# Derivatives of Rankin–Selberg L-functions via the Goldfeld–Zhang method

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**Abstract**

We use a method by Goldfeld and Zhang to compute values and derivatives of the Rankin–Selberg $L$-function associated to a pair of modular forms. Central to this approach is the holomorphic kernel, which relates the $L$-function to an inner product of modular forms. Using the Petersson formula, the Fourier coefficients of the holomorphic kernel can be understood. We solidify the Goldfeld–Zhang method by providing extra details to proofs and making justified adjustments to their formulas. In particular, we believe that an additional term has been overlooked in their main theorem. For weight $k > 2$, we verify their final claim that it is possible to recover the analytic part of the Gross–Zagier formulas for heights of Heegner cycles. More generally, we recover the analytic results in recent work by Lilienfeldt and Shnidman on heights of generalized Heegner cycles.

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# 1 Introduction

## 1.1 The arithmetic of elliptic curves

Elliptic curves have long been an object of study. In some ways, we understand them well: over a number field $K$, the $K$-rational points $E(K)$ of an elliptic curve form a finitely generated group, and $E(K)$ is thus of the form

$$E(K) \cong \mathbf{Z}^r \times E(K)_{\mathrm{tors}},$$

with $r \in \mathbf{Z}_{\geq 0}$ the algebraic rank of $E$ over $K$ and $E(K)_{\mathrm{tors}}$ the finite torsion subgroup. If $K = \mathbf{Q}$, we even know that $E(\mathbf{Q})_{\mathrm{tors}}$ has at most 16 elements. This is a result of Mazur [Maz78] and has been extended by Merel [Mer96] to a bound on $E(K)_{\mathrm{tors}}$ for any number field $K$ in terms of the degree of $K$ over $\mathbf{Q}$. The rank $r$ is somewhat more mysterious. Computing the rank of a given elliptic curve is in general hard, and we do not know whether the rank of all elliptic curves over some number field $K$ (or even $\mathbf{Q}$) is bounded.

A similar object with many open questions surrounding it is the Hasse–Weil $L$-function $L(E/K, s)$ associated to an elliptic curve $E$ over $K$, given by (A.1). Due to the (very deep) modularity theorem, first proven for semistable elliptic curves by Wiles [Wil95] and then in general by Breuil, Conrad, Diamond and Taylor [Bre+01] (see also Theorem A.6), we know that $L(E/\mathbf{Q}, s)$ extends to the whole complex plane for any elliptic curve $E$ over $\mathbf{Q}$. As such, one can consider the order of vanishing of $L(E/\mathbf{Q}, s)$ at $s = 1$, which is called the analytic rank of $E$. In the 1960s, at which time the modularity theorem had not yet been proven, Birch and Swinnerton-Dyer came up with a conjecture based on extensive numerical evidence. In a modern phrasing, it can be stated as: the algebraic rank and the analytic rank of an elliptic curve agree. At the moment, it is one of the six remaining open Millennium Prize Problems.

In 1986, a partial proof of the conjecture over $\mathbf{Q}$ was given by Gross and Zagier [GZ86]. Together with a later result of Kolyvagin [Kol88], it follows that the algebraic and analytic rank agree if the analytic rank is at most one. To date, there have been no successful ideas for proving the conjecture for curves of higher ranks, and so the conjecture still remains wide open, even over $\mathbf{Q}$.

### The Gross–Zagier strategy

The paper by Gross and Zagier can be divided into two parts. In the algebraic part, they compute height pairings of a divisor associated to a Heegner point. Heegner points are special points on an elliptic curve that are constructed from imaginary quadratic points in the complex upper half-plane using the modularity theorem. In the analytic part, Gross and Zagier compute the value and derivative of certain Rankin–Selberg $L$-functions at the center. The Rankin–Selberg $L$-function associated to two modular forms $f \in S_k(\Gamma_0(N))$ and $g \in M_\ell(\Gamma_0(D), \chi)$ is given by

$$L(f \otimes g, s) = \sum_{n=1}^{\infty} \frac{a(n)b(n)}{n^s},$$

where $a(n)n^{\frac{k-1}{2}}$ and $b(n)n^{\frac{\ell-1}{2}}$ are the Fourier coefficients of $f$ and $g$. It converges absolutely for $\Re(s) > \frac{\ell+1}{2}$ and has a meromorphic continuation to the whole complex plane. For fixed $g$ and $s$, the map $f \mapsto L(f \otimes g, s)$ is linear, and so there exists a holomorphic kernel $\Phi_{\bar{s},g} \in S_k(\Gamma_0(N))$ with

$$L(f \otimes g, s) = \langle f, \Phi_{\bar{s},g} \rangle \qquad \text{for all } f \in S_k(\Gamma_0(N)).$$

Here $\langle \cdot, \cdot \rangle$ is the Petersson inner product on $S_k(\Gamma_0(N))$ as in (2.3). Gross and Zagier consider the Rankin–Selberg $L$-function when $\ell = 1$ and $g$ is a theta series associated to an imaginary quadratic field $K$ of square-free discriminant. When $k = 2$, by the modularity theorem, there is some newform $f \in S_k^{\text{new}}(\Gamma_0(N))$ such that the two $L$-functions $L(f \otimes g, s)$ and $L(E/K, s)$ are closely related. By computing the Fourier coefficients of the holomorphic kernel, Gross and Zagier are, under the right conditions, able to relate the central derivative $L'(E/K, 1)$ to the height of a Heegner point and deduce a lower bound for the algebraic rank of $E$ over $K$. See Appendix A for more details.

In order to compute the Fourier coefficients of the holomorphic kernel, Gross and Zagier use Rankin's method, also known as the Rankin–Selberg method, to derive formulas for the Fourier coefficients of the holomorphic kernel. This method was independently discovered and used by Rankin [Ran39] and Selberg [Sel40] to study $L$-functions of the form $L(f \otimes \bar{g}, s)$. After using Rankin's method, non-holomorphic modular forms of different levels appear and so Gross and Zagier need to use traces and holomorphic projections to obtain a final formula in terms of holomorphic modular forms. We elaborate on this in Section 3.3.

#### The Goldfeld–Zhang method

Around the turn of the millennium, Goldfeld and Zhang published a paper with a different method for deriving formulas for the Fourier coefficients of the holomorphic kernel $\Phi_{s,g}$ [GZ99]. This method, which we will refer to as the Goldfeld–Zhang method, computes these coefficients using the Petersson formula for Poincaré series (Proposition 2.9). In particular, this method avoids taking traces and holomorphic projections. It could, in this sense, be interpreted as a more direct method for computing the holomorphic kernel. This method has since been generalized by Nelson, allowing $f$, and thus $\Phi_{s,g}$, to have any nebentypus $\varepsilon$ [Nel13].

Goldfeld and Zhang end their paper with a remark claiming that one can recover the analytic results in [GZ86] by evaluating their final formula at specific points. In this thesis, we will take a close look at the Goldfeld–Zhang method and verify this claim.

## 1.2 Contributions

Broadly speaking, with this thesis we have two main goals. Our first aim is to solidify the work by Goldfeld and Zhang. In Chapter 4, we follow the approach in their paper to obtain a formula for the Fourier coefficients of the holomorphic kernel of the Rankin–Selberg $L$-function. We elaborate on their proofs and, where necessary, we adjust their formulas and statements. Most notably, we believe that they have missed an additional term that should be present in all main formulas. This additional term comes from the residue in Lemma 4.1, which appears because the twisted $L$-series $L_g(s, \frac{a}{c})$ given by (3.8) has a pole at $s = \frac{\ell+1}{2}$ if $g$ is not a cusp form (Proposition 3.5). We have proceeded methodically and have justified all our steps. We will not explicitly mention all adjustments we made in the body of the thesis. Instead, the interested reader can find an overview of the changes in Appendix B.

The main result for the coefficients of the holomorphic kernel is Theorem 4.4 and corresponds to [GZ99, Thm. 6.5]. The Fourier coefficients for the holomorphic kernel modified suitably by adding oldforms are given in Proposition 4.11 and this corresponds to [GZ99, Prop. 9.1]. Finally, we compute the Fourier coefficients of the modified holomorphic kernel associated to a theta series in Theorem 4.13, which corresponds to [GZ99, Thm. 11.5]. We use these formulas to deduce a functional equation for the Rankin–Selberg $L$-function associated to a cusp form and a theta series (Theorem 4.16). For completeness, we also give a corrected version of [GZ99, Thm. 10.1] in Appendix C.

Our second aim is to recover parts of the analytic results of the influential paper by Gross and Zagier [GZ86] and thereby verifying the last claim made in [GZ99]. We only verify the claim for weight $k > 2$, as it is not possible to obtain results for $k = 2$ without modifying the Goldfeld–Zhang method in a major way. In Chapter 5, we derive an expression for special values (Theorem 5.8), the central value (Theorem 5.9) and the derivative (Theorem 5.10) of the Rankin–Selberg $L$-function associated to a cusp form and a theta series of an unramified Hecke character of possibly infinite order. These theorems are generalizations of [GZ86, Thm. IV.5.5],[GZ86, Thm. IV.5.6] and [GZ86, Thm. IV.5.8], respectively, which only consider finite order Hecke characters. With Theorem 5.10, we also recover the analytic part of the recent paper by Lilienfeldt and Shnidman [LS24, Thm. 3.6], in which they prove a generalization of the Gross–Zagier formula.

**Remark.** In [Nel13, p. 2602-2603], of which we were not aware while writing the thesis, Nelson gives a list of errors in the Goldfeld–Zhang paper. Their findings agree with ours. The aim of their paper is finding a formula for twisted first moments of the form

$$\sum_{f\in S_k(\Gamma_0(N),\varepsilon)} \frac{L(f\otimes g,s)}{\langle f,f\rangle}\overline{\lambda_m(f)},$$

with $\lambda_m(f)m^{\frac{k-1}{2}}$ the $m^{\text{th}}$ Fourier coefficient of $f$ and the sum taken over an orthogonal basis of $S_k(\Gamma_0(N),\varepsilon)$ for some nebentypus $\varepsilon$. As they only consider the case where $g$ is a cusp form, they do not derive a formula for the term that is missing from the formulas of Goldfeld and Zhang. They also do not give corrected versions of the main formulas in [GZ99], but they do give a variant of [GZ99, Prop. 9.1] in the special case where $\gcd(D,N) = 1$ and $g$ is a cusp form.

