of $\mathrm{Isog}^{\mathfrak{N}}(A)$

$$\{(\varphi_{q,\beta}, A_{q,\beta}) \mid \beta \in \mathbf{P}^1(\mathbb{F}_q)\} = \{(\psi_{q,\beta}, \mathbf{C}/\Lambda^t_{q,\beta})) \mid \beta \in \mathbf{P}^1(\mathbb{F}_q)\}, \tag{2}$$

and $\mathrm{Gal}(H_q/H)$ acts simply transitively on this set when q is inert in K, as proved in the proof of Theorem 2.

The rest of the proof of Theorem 2 goes through when replacing τ by $\gamma(\tau)$ and upon noting (page 412, line 37) that the isogeny $\varphi_{p,q,\beta}$ of Section A.1 now corresponds to the subgroup

$$\left\langle \xi\left(\frac{\tau + c^{-1}d}{p}\right), \ker(\psi_{q,\beta})\right\rangle \subset A(\bar{H}).$$

Remark 4. On (page 413, line 15), between equations (96) and (97), when identifying $\mathbf{Q}[G_\ell]$ with $\mathbf{Q} \times \mathbf{Q}(\zeta_\ell)$, the first coordinate of the map should be the sum of the coefficients $\sum_{i=0}^{\ell-1}\lambda_i$ instead of λ_0.