## 1.3 Thesis overview

In Chapter 2, we give a quick overview of the theory of modular forms, and we describe two important examples of modular forms: Poincaré series and theta series. These will be used to derive the main results of the thesis. In Chapter 3, we define various $L$-series and recall or derive their analytic properties. We also illustrate Rankin's method, which is used in the paper by Gross and Zagier. In Chapter 4, we use the Goldfeld–Zhang method to derive a formula for the Fourier coefficients of the holomorphic kernel of the Rankin–Selberg $L$-function. We end the chapter by considering the holomorphic kernel in the context of theta series and obtaining specialized coefficients. In Chapter 5, we compute these coefficients and their derivatives at special points, thereby deducing [GZ86, Thms. IV.5.5, IV.5.6, IV.5.8] and [LS24, Thm. 3.6]. Finally, in Chapter 6, we reflect on the thesis and give suggestions for further research.

In Appendix A, we cover some background on elliptic curves, modularity, and the famous Birch and Swinnerton-Dyer conjecture. We also give a broad overview of the Gross–Zagier paper. This appendix is meant to provide context for the interested reader. In Appendix B, we have compiled a list of adjustments we made to the Goldfeld–Zhang method. In Appendix C, we give a version of [GZ99, Thm. 10.1] that takes the additional missing factor into account.

# 2 Modular forms

In this chapter, we give a brief overview of the theory of modular forms. We state the definition of a modular form and mention general results that are needed for this thesis in Section 2.1. For a more detailed approach, see [DS05]. In Section 2.2 and Section 2.3, we give the definitions of two important types of modular forms: Poincaré series and theta series. These are needed to obtain the main result of this thesis.

## 2.1 Modular forms

### Congruence subgroups

For an integer $N \geq 1$, we define the subgroup $\Gamma(N) \subseteq \mathrm{SL}_2(\mathbf{Z})$ by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \;\middle|\; \begin{matrix} a \equiv d \equiv 1 \pmod{N} \\ b \equiv c \equiv 0 \pmod{N} \end{matrix} \right\}.$$

Note that $\Gamma(N)$ is a normal subgroup of finite index, as it is the kernel of the reduction map $\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$. Given a subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$, we say that $\Gamma$ is a *congruence subgroup* if $\Gamma(N) \subseteq \Gamma$ for some $N \geq 1$. The smallest $N$ for which such an inclusion holds is called the *level* of $\Gamma$. Two important examples of congruence subgroups of level $N$ are

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \;\middle|\; \begin{matrix} a \equiv d \equiv 1 \pmod{N} \\ c \equiv 0 \pmod{N} \end{matrix} \right\},$$
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}.$$

It is clear that there are inclusions

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbf{Z}),$$

and these are strict for $N \geq 3$.

### Slashing operators

Given a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbf{R})$ and a point $z \in \mathcal{H}$, it holds that

$$\gamma z := \frac{az+b}{cz+d} \in \mathcal{H},$$

and it turns out that this defines an action of $\mathrm{GL}_2^+(\mathbf{R})$ on $\mathcal{H}$. We define the *factor of automorphy* $j(\gamma, z)$ by $cz + d$. A simple calculation reveals that it satisfies the following two identities:

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z) \cdot j(\gamma_2, z), \tag{2.1}$$
$$\Im(\gamma z) = \frac{\det(\gamma) \cdot \Im(z)}{|j(\gamma, z)|^2}. \tag{2.2}$$

We can now define an action of $\mathrm{GL}_2^+(\mathbf{R})$ on the group of holomorphic functions from $\mathcal{H}$ to $\mathbf{C}$. Fix an integer $k$. Given $f : \mathcal{H} \to \mathbf{C}$ and $\gamma \in \mathrm{GL}_2^+(\mathbf{R})$, we define $(f|_k\gamma)(z) = \frac{\det(\gamma)^{k/2}}{j(\gamma,z)^k} f(\gamma z)$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$, this reduces to $(f|_k\gamma)(z) = (cz+d)^{-k} f(\gamma z)$. It is readily verified that this satisfies the properties of an action. This action is called the *weight $k$ slashing operator*.

## Modular forms

Given a congruence subgroup $\Gamma$ of level $N$, we say that a holomorphic function $f : \mathcal{H} \to \mathbf{C}$ is *weakly modular of weight $k$ for* $\Gamma$ if $f|_k\gamma = f$ for all $\gamma \in \Gamma$. As $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, it follows that $f(z+N) = f(z)$ for all $z \in \mathcal{H}$, and so $f$ is periodic. More precisely, define

$$\mathrm{SL}_2(\mathbf{Z})_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbf{Z} \right\},$$

and

$$\Gamma_\infty = \Gamma \cap \mathrm{SL}_2(\mathbf{Z})_\infty.$$

Let $h_\infty \geq 1$ be the smallest positive integer such that $\left(\begin{smallmatrix} 1 & h_\infty \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_\infty$. It then follows that $f(z+h_\infty) = f(z)$ for all $z \in \mathcal{H}$ and so there exists some analytic function $\tilde{f} : \mathbf{D} \setminus \{0\} \to \mathbf{C}$ on the punctured unit disc such that $f(z) = \tilde{f}(\exp(2\pi i z/h_\infty))$. If $\tilde{f}$ can be continued analytically to $\mathbf{D}$, we say that $f$ is *holomorphic at* $\infty$. Then $\tilde{f}(z)$ has a Taylor expansion at $z = 0$ and so $f$ has a Fourier expansion of the following form:

$$f(z) = a(0) + \sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} e^{2\pi i n z/h_\infty}.$$

Here we have normalized the coefficients $a(n)$. In the special case that $\Gamma = \Gamma_0(N)$ or $\Gamma = \Gamma_1(N)$, it holds that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ and so $h_\infty = 1$.

More generally, consider the quotient of $\mathbf{P}^1(\mathbf{Q}) = \mathbf{Q} \cup \{\infty\}$ by $\Gamma$, using the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} q = \frac{aq+b}{cq+d}.$$

Here we define $\frac{*}{0} = \infty$ and $\frac{a\cdot\infty+b}{c\cdot\infty+d} = \frac{a}{c}$. We write $\mathrm{Cusps}(\Gamma) = \Gamma\backslash\mathbf{P}^1(\mathbf{Q})$ for this quotient and an equivalence class is called a *cusp* of $\Gamma$. Note that the action is transitive in the case of $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, and so $\mathrm{SL}_2(\mathbf{Z})$ only has one cusp. Given a cusp $\mathfrak{c} \in \mathrm{Cusps}(\Gamma)$, we can thus choose a matrix $\gamma_\mathfrak{c} \in \mathrm{SL}_2(\mathbf{Z})$ such that $[\gamma_\mathfrak{c}\infty] = \mathfrak{c}$. Now consider $f|_k\gamma_\mathfrak{c}$ and

$$\Gamma_\mathfrak{c} = (\gamma_\mathfrak{c}^{-1}\Gamma\gamma_\mathfrak{c}) \cap \mathrm{SL}_2(\mathbf{Z})_\infty.$$

Note that $f|_k\gamma_\mathfrak{c}$ does not depend on the choice of $\gamma_\mathfrak{c}$, as $\gamma_\mathfrak{c}$ is determined uniquely up to an element in $\Gamma$ and $f$ is weakly modular of weight $k$ for $\Gamma$. There exists a minimal $h_\mathfrak{c} \geq 1$ such that $\left(\begin{smallmatrix} 1 & h_\mathfrak{c} \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_\mathfrak{c}$, and then $(f|_k\gamma_\mathfrak{c})(z+h_\mathfrak{c}) = (f|_k\gamma_\mathfrak{c})(z)$, which gives us an analytic function $\tilde{f}_\mathfrak{c} : \mathbf{D} \setminus \{0\} \to \mathbf{C}$ with $(f|_k\gamma_\mathfrak{c})(z) = \tilde{f}_\mathfrak{c}(\exp(2\pi i z/h_\mathfrak{c}))$. If $\tilde{f}_\mathfrak{c}$ has an analytic extension to $\mathbf{D}$, we say that $f$ is *holomorphic at the cusp* $\mathfrak{c}$. In that case, we have a Fourier expansion

$$(f|_k\gamma_\mathfrak{c})(z) = a_\mathfrak{c}(0) + \sum_{n=1}^{\infty} a_\mathfrak{c}(n) n^{\frac{k-1}{2}} e^{2\pi i n z/h_\mathfrak{c}}.$$

A holomorphic map $f : \mathcal{H} \to \mathbf{C}$ that is weakly modular of weight $k$ for $\Gamma$ and is holomorphic at all cusps of $\Gamma$ is called a *modular form* (of weight $k$ for $\Gamma$). We will write $M_k(\Gamma)$ for the space of all such modular forms. If, in addition, $f$ vanishes at every cusp, i.e., $a_\mathfrak{c}(0) = 0$ for every $\mathfrak{c} \in \mathrm{Cusps}(\Gamma)$, then we say that $f$ is a *cusp form*, and we write $S_k(\Gamma)$ for the subspace of cusp forms. $M_k(\Gamma)$ and $S_k(\Gamma)$ are complex vector spaces of finite dimension.

An important fact about modular forms that we will need is the following.

**Lemma 2.1.** *Let $f \in M_k(\Gamma)$ be a modular form. Let $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$ be a matrix and define*

$$\Gamma' = (\alpha^{-1}\Gamma\alpha) \cap \Gamma,$$

*and*

$$f_\alpha = \sum_{\gamma \in \Gamma' \backslash \Gamma} f|_k \alpha\gamma.$$

*Then this is a well-defined finite sum, and $f_\alpha \in M_k(\Gamma)$. Moreover, if $f$ is a cusp form, then so is $f_\alpha$.*

*Proof.* See [DS05, Section 5.1]. □

### Modular forms with a nebentypus

Let $N \geq 1$ be a level and let $\chi$ be a Dirichlet character modulo $N$. We can then consider the subspace $M_k(\Gamma_0(N), \chi) \subseteq M_k(\Gamma_1(N))$ consisting of modular forms $f$ that satisfy

$$f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d) \cdot f \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

If $f \in M_k(\Gamma_0(N), \chi)$, we call $\chi$ the *nebentypus* of $f$. In the case that $\chi$ is the trivial character, we have an equality $M_k(\Gamma_0(N), \chi) = M_k(\Gamma_0(N))$. It turns out that $M_k(\Gamma_1(N))$ decomposes as a direct sum of these subspaces:

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi \in \widehat{(\mathbf{Z}/N\mathbf{Z})^\times}} M_k(\Gamma_0(N), \chi).$$

Similarly, we can define a subspace $S_k(\Gamma_0(N), \chi) \subseteq S_k(\Gamma_1(N))$. There is an obvious inclusion $S_k(\Gamma_0(N), \chi) \subseteq M_k(\Gamma_0(N), \chi)$, and as above, $S_k(\Gamma_1(N))$ decomposes as a direct sum of the subspaces $S_k(\Gamma_0(N), \chi)$.

### Petersson inner product

On $S_k(\Gamma)$, we can define the Petersson inner product. Given $f, g \in S_k(\Gamma)$, we define

$$\langle f, g \rangle = \int_{\Gamma \backslash \mathcal{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}. \tag{2.3}$$

This allows us to talk about orthogonality in the space of cusp forms, which is needed to construct the so-called newforms.

### Oldforms and newforms

Given a level $N$ and a proper divisor $M$ of $N$, we have an inclusion $\Gamma_0(N) \subseteq \Gamma_0(M)$. It follows that a holomorphic function that is weakly modular of weight $k$ for $\Gamma_0(M)$ is also weakly modular of weight $k$ for $\Gamma_0(N)$. As a result we obtain an embedding $S_k(\Gamma_0(M)) \hookrightarrow S_k(\Gamma_0(N))$. In fact, something stronger is true: given a divisor $d | \frac{N}{M}$, we have a linear map

$$\begin{aligned} i_{M,N,d} : S_k(\Gamma_0(M)) &\to S_k(\Gamma_0(N)), \\ f(z) &\mapsto f(dz). \end{aligned}$$

The space generated by the images of all these maps is called the space of oldforms:

$$S_k^{\mathrm{old}}(\Gamma_0(N)) := \bigoplus_{\substack{M|N \\ M\neq N}} \bigoplus_{d|\frac{N}{M}} i_{M,N,d}(S_k(\Gamma_0(M))).$$

We then define the space of newforms to be the orthogonal complement of the oldforms:

$$S_k^{\mathrm{new}}(\Gamma_0(N)) := S_k^{\mathrm{old}}(\Gamma_0(N))^{\perp}. \tag{2.4}$$

These are the modular forms of level $N$ that do not arise from a modular form of a lower level. Results about modular forms can often be reduced to results about newforms.

For any congruence subgroup $\Gamma$, one can define a decomposition

$$M_k(\Gamma) = \mathcal{E}_k(\Gamma) \oplus S_k(\Gamma),$$

where $\mathcal{E}_k(\Gamma)$ is called the Eisenstein subspace, see [DS05, Section 5.11]. Using an explicit basis for $\mathcal{E}_k(\Gamma(N))$ and a deep result of Deligne [Del74], one can obtain the following bounds for the Fourier coefficients of a modular form with nebentypus.

**Lemma 2.2.** *Let $k, N \geq 1$ be integers and $\chi$ a Dirichlet character modulo $N$. Let $f \in M_k(\Gamma_0(N), \chi)$ be a modular form and consider its Fourier expansion*

$$f(z) = a(0) + \sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} e^{2\pi i n z}.$$

*If $f \in \mathcal{E}_k(\Gamma_0(N), \chi)$, then $a(n) = O(n^{\frac{k-1}{2}+\varepsilon})$ as $n \to \infty$, for any $\varepsilon > 0$. If $f \in S_k(\Gamma_0(N), \chi)$, then $a(n) = O(n^{\varepsilon})$ as $n \to \infty$, for any $\varepsilon > 0$. In particular, for general $f \in M_k(\Gamma_0(N), \chi)$, one has $a(n) = O(n^{\gamma})$ as $n \to \infty$, for any $\gamma > \frac{k-1}{2}$.*

*Proof.* For the case $f \in \mathcal{E}_k(\Gamma_0(N), \chi)$, note that $\mathcal{E}_k(\Gamma_0(N), \chi) \subseteq \mathcal{E}_k(\Gamma(N))$ and use the explicit Fourier expansion of a basis for $\mathcal{E}_k(\Gamma(N))$ given by [DS05, Thm. 4.2.3]. For the case $f \in S_k(\Gamma_0(N), \chi)$, this is a consequence of the proof of the Weil conjectures by Deligne, see also [IK04, Section 14.9]. □

### Atkin–Lehner operators

Let $D \geq 1$ be a square-free integer and $\chi$ a Dirichlet character modulo $D$. Let $g \in M_\ell(\Gamma_0(D), \chi)$ be a modular form and fix a decomposition $D = \delta \cdot \delta'$. As $\gcd(\delta, \delta') = 1$, we can find integers $x, y \in \mathbf{Z}$ with $x\delta - y\delta' = 1$ and define the matrix

$$\omega_\delta = \begin{pmatrix} x & y \\ \delta' & \delta \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x\delta & y \\ D & \delta \end{pmatrix}.$$

Note that $\omega_\delta$ has determinant $\delta$. We now define

$$g^{\delta} = g|_\ell \omega_\delta, \tag{2.5}$$

and

$$\chi^{\delta} = \chi_\delta^{-1} \cdot \chi_{\delta'}.$$

Here $\chi_\delta$ and $\chi_{\delta'}$ denote the unique Dirichlet characters modulo $\delta$ and $\delta'$, respectively, such that $\chi = \chi_\delta \cdot \chi_{\delta'}$. We note that $\omega_\delta^2 = \delta \cdot \gamma$ for some $\gamma \in \Gamma_0(D)$ and so $(g^\delta)^\delta$ is again equal to $g$ up to multiplication by its nebentypus. Slashing with $\omega_\delta$ is an example of an Atkin–Lehner operator. If $g$ is a modular form for $\Gamma_0(D)$, then it has a trivial nebentypus and hence $(g^\delta)^\delta = g$. In this case, the operator is an involution.

We will show that $g^\delta \in M_\ell(\Gamma_0(D), \chi^\delta)$ using the following lemma.

**Lemma 2.3.** *$\omega_\delta$ normalizes $\Gamma_0(D)$ and $\Gamma_1(D)$.*

*Proof.* Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(D)$ be given. A simple calculation shows that

$$\omega_\delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} \omega_\delta^{-1} = \begin{pmatrix} ax\delta + cy - bxD - dy\delta' & * \\ aD + c\delta - bD\delta' - dD & -ay\delta' - cy + bxD + dx\delta \end{pmatrix}.$$

As $D$ divides $c$ by assumption, it follows that $\omega_\delta \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \omega_\delta^{-1} \in \Gamma_0(D)$. If furthermore $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(D)$, then $\omega_\delta \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \omega_\delta^{-1} \in \Gamma_1(D)$, as we have

$$\begin{aligned} ax\delta + cy - bDx - dy\delta' &\equiv \delta x - y\delta' = 1 \pmod D, \\ -ay\delta' - cy + bDx + dx\delta &\equiv -y\delta' + x\delta = 1 \pmod D. \end{aligned}$$

This proves the inclusion $\omega_\delta \Gamma_i(D) \omega_\delta^{-1} \subseteq \Gamma_i(D)$. A similar argument shows that $\omega_\delta^{-1} \Gamma_i(D) \omega_\delta \subseteq \Gamma_i(D)$, from which the lemma follows. ☐

As a result, we obtain the following proposition.

**Proposition 2.4.** *Let $g \in M_\ell(\Gamma_0(D), \chi)$ with $D$ square-free and fix a decomposition $D = \delta \cdot \delta'$. Then $g^\delta \in M_\ell(\Gamma_0(D), \chi^\delta)$. Moreover, $g^\delta$ is a cusp form if and only if $g$ is a cusp form.*

*Proof.* By Lemma 2.1 and Lemma 2.3, it immediately follows that $g^\delta \in M_\ell(\Gamma_1(D))$ and that $g^\delta \in S_\ell(\Gamma_1(D))$ if and only if $g \in S_\ell(\Gamma_1(D))$. It remains to show that $g^\delta|_\ell \gamma = \chi^\delta(d) g^\delta$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(D)$.

Given such $\gamma$, consider $\gamma' = \omega_\delta \gamma \omega_\delta^{-1} \in \Gamma_0(D)$, so that $\omega_\delta \gamma = \gamma' \omega_\delta$. By Lemma 2.3, we know that the lower right entry of $\gamma'$ is given by $-ay\delta' - cy + bDx + dx\delta$. Now note that

$$\begin{aligned} \chi(-ay\delta' - cy + bDx + dx\delta) &= \chi(dx\delta - ay\delta') = \chi_\delta(-ay\delta') \cdot \chi_{\delta'}(dx\delta) \\ &= \chi_\delta(a) \cdot \chi_\delta(-y\delta') \cdot \chi_{\delta'}(d) \cdot \chi_{\delta'}(x\delta) = \chi_\delta(a) \cdot \chi_{\delta'}(d) = \chi_\delta^{-1}(d) \cdot \chi_{\delta'}(d) = \chi^\delta(d). \end{aligned}$$

Here we used that $x\delta - y\delta' = 1$ and that $ad \equiv 1 \pmod D$. We conclude that

$$g^\delta|_\ell \gamma = g|_\ell \omega_\delta \gamma = g|_\ell \gamma' \omega_\delta = (\chi(-ay\delta' - cy + bDx + dx\delta) g)|_\ell \omega_\delta = \chi^\delta(d) g^\delta.$$

☐

As a consequence, $g^\delta$ has a Fourier expansion and we will write it as

$$g^\delta(z) = b^\delta(0) + \sum_{n=1}^\infty b^\delta(n) n^{\frac{\ell-1}{2}} e^{2\pi i n z}. \tag{2.6}$$

**Remark.** The modular form $g^\delta$ is related to the behavior of $g$ at the cusps of $\Gamma_1(D)$ of the form $\frac{a}{c}$ with $\delta' = \gcd(D, c)$, as can be seen in Lemma 3.6.

## 2.2 Poincaré series

An important example of a modular form, and one that will play a key role in our approach, is the Poincaré series $P_m$. In this section, we give the definition of a Poincaré series and state a formula for its Fourier coefficients (Proposition 2.8). We also state the Petersson formula (Proposition 2.9), which relates the Petersson inner product $\langle f, P_m \rangle$ to the $m^{\text{th}}$ coefficient of a modular form $f$.

### Definition

Fix a level $N \geq 1$ and an even weight $k \geq 4$. Let $m \geq 1$ be an integer. We then define the Poincaré series of order $m$ (of weight $k$ and level $N$) by

$$P_m(z) = m^{\frac{k-1}{2}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} j(\gamma, z)^{-k} e^{2\pi i m \gamma z}. \tag{2.7}$$

This sum converges absolutely and uniformly on compact subsets of $\mathcal{H}$ and thus defines an analytic function there. By (2.1), it is not hard to see that $P_m(z)$ is weakly modular of weight $k$ for $\Gamma_0(N)$. It turns out that $P_m(z)$ is also holomorphic at the cusps of $\Gamma_0(N)$ and hence defines a modular form of weight $k$ and level $N$.

### Fourier coefficients

**Definition 2.5.** Let $m, n, c \geq 1$ be integers. We define their *Kloosterman sum* as

$$K(m, n; c) = \sum_{r \in (\mathbf{Z}/c\mathbf{Z})^\times} e^{2\pi i \frac{mr + n\bar{r}}{c}},$$

where $\bar{r}$ denotes the inverse of $r$ modulo $c$.

**Lemma 2.6.** *For $m, n, c \geq 1$, the Kloosterman sum $K(m, n; c)$ satisfies the following bound:*

$$|K(m, n; c)| \leq \sigma_0(c) \sqrt{\gcd(m, n, c)} \sqrt{c},$$

*where $\sigma_0(c)$ denotes the number of divisors of $c$.*

*Proof.* The bound can be reduced to the case that $c$ is prime. The case $c$ prime is proven in [Wei48, p. 207]. □

There are various ways to define the Bessel function of the first kind. See, for example, [Erd+81b, Ch. VII], where many equivalent definitions are given. The following definition corresponds to definition (34) on page 21 after a change of variables.

**Definition 2.7.** Fix an integer $k > 0$ and a real number $0 < \epsilon < \frac{k-1}{2}$. Then we define the Bessel function of the first kind for $x > 0$ by

$$J_{k-1}(x) = \frac{1}{2\pi i} \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \left(\frac{x}{2}\right)^{-2w} dw.$$

The function $J_{k-1}(x)$ does not depend on the choice of the value $\epsilon$. Using Stirling's approximation for the gamma function, it follows that the integral above converges absolutely for all $x > 0$. The approximation states that for fixed $\sigma \in \mathbf{R}$,

$$|\Gamma(\sigma + it)| = \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi|t|}(1 + o(1)) \qquad \text{as} \quad |t| \to \infty, \tag{2.8}$$

and so for $w = \epsilon - \frac{k-1}{2} + it$, the integrand is $O(|t|^{2\epsilon-k})$ as $|t| \to \infty$.

**Remark.** One can also express $J_{k-1}(x)$ as an infinite sum [Erd+81b, Eq. 7.2 (2)]:

$$J_{k-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{\Gamma(n+k)} \left(\frac{x}{2}\right)^{2n+k-1}. \tag{2.9}$$

This expression can be obtained from the integral representation in Definition 2.7 by using the residue theorem and taking a contour integral consisting of a vertical line segment at $\Re(w) = \epsilon - \frac{k-1}{2}$ and a large semicircle to the left of this segment. In the limit, this contour will pick up all poles of the integrand, which correspond to the poles of $\Gamma(\frac{k-1}{2} + w)$ at $w = -n - \frac{k-1}{2}$ for $n \in \mathbf{Z}_{\geq 0}$. One can show that the integral over the semicircle vanishes as the radius increases and by taking the limit one obtains the equality above.

The Fourier coefficients of the Poincaré series can be given in terms of Kloosterman sums and Bessel functions of the first kind.

**Proposition 2.8.** *Let $k \geq 4$ be even and $m, N \geq 1$ be integers. Then $P_m(z)$, as defined by (2.7), lies in $S_k(\Gamma_0(N))$. Furthermore, the coefficients $p_m(n)$ in the Fourier expansion*

$$P_m(z) = \sum_{n=1}^{\infty} p_m(n) n^{\frac{k-1}{2}} e^{2\pi i n z},$$

*are given by*

$$p_m(n) = \delta_{m,n} + 2\pi i^k \sum_{\substack{c=1 \\ N|c}}^{\infty} \frac{K(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

*where $\delta_{m,n}$ is the Kronecker delta function that is 1 if $m = n$ and 0 otherwise.*

*Proof.* See [IK04, Lemma 14.2]. Note the difference in normalization. □

**Remark.** There are no non-trivial cusp forms of weight $4 \leq k \leq 10$ and level $N = 1$, and so $p_m(n) = 0$ for all $m, n \geq 1$ in that case.

**Remark.** It is possible to define the Poincaré series for weight 2. In that case, the infinite sum in (2.7) still converges, but not absolutely [IK04, p. 358]. Alternatively, $P_m$ can be defined as a limit of automorphic forms

$$P_{m,s}(z) = \sqrt{m} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{(\Im(\gamma z))^s}{j(\gamma, z)^2} e^{2\pi i m \gamma z}. \tag{2.10}$$

This series converges absolutely for $\Re(s) > 0$ and, for fixed $z \in \mathcal{H}$, extends meromorphically in $s$ to the whole complex plane [GZ99, Section 2]. One can then define $P_m(z) = \lim_{s\to 0} P_{m,s}(z)$. The formulas for the Fourier coefficients of $P_m \in S_2(\Gamma_0(N))$ are still as in Proposition 2.8.

**Petersson formula**

An important property of the Poincaré series is that they can isolate the Fourier coefficients of other modular forms of the same weight and level, as is made precise by the following proposition.

**Proposition 2.9** (Petersson Formula)**.** *Let $f \in S_k(\Gamma_0(N))$ be a cusp form. Write its Fourier expansion as*

$$f(z) = \sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} e^{2\pi i n z}.$$

*Then for all $m \geq 1$, we have*

$$a(m) = \frac{(4\pi)^{k-1}}{(k-2)!} \langle f, P_m \rangle.$$

*Proof.* We will write $\Gamma = \Gamma_0(N)$ and $e(t) = e^{2\pi i t}$. We first use (2.2), and the fact that $f$ is modular:

$$\begin{aligned}
\langle f, P_m \rangle &= \int_{\Gamma\backslash\mathcal{H}} f(z) \overline{P_m(z)} y^k \frac{dxdy}{y^2} \\
&= m^{\frac{k-1}{2}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\Gamma\backslash\mathcal{H}} f(z) \overline{e(m\gamma z)(j(\gamma, z)^{-k})} y^k \frac{dxdy}{y^2} \\
&= m^{\frac{k-1}{2}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\Gamma\backslash\mathcal{H}} f(\gamma z) \overline{e(m\gamma z)} (\Im(\gamma z))^k \frac{dxdy}{y^2} \\
&= m^{\frac{k-1}{2}} \int_{\Gamma_\infty\backslash\mathcal{H}} f(z) \overline{e(mz)} y^k \frac{dxdy}{y^2}.
\end{aligned}$$

Now note that $\Gamma_\infty\backslash\mathcal{H}$ is represented by the vertical strip $0 \leq \Re(s) < 1$ with $\Im(s) > 0$. Using the Fourier expansion of $f$, we obtain:

$$\begin{aligned}
m^{\frac{k-1}{2}} \int_{\Gamma_\infty\backslash\mathcal{H}} f(z) \overline{e(mz)} y^k \frac{dxdy}{y^2} &= m^{\frac{k-1}{2}} \int_0^\infty \int_0^1 \sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} e(nz) \overline{e(mz)} y^k \frac{dxdy}{y^2} \\
&= m^{\frac{k-1}{2}} \int_0^\infty \sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} e^{-2\pi(m+n)y} y^{k-2} \int_0^1 e((n-m)x) dxdy \\
&= m^{\frac{k-1}{2}} \int_0^\infty a(m) m^{\frac{k-1}{2}} e^{-4\pi m y} y^{k-2} dy \\
&= \frac{a(m)}{(4\pi)^{k-1}} \int_0^\infty t^{k-2} e^{-t} dt.
\end{aligned}$$

Noting that the integral represents $\Gamma(k-1) = (k-2)!$, we immediately deduce:

$$a(m) = \frac{(4\pi)^{k-1}}{(k-2)!} \langle f, P_m \rangle.$$

□

**Remark.** If $f \in S_k(\Gamma_0(N))$ is orthogonal to every Poincaré series $P_m \in S_k(\Gamma_0(N))$, then by the Petersson formula, all of the Fourier coefficients of $f$ must be zero, and so $f$ must be zero. As $S_k(\Gamma_0(N))$ is finite dimensional, it follows that the Poincaré series span $S_k(\Gamma_0(N))$.

## 2.3 Theta series

A second type of modular form that is of interest to us is a theta series. In this section, we introduce notation related to imaginary quadratic fields and state the definition of a theta series. In Proposition 2.14, we give a formula for slashing a theta series with any matrix in $\mathrm{SL}_2(\mathbf{Z})$. This is a generalization of [GZ86, Lemma IV.2.3]. Using this proposition, we deduce formulas for the Fourier coefficients of a theta series and its Atkin–Lehner translates (Proposition 2.15).

### Definitions

Let $D$ be a positive and square-free integer and consider the imaginary quadratic field $K = \mathbf{Q}(\sqrt{-D})$, viewed as a subfield of $\mathbf{C}$. We will assume that $D \equiv 3 \pmod 4$, so that $\Delta_K = -D$ and $\mathcal{O}_K = \mathbf{Z}[\frac{1+\sqrt{-D}}{2}]$. We will write $w = |\mathcal{O}_K^\times|$, $u = \frac{w}{2}$, $h = |\mathrm{Cl}_K|$ and $\mathrm{N}(\cdot) = \mathrm{N}_{K/\mathbf{Q}}(\cdot)$. Attached to this field is a Dirichlet character $\chi$ modulo $D$, that is given by the Kronecker symbol $\chi(\cdot) = \left(\frac{\Delta_K}{\cdot}\right)$. For a prime number $p$, it satisfies

$$\chi(p) = \begin{cases} 1 & \text{if } p \text{ is split,} \\ -1 & \text{if } p \text{ is inert,} \\ 0 & \text{if } p \text{ is ramified.} \end{cases}$$

If we let $r_K(n)$ denote the number of ideals in $\mathcal{O}_K$ of norm $n$, then

$$r_K(n) = \sum_{d|n} \chi(d).$$

In particular, we find that $|r_K(n)| \leq \sigma_0(n)$, where $\sigma_0(n)$ denotes the number of divisors of $n$. It is well known that for any $\varepsilon > 0$, we have a bound $\sigma_0(n) = O(n^\varepsilon)$ as $n \to \infty$. We deduce:

**Lemma 2.10.** *The number of ideals in $\mathcal{O}_K$ of norm $n$ satisfies the asymptotic bound $r_K(n) = O(n^\varepsilon)$ as $n \to \infty$ for any $\varepsilon > 0$.* □

We will write $I_K$ for the group of fractional ideals of $\mathcal{O}_K$. Let $\psi : I_K \to \mathbf{C}^\times$ be a homomorphism and assume that there exists a positive integer $t$ such that $\psi((\alpha)) = \alpha^{t-1}$ for all $\alpha \in K^\times$. Note that this forces $t$ to be odd (and $t \equiv 1 \pmod 6$ if $D = 3$). This is an example of an *unramified Hecke character of infinity type* $(t-1, 0)$. In this thesis, we will not consider other types of Hecke characters. For the definition of a Hecke character in general, see [Neu99, Section VII.6]. In the case that $t = 1$, $\psi$ maps principal ideals to 1 and thus can be considered as a homomorphism $\psi : \mathrm{Cl}_K \to \mathbf{C}^\times$. In this case, $\psi$ is a finite order character. Otherwise, when $t > 1$, $\psi$ has infinite order.

Let $\mathcal{A} \in \mathrm{Cl}_K$ be some ideal class. Then we define coefficients $r_{\mathcal{A},\psi}(n)$ for $n \geq 1$ by

$$r_{\mathcal{A},\psi}(n) = \sum_{\substack{\mathfrak{a} \in \mathcal{A} \\ \mathrm{N}(\mathfrak{a}) = n}} \psi(\mathfrak{a}).$$

We also define

$$r_{\mathcal{A},\psi}(0) = \begin{cases} \frac{\psi(\mathcal{A})}{w} & \text{if } t = 1, \\ 0 & \text{if } t > 1. \end{cases}$$

These coefficients are the (unnormalized) Fourier coefficients of a modular form

$$\theta_{\mathcal{A},\psi}(z) = r_{\mathcal{A},\psi}(0) + \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_K \\ [\mathfrak{a}]=\mathcal{A}}} \psi(\mathfrak{a}) e^{2\pi i \mathrm{N}(\mathfrak{a})z} = \sum_{n=0}^{\infty} r_{\mathcal{A},\psi}(n) e^{2\pi i n z} \in M_t(\Gamma_0(D),\chi),$$

which is a cusp form for $t > 1$. This can be proven using the Poisson summation formula. For the case $t = 1$, see [IK04, Section 14.3]. We call $\theta_{\mathcal{A},\psi}$ the theta series associated to $\mathcal{A}$ and $\psi$. Given some ideal $\mathfrak{a} \in \mathcal{A}$, we have a third representation of $\theta_{\mathcal{A},\psi}$ given by

$$\theta_{\mathcal{A},\psi}(z) = \frac{\psi(\mathfrak{a})}{w} \sum_{\lambda\in\mathfrak{a}^{-1}} \lambda^{t-1} e^{2\pi i \mathrm{N}(\lambda\mathfrak{a})z}, \tag{2.11}$$

as any integral ideal $\mathfrak{b} \in \mathcal{A}$ is of the form $\mathfrak{b} = \lambda\mathfrak{a}$ with $\lambda \in \mathfrak{a}^{-1}$, and $\psi(\lambda\mathfrak{a}) = \lambda^{t-1}\psi(\mathfrak{a})$.

**Remark.** The reason that we restrict ourselves to unramified Hecke characters is so that (2.11) holds. This equality plays a vital role in the proof of Proposition 2.14.

**Proposition 2.11.** *The coefficients $r_{\mathcal{A},\psi}(n)$ satisfy a bound of the form $r_{\mathcal{A},\psi}(n) = O(n^{\frac{t-1}{2}+\varepsilon})$ as $n \to \infty$ for any $\varepsilon > 0$.*

*Proof.* Note that

$$|r_{\mathcal{A},\psi}(n)| \le \sum_{\substack{\mathfrak{a}\in\mathcal{A} \\ \mathrm{N}(\mathfrak{a})=n}} |\psi(\mathfrak{a})| \le \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_K \\ \mathrm{N}(\mathfrak{a})=n}} |\psi(\mathfrak{a})| = n^{\frac{t-1}{2}} \cdot r_K(n).$$

Here we used that $|\psi(\mathfrak{a})|^2 = \psi(\mathrm{N}(\mathfrak{a})) = \mathrm{N}(\mathfrak{a})^{t-1}$. Now apply Lemma 2.10. ☐

### $\mathrm{SL}_2(\mathbf{Z})$-translates

We are interested in the behavior of slashing $\theta_{\mathcal{A},\psi}$ with arbitrary matrices in $\mathrm{SL}_2(\mathbf{Z})$. In the works of Gross–Zagier, they give the behavior for weight 1 theta series, see [GZ86, Lemma IV.2.3]. We will generalize this lemma in Proposition 2.14, but first we need a few definitions.

Given a decomposition $D = \delta_1 \cdot \delta_2$, we can also decompose $\Delta_K = D_1 \cdot D_2$ as a product of discriminants with $|D_i| = \delta_i$ and $D_i \equiv 1 \pmod 4$. Given such a decomposition, we can define a character $\chi_{D_1\cdot D_2} : I_K \to \mathbf{C}^\times$ on $\mathfrak{a} \subseteq \mathcal{O}_K$ by

$$\chi_{D_1\cdot D_2}(\mathfrak{a}) = \begin{cases} \chi_{D_1}(\mathrm{N}(\mathfrak{a})) & \text{if } \gcd(\mathrm{N}(\mathfrak{a}), D_1) = 1, \\ \chi_{D_2}(\mathrm{N}(\mathfrak{a})) & \text{if } \gcd(\mathrm{N}(\mathfrak{a}), D_2) = 1. \end{cases} \tag{2.12}$$

Here $\chi_{D_i}$ is the unique Dirichlet character modulo $|D_i|$ such that $\chi = \chi_{D_1} \cdot \chi_{D_2}$. It turns out that (2.12) yields a well-defined character, and that $\chi_{D_1\cdot D_2}(\mathfrak{a}) = 1$ for any principal ideal $\mathfrak{a}$. It follows that $\chi_{D_1\cdot D_2}$ induces a class group character, i.e., a map $\mathrm{Cl}_K \to \mathbf{C}^\times$. We will use the notation $\chi_{D_1\cdot D_2}$ for both the character on ideals and the character on classes, and we will also write $\chi_{\delta_1\cdot\delta_2}$ instead of $\chi_{D_1\cdot D_2}$, when it is more convenient to do so.

Given a divisor $\delta | D$, we define

$$\kappa(\delta) = \begin{cases} 1 & \text{if } \chi_\delta(-1) = 1 \\ i & \text{if } \chi_\delta(-1) = -1 \end{cases} = \begin{cases} 1 & \text{if } \delta \equiv 1 \pmod 4, \\ i & \text{if } \delta \equiv 3 \pmod 4. \end{cases}$$

We also need a result on Fourier transforms.

**Lemma 2.12.** *Let $a \in \mathcal{H}$, $t \in \mathbf{Z}_{\geq 0}$ and define $f_t : \mathbf{C} \to \mathbf{C}$ by $f_t(z) = z^t \exp(2\pi i |z|^2 a)$. Then the Fourier transform of $f_t$ is given by*

$$\hat{f}_t(z) = \frac{i}{(2a)^{t+1}} z^t \exp\left(-\frac{\pi i |z|^2}{2a}\right).$$

*Sketch of proof.* Reduce to the case $t = 0$ using [SW71, (1.9)]. Write the integral defining the Fourier transform as a product of two integrals and use the fact that the $e^{-\pi x^2}$ is its own Fourier transform. □

Using this fact, together with the Poisson summation formula, we can prove the following lemma.

**Lemma 2.13.** *Let $\mathfrak{b} \subseteq K$ be a fractional ideal. Let $\lambda \in K$, $a \in \mathcal{H}$ and $t \geq 1$ an integer. Then*

$$\sum_{\mu \in \mathfrak{b}} (\lambda + \mu)^{t-1} e\left(\mathrm{N}(\lambda + \mu) a\right) = \frac{i D^{-\frac{1}{2}}}{\mathrm{N}(\mathfrak{b}) a^t} \sum_{\nu \in \bar{\mathfrak{b}}^{-1} \mathfrak{d}^{-1}} \nu^{t-1} e\left(-\frac{\mathrm{N}(\nu)}{a}\right) e\left(\mathrm{Tr}(\bar{\lambda}\nu)\right),$$

*where we write $e(x)$ for $e^{2\pi i x}$.*

*Proof.* We interpret $L = \mathfrak{b}$ as a lattice in $\mathbf{C}$. The lattice dual to $L$ is given by

$$L^\vee = \{u \in \mathbf{R}^2 \mid \forall v \in L, \langle u, v \rangle \in \mathbf{Z}\},$$

where $\langle u, v \rangle$ is the standard scalar product on $\mathbf{R}^2$. This dual lattice can be identified with

$$L^\vee = \{u \in K \mid \forall v \in \mathfrak{b}, \mathrm{Tr}_{K/\mathbf{Q}}(u\bar{v}) \in 2\mathbf{Z}\} = 2\{u \in K \mid \mathrm{Tr}_{K/\mathbf{Q}}(u\bar{\mathfrak{b}}) \subseteq \mathbf{Z}\} = 2\bar{\mathfrak{b}}^{-1}\mathfrak{d}^{-1}.$$

Here $\mathfrak{d}$ is the different of $K$ and it is the unique ideal of norm $D$. As the co-volume of $\mathfrak{b}$ is given by $\frac{\sqrt{D}}{2} \cdot \mathrm{N}(\mathfrak{b})$, we conclude using the Poisson summation formula [SW71, Thm. VII.2.4] that

$$\begin{aligned}
\sum_{\mu \in \mathfrak{b}} (\lambda + \mu)^{t-1} e\left(\mathrm{N}(\lambda + \mu) a\right) &= \sum_{\mu \in \mathfrak{b}} f_{t-1}(\lambda + \mu) \\
&= \frac{2}{D^{\frac{1}{2}} \mathrm{N}(\mathfrak{b})} \sum_{\nu \in 2\bar{\mathfrak{b}}^{-1}\mathfrak{d}^{-1}} \hat{f}_{t-1}(\nu) \cdot e^{2\pi i \Re(\bar{\lambda}\nu)} \\
&= \frac{i}{D^{\frac{1}{2}} \mathrm{N}(\mathfrak{b}) 2^{t-1} a^t} \sum_{\nu \in 2\bar{\mathfrak{b}}^{-1}\mathfrak{d}^{-1}} \nu^{t-1} \exp\left(-\frac{\pi i \mathrm{N}(\nu)}{2a}\right) e\left(\Re(\bar{\lambda}\nu)\right) \\
&= \frac{i D^{-\frac{1}{2}}}{\mathrm{N}(\mathfrak{b}) a^t} \sum_{\nu \in \bar{\mathfrak{b}}^{-1}\mathfrak{d}^{-1}} \nu^{t-1} e\left(-\frac{\mathrm{N}(\nu)}{a}\right) e\left(\mathrm{Tr}(\bar{\lambda}\nu)\right),
\end{aligned}$$

□

We can now state and prove the generalization of [GZ86, Lemma IV.2.3]. Most of their proof generalizes nicely, but as they are able make a few simplifications that do not hold in the general case, we will write out the entire proof, except for the reduction step at the start. We follow their proof closely.

**Proposition 2.14.** *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$. Let $\delta_2 = \gcd(c, D)$ and $\delta_1 = D/\delta_2$. Let $\Delta_K = D_1 \cdot D_2$ be the corresponding decomposition of fundamental discriminants, i.e., $|D_i| = \delta_i$ and $D_i \equiv 1 \pmod 4$. Lastly, let $\mathcal{D}_i \in \mathrm{Cl}_K$ denote the class of the unique ideal $\mathfrak{d}_i \subseteq \mathcal{O}_K$ with $\mathfrak{d}_i^2 = (D_i)$. Then*

$$(\theta_{\mathcal{A},\psi}|_t\gamma)(z) = \frac{\chi_{D_1}(c/\delta_2)\chi_{D_2}(d)}{\delta_1^{\frac{1}{2}}\kappa(\delta_1)\psi(\mathfrak{d}_1)}\chi_{D_1\cdot D_2}(\mathcal{A})\theta_{\mathcal{A}\mathcal{D}_1,\psi}\left(\frac{z+\bar{c}d}{\delta_1}\right),$$

*where $\bar{c}$ is an inverse of $c$ modulo $\delta_1$.*

*Proof.* Analogous to what is shown in [GZ86, Lemma IV.2.3], we may reduce to the case $c = \delta_2$ and we will use $c$ and $\delta_2$ interchangeably.

Let $\zeta = -\frac{1}{c(cz+d)}$ and let $\mathfrak{a} \subseteq \mathcal{O}_K$ be some ideal with $[\mathfrak{a}] = \mathcal{A}$. Let $A = \mathrm{N}(\mathfrak{a})$. Then by (2.11),

$$\theta_{\mathcal{A},\psi}\left(\frac{az+b}{cz+d}\right) = \theta_{\mathcal{A},\psi}\left(\frac{a}{c}+\zeta\right) = \frac{\psi(\mathfrak{a})}{w}\sum_{\lambda\in\mathfrak{a}^{-1}}\lambda^{t-1}e\left(A\mathrm{N}(\lambda)\left(\frac{a}{c}+\zeta\right)\right),$$

where $e(x) = e^{2\pi i x}$ and $w = \#\mathcal{O}_K^\times$. For $\lambda \in \mathfrak{a}^{-1}$ and $\mu \in \mathfrak{a}^{-1}\mathfrak{d}_2$, note that

$$A\mathrm{N}(\lambda+\mu) = A\mathrm{N}(\lambda) + A\mathrm{N}(\mu) + A\mathrm{Tr}(\lambda\bar{\mu}),$$

that $\delta_2 | A\mathrm{N}(\mu)$ and that $\lambda\bar{\mu} \in \mathfrak{a}^{-1}\cdot\overline{\mathfrak{a}^{-1}\mathfrak{d}_2} = \frac{\delta_2}{A}\mathfrak{d}_2^{-1} \subseteq \frac{\delta_2}{A}\mathfrak{d}^{-1}$. As $\mathfrak{d}$ is the different of $K$, we find that $A\mathrm{Tr}(\lambda\bar{\mu}) \in \delta_2\mathbf{Z}$. It follows that $A\mathrm{N}(\lambda+\mu) \equiv A\mathrm{N}(\lambda) \pmod{\delta_2}$ for all $\mu \in \mathfrak{a}^{-1}\mathfrak{d}_2$, and so we can rewrite the sum as

$$\theta_{\mathcal{A},\psi}\left(\frac{az+b}{cz+d}\right) = \frac{\psi(\mathfrak{a})}{w}\sum_{\lambda\in\mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{d}_2} e\left(A\mathrm{N}(\lambda)\frac{a}{c}\right)\sum_{\mu\in\mathfrak{a}^{-1}\mathfrak{d}_2}(\lambda+\mu)^{t-1}e(A\mathrm{N}(\lambda+\mu)\zeta).$$

Using Lemma 2.13, we find that

$$\sum_{\mu\in\mathfrak{a}^{-1}\mathfrak{d}_2}(\lambda+\mu)^{t-1}e(A\mathrm{N}(\lambda+\mu)\zeta) = \frac{(-1)^t i\delta_2^{t-1}(cz+d)^t}{D^{\frac{1}{2}}A^{t-1}}\sum_{\nu\in\overline{\mathfrak{a}\mathfrak{d}_2^{-1}}\mathfrak{d}^{-1}}\nu^{t-1}e\left(\frac{\mathrm{N}(\nu)}{A}c(cz+d)\right)e(\mathrm{Tr}(\bar{\lambda}\nu)).$$

We note that $(-1)^t = -1$, because $t$ must be odd. As $\overline{\mathfrak{a}\mathfrak{d}_2^{-1}}\mathfrak{d}^{-1} = \delta_2^{-1}\cdot\bar{\mathfrak{a}}\mathfrak{d}_1^{-1}$ and $\bar{\mathfrak{d}}_1 = \mathfrak{d}_1$, we can substitute $\nu$ with $\delta_2^{-1}\bar{\nu}$ to obtain

$$\begin{aligned}(\theta_{\mathcal{A},\psi}|_t\gamma)(z) &= \frac{-\psi(\mathfrak{a})i}{wD^{\frac{1}{2}}A^{t-1}}\sum_{\lambda\in\mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{d}_2}e\left(A\mathrm{N}(\lambda)\frac{a}{c}\right)\sum_{\nu\in\mathfrak{a}\mathfrak{d}_1^{-1}}\bar{\nu}^{t-1}e\left(\frac{\mathrm{N}(\nu)}{A}\left(z+\frac{d}{c}\right)\right)e\left(\frac{\mathrm{Tr}(\lambda\nu)}{c}\right)\\ &= \frac{-\psi(\mathfrak{a})i}{wD^{\frac{1}{2}}A^{t-1}}\sum_{\nu\in\mathfrak{a}\mathfrak{d}_1^{-1}}\bar{\nu}^{t-1}C(\nu)e\left(\frac{\mathrm{N}(\nu)}{A}\left(z+\frac{d}{c}\right)\right),\end{aligned}$$

where

$$C(\nu) = \sum_{\lambda\in\mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{d}_2}e_c(aA\mathrm{N}(\lambda)+\mathrm{Tr}(\lambda\nu)) \qquad \text{with} \qquad e_c(x) = e\left(\frac{x}{c}\right).$$

Now choose $\lambda_0 \in \mathfrak{a}^{-1}$ such that $(\lambda_0)\mathfrak{a}$ and $\mathfrak{d}_2$ are coprime. Then there is a one to one correspondence between $\mathcal{O}_K/\mathfrak{d}_2$ and $\mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{d}_2$ given by $\mu \leftrightarrow \lambda_0\mu$, and so if we write $R = a \cdot A \cdot \mathrm{N}(\lambda_0)$, then

$$C(\nu) = \sum_{\mu \in \mathcal{O}_K/\mathfrak{d}_2} e_c(R\mathrm{N}(\mu) + \mathrm{Tr}(\lambda_0\mu\nu)).$$

The trace $\mathrm{Tr}(\lambda_0\mu\nu)$ lies in $\mathbf{Z}$, as $\lambda_0\mu\nu \in \mathfrak{d}_1^{-1} \subseteq \mathfrak{d}^{-1}$. Moreover, $R$ is prime to $\delta_2$. If we let $R^*$ denote an inverse of $R$ modulo $\delta_2$ that is divisible by $\delta_1$, then it follows that

$$\begin{aligned} e_c(R\mathrm{N}(\mu) + \mathrm{Tr}(\lambda_0\mu\nu)) &= e_c(R\mathrm{N}(\mu) + R\mathrm{Tr}(R^*\lambda_0\mu\nu)) \\ &= e_c(R\mathrm{N}(\mu + R^*\overline{\lambda_0\nu}) - R\mathrm{N}(R^*\lambda_0\nu)) \\ &= e_c(R\mathrm{N}(\mu + R^*\overline{\lambda_0\nu}) - R^*\mathrm{N}(\lambda_0\nu)), \end{aligned}$$

where we use the equality $\mathrm{N}(x+y) = \mathrm{N}(x) + \mathrm{N}(y) + \mathrm{Tr}(x\bar{y})$ for all $x, y \in K$. We deduce that

$$\begin{aligned} C(\nu) &= e_c(-R^*\mathrm{N}(\lambda_0\nu)) \sum_{\mu \in \mathcal{O}_K/\mathfrak{d}_2} e_c(R\mathrm{N}(\mu + R^*\overline{\lambda_0\nu})) \\ &= e_c(-R^*\mathrm{N}(\lambda_0\nu)) \sum_{\mu \in \mathcal{O}_K/\mathfrak{d}_2} e_c(R\mathrm{N}(\mu)), \end{aligned}$$

where we use that $R^*\overline{\lambda_0\nu} \in \mathcal{O}_K$, as $\lambda_0\nu \in \mathfrak{d}_1^{-1} = \delta_1^{-1} \cdot \mathfrak{d}_1$ and $\delta_1 | R^*$. Using $\mathbf{Z}/\delta_2\mathbf{Z}$ as representatives for $\mathcal{O}_K/\mathfrak{d}_2$, we obtain

$$\sum_{\mu \in \mathcal{O}_K/\mathfrak{d}_2} e_c\left(R\mathrm{N}(\mu)\right) = \sum_{n \in \mathbf{Z}/\delta_2\mathbf{Z}} e_c\left(Rn^2\right) = \kappa(\delta_2)\delta_2^{\frac{1}{2}}\chi_{D_2}(R) = \kappa(\delta_2)\delta_2^{\frac{1}{2}}\chi_{D_2}(d)\chi_{D_1\cdot D_2}(\mathcal{A}).$$

Here we used that $\chi_{D_2}(R) = \chi_{D_2}(d)\chi_{D_1\cdot D_2}(\mathcal{A})$, as $R = a\mathrm{N}(\lambda_0\mathfrak{a})$. Finally, note that

$$\begin{aligned} e_c(-R^*\mathrm{N}(\lambda_0\nu))e\left(\frac{\mathrm{N}(\nu)}{A}\left(z + \frac{d}{c}\right)\right) &= e\left(\frac{\mathrm{N}(\nu)}{A}\left(z + \frac{d - AR^*\mathrm{N}(\lambda_0)}{c}\right)\right) \\ &= e\left(\frac{\mathrm{N}(\nu)}{A}(z + \bar{c}d)\right), \end{aligned}$$

because $(d - AR^*\mathrm{N}(\lambda_0))/c \equiv \bar{c}d \pmod{\delta_1}$ and $\frac{\mathrm{N}(\nu)}{A} = \frac{r}{\delta_1}$ for some $r \in \mathbf{Z}_{\geq 0}$. As $\bar{\mathcal{A}}^{-1} = \mathcal{A}$ and $\kappa(\delta_1) \cdot \kappa(\delta_2) = i$, we conclude that

$$\begin{aligned} (\theta_{\mathcal{A},\psi}|_t\gamma)(z) &= \frac{\psi(\mathfrak{a}) \cdot -i}{wD^{\frac{1}{2}}A^{t-1}}\kappa(\delta_2)\delta_2^{\frac{1}{2}}\chi_{D_2}(d)\chi_{D_1\cdot D_2}(\mathcal{A}) \sum_{\nu \in \mathfrak{a}\mathfrak{d}_1^{-1}} \bar{\nu}^{t-1} e\left(\mathrm{N}(\mathfrak{a}^{-1}\mathfrak{d}_1)\mathrm{N}(\nu)\left(\frac{z + \bar{c}d}{\delta_1}\right)\right) \\ &= \frac{1}{\delta_1^{\frac{1}{2}}\kappa(\delta_1)}\chi_{D_2}(d)\chi_{D_1\cdot D_2}(\mathcal{A})\frac{\psi(\mathfrak{a})}{wA^{t-1}} \sum_{\nu \in \bar{\mathfrak{a}}\mathfrak{d}_1^{-1}} \nu^{t-1} e\left(\mathrm{N}(\bar{\mathfrak{a}}^{-1}\mathfrak{d}_1)\mathrm{N}(\nu)\left(\frac{z + \bar{c}d}{\delta_1}\right)\right) \\ &= \frac{1}{\delta_1^{\frac{1}{2}}\kappa(\delta_1)}\chi_{D_2}(d)\chi_{D_1\cdot D_2}(\mathcal{A})\frac{\psi(\mathfrak{a})\psi(\bar{\mathfrak{a}})}{\psi(\mathfrak{d}_1)A^{t-1}}\theta_{\bar{\mathcal{A}}^{-1}\mathcal{D}_1,\psi}\left(\frac{z + \bar{c}d}{\delta_1}\right) \\ &= \frac{\chi_{D_2}(d)}{\delta_1^{\frac{1}{2}}\kappa(\delta_1)\psi(\mathfrak{d}_1)}\chi_{D_1\cdot D_2}(\mathcal{A})\theta_{\mathcal{A}\mathcal{D}_1,\psi}\left(\frac{z + \bar{c}d}{\delta_1}\right). \end{aligned}$$

□

**Fourier coefficients**

We will denote the normalized Fourier coefficients of $\theta_{\mathcal{A},\psi}(z)$ by $b_{\mathcal{A},\psi}(n)$, so that

$$\theta_{\mathcal{A},\psi}(z) = b_{\mathcal{A},\psi}(0) + \sum_{n=1}^{\infty} b_{\mathcal{A},\psi}(n) n^{\frac{t-1}{2}} e^{2\pi i n z}.$$

Recall from (2.5) that we defined the Atkin–Lehner operator for a decomposition $D = \delta \cdot \delta'$ by

$$\theta_{\mathcal{A},\psi}^{\delta} = \theta_{\mathcal{A},\psi}|_t \omega_\delta \qquad \text{with} \qquad \omega_\delta = \begin{pmatrix} x & y \\ \delta' & \delta \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix},$$

where $x$ and $y$ are integers satisfying $x\delta - y\delta' = 1$. Moreover, we defined the coefficients $b_{\mathcal{A},\psi}^{\delta}(n)$ via

$$\theta_{\mathcal{A},\psi}^{\delta}(z) = b_{\mathcal{A},\psi}^{\delta}(0) + \sum_{n=1}^{\infty} b_{\mathcal{A},\psi}^{\delta}(n) n^{\frac{t-1}{2}} e^{2\pi i n z}.$$

**Proposition 2.15.** *The Fourier coefficients $b_{\mathcal{A},\psi}(n)$ and $b_{\mathcal{A},\psi}^{\delta}(n)$ as above satisfy the following properties:*

*(1)* $b_{\mathcal{A},\psi}(nD) = (-1)^{\frac{t-1}{2}} \cdot b_{\mathcal{A},\psi}(n)$ *for all* $n \geq 1$.

*(2)* $b_{\mathcal{A},\psi}(n\delta^2) = b_{\mathcal{A},\psi}(n)$ *for all* $n \geq 1$ *and* $\delta | D$.

*(3)* $b_{\mathcal{A},\psi}(n) = 0$ *if* $\mathcal{O}_K$ *has no ideals of norm* $n$.

*(4)* $b_{\mathcal{A},\psi}^{\delta}(n) = \kappa(\delta)^{-1}\chi_{\delta'}(\delta)\chi_{\delta\cdot\delta'}(\mathcal{A}) b_{\mathcal{A},\psi}(n\delta)$ *for all* $n \geq 1$ *and decompositions* $D = \delta \cdot \delta'$.

*(5)* $b_{\mathcal{A},\psi}^{\delta}(0) = \delta^{\frac{t-1}{2}} \kappa(\delta)^{-1}\chi_{\delta'}(\delta)\chi_{\delta\cdot\delta'}(\mathcal{A}) b_{\mathcal{A},\psi}(0)$ *for all decompositions* $D = \delta \cdot \delta'$.

*(6)* $b_{\mathcal{A},\psi}^{\delta}(n) = O(n^{\varepsilon})$ *as* $n \to \infty$, *for any* $\varepsilon > 0$ *and divisor* $\delta | D$.

*Proof.* (1) There is a unique ideal of norm $D$ given by $\mathfrak{d} = (\sqrt{-D})$. As $\psi(\mathfrak{d}) = (-D)^{\frac{t-1}{2}}$, it follows that

$$(nD)^{\frac{t-1}{2}} b_{\mathcal{A},\psi}(nD) = \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_K \\ [\mathfrak{a}] = \mathcal{A} \\ \mathrm{N}(\mathfrak{a}) = nD}} \psi(\mathfrak{a}) = (-D)^{\frac{t-1}{2}} \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_K \\ [\mathfrak{a}] = \mathcal{A} \\ \mathrm{N}(\mathfrak{a}) = n}} \psi(\mathfrak{a}) = (-nD)^{\frac{t-1}{2}} b_{\mathcal{A},\psi}(n).$$

(2) Note that $(\delta)$ is the unique ideal of norm $\delta^2$. As $\psi((\delta)) = \delta^{t-1}$, it follows that

$$(n\delta^2)^{\frac{t-1}{2}} b_{\mathcal{A},\psi}(n\delta^2) = \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_K \\ [\mathfrak{a}] = \mathcal{A} \\ \mathrm{N}(\mathfrak{a}) = n\delta^2}} \psi(\mathfrak{a}) = \delta^{t-1} \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_K \\ [\mathfrak{a}] = \mathcal{A} \\ \mathrm{N}(\mathfrak{a}) = n}} \psi(\mathfrak{a}) = (n\delta^2)^{\frac{t-1}{2}} b_{\mathcal{A},\psi}(n).$$

(3) Clear.

(4) Let $D = \delta \cdot \delta'$ be a decomposition and let $\Delta_K = D_1 \cdot D_2$ be the corresponding decomposition of fundamental discriminants. Recall that $\theta^{\delta}_{\mathcal{A},\psi}$ is defined as $\theta_{\mathcal{A},\psi}|_t w_\delta$, with $w_\delta = \left(\begin{smallmatrix} x & y \\ \delta' & \delta \end{smallmatrix}\right)\left(\begin{smallmatrix} \delta & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $x, y \in \mathbf{Z}$ are chosen such that $x\delta - y\delta' = 1$. We will write

$$\gamma_1 = \begin{pmatrix} x & y \\ \delta' & \delta \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}.$$

Applying Proposition 2.14 with $\gamma_1$ yields

$$(\theta_{\mathcal{A},\psi}|_t\gamma_1)(z) = \frac{\chi_{D_2}(\delta)\chi_{D_1\cdot D_2}(\mathcal{A})}{\delta^{\frac{1}{2}}\psi(\mathfrak{d}_1)\kappa(\delta)}\theta_{\mathcal{A}\mathcal{D}_1,\psi}\left(\frac{z}{\delta}\right),$$

where we write $\mathfrak{d}_1$ for the unique ideal of norm $\delta$ and $\mathcal{D}_1$ for the ideal class of $\mathfrak{d}_1$. It follows that

$$\theta^{\delta}_{\mathcal{A},\psi}(z) = (\theta_{\mathcal{A},\psi}|_t\gamma_1\gamma_2)(z) = \delta^{\frac{t}{2}}(\theta_{\mathcal{A},\psi}|_t\gamma_1)(\delta z) = \delta^{\frac{t-1}{2}}\frac{\chi_{D_2}(\delta)\chi_{D_1\cdot D_2}(\mathcal{A})}{\psi(\mathfrak{d}_1)\kappa(\delta)}\theta_{\mathcal{A}\mathcal{D}_1,\psi}(z). \tag{2.13}$$

Comparing coefficients, we find that

$$\begin{aligned} b^{\delta}_{\mathcal{A},\psi}(n) &= \delta^{\frac{t-1}{2}}\psi(\mathfrak{d}_1)^{-1}\chi_{D_2}(\delta)\kappa(\delta)^{-1}\chi_{D_1\cdot D_2}(\mathcal{A})b_{\mathcal{A}\mathcal{D}_1,\psi}(n) \\ &= \delta^{t-1}\psi(\mathfrak{d}_1)^{-2}\chi_{D_2}(\delta)\kappa(\delta)^{-1}\chi_{D_1\cdot D_2}(\mathcal{A})b_{\mathcal{A},\psi}(n\delta) \\ &= \chi_{D_2}(\delta)\kappa(\delta)^{-1}\chi_{D_1\cdot D_2}(\mathcal{A})b_{\mathcal{A},\psi}(n\delta). \end{aligned}$$

Here we used that $b_{\mathcal{A}\mathcal{D}_1,\psi}(n) = \delta^{\frac{t-1}{2}}\psi(\mathfrak{d}_1)^{-1}b_{\mathcal{A},\psi}(n\delta)$, as

$$\psi(\mathfrak{d}_1)r_{\mathcal{A}\mathcal{D}_1,\psi}(n) = \psi(\mathfrak{d}_1)\sum_{\substack{\mathfrak{a}\in\mathcal{A}\mathcal{D}_1 \\ \mathrm{N}(\mathfrak{a})=n}}\psi(\mathfrak{a}) = \sum_{\substack{\mathfrak{a}\in\mathcal{A}\mathcal{D}_1^2 \\ \mathrm{N}(\mathfrak{a})=n\delta}}\psi(\mathfrak{a}) = r_{\mathcal{A}\mathcal{D}_1^2,\psi}(n\delta) = r_{\mathcal{A},\psi}(n\delta).$$

(5) This follows from (2.13). In the only non-trivial case $t = 1$, note that the constant coefficient of $\theta_{\mathcal{A}\mathcal{D}_1,\psi}$ is $\frac{\psi(\mathcal{A}\mathcal{D}_1)}{w}$, so that the factor $\psi(\mathfrak{d}_1)$ cancels.

(6) Given a divisor $\delta|D$, we know by (2.13) that $\theta^{\delta}_{\mathcal{A},\psi}$ is, up to scaling, a theta series. Now use Proposition 2.11.

□

# 3 L-series

In this chapter, we will give the definitions and properties of various $L$-series. In Section 3.1, we consider the $L$-series associated to a Dirichlet character. We give its functional equation and state some results in the case that the character is associated to an imaginary quadratic field. In Section 3.2, we consider the (twisted) $L$-function associated to a modular form of square-free level. We prove that it can be extended meromorphically and that it has a functional equation. We also give a way to bound the $L$-function on vertical strips of the complex plane. Lastly, in Section 3.3, we define the Rankin–Selberg $L$-function of a pair of modular forms and illustrate Rankin's method.

## 3.1 Dirichlet L-series

Let $D \geq 1$ be an integer and let $\chi$ be a Dirichlet character modulo $D$. We define its associated $L$-series $L(\chi, s)$ by

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

It converges absolutely for $\Re(s) > 1$ and defines an analytic function there. It has a meromorphic continuation to the complex plane and is entire if $\chi$ is not principal. In the case that $D = 1$, $\chi$ must be the trivial character, and it follows that $L(\chi, s)$ is given by $\zeta(s)$, the Riemann zeta function. For $\Re(s) > 1$, $L(\chi, s)$ satisfies the identity

$$L(\chi, s) = \prod_{p \text{ prime}} \frac{1}{1 - \chi(p)p^{-s}}. \tag{3.1}$$

In particular, $L(\chi, s)$ does not vanish for $\Re(s) > 1$. Given some integer $N \geq 1$, we define

$$L^{(N)}(\chi, s) = \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} \frac{\chi(n)}{n^s}. \tag{3.2}$$

Note that this is the Dirichlet $L$-series of $\chi$ lifted to a higher modulus. By (3.1), it now follows that

$$L^{(N)}(\chi, s) = L(\chi, s) \prod_{\substack{p \text{ prime} \\ p \mid N}} (1 - \chi(p)p^{-s}). \tag{3.3}$$

We also have for $\Re(s) > 1$ that

$$\frac{1}{L^{(N)}(\chi, s)} = \prod_{\substack{p \text{ prime} \\ p \nmid N}} (1 - \chi(p)p^{-s}) = \prod_{\substack{p \text{ prime} \\ p \nmid N}} \sum_{n=0}^{\infty} \mu(p^n)\chi(p^n)p^{-ns} = \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} \frac{\mu(n)\chi(n)}{n^s}. \tag{3.4}$$

### Functional equation

Define the Gauss sum

$$\tau(\chi) = \sum_{r \in (\mathbf{Z}/D\mathbf{Z})^\times} \chi(r) e^{2\pi i \frac{r}{D}}, \tag{3.5}$$

and take $\epsilon \in \{0, 1\}$ such that $\chi(-1) = (-1)^\epsilon$. If $\chi$ is primitive, $L(\chi, s)$ satisfies the following functional equation.

**Theorem 3.1.** *Let $\chi$ be a primitive Dirichlet character modulo $D$. Then*

$$L(\chi, s) = \frac{\tau(\chi)}{i^{\epsilon}\sqrt{D}} \frac{(2\pi)^s}{\pi} D^{\frac{1}{2}-s} \sin(\tfrac{\pi}{2}(s+\epsilon))\Gamma(1-s)L(\bar{\chi}, 1-s).$$

*Proof.* This follows from [Neu99, Thm. VII.2.8]. Their theorem is stated in terms of the completed Dirichlet $L$-series. Unpacking the definition and using that

$$\frac{\Gamma(\frac{s+\epsilon}{2})}{\Gamma(\frac{1+\epsilon-s}{2})} = \frac{2^s}{\sqrt{\pi}} \sin(\tfrac{\pi}{2}(s+\epsilon))\Gamma(1-s),$$

proves this functional equation. □

In particular, by taking $D = 1$ we obtain the functional equation for $\zeta(s)$:

$$\zeta(s) = \frac{(2\pi)^s}{\pi} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s). \tag{3.6}$$

### Imaginary quadratic fields

We will now consider the special case where $\chi$ is the real character associated to some imaginary quadratic field $K$ of discriminant $-D$, as in Section 2.3. As $\chi$ is primitive, $L(\chi, s)$ can be continued analytically to the entire complex plane. We can explicitly give its value at $s = 1$ in terms of invariants of $K$.

**Proposition 3.2.** *Let $K$ be an imaginary quadratic field with discriminant $-D$. Let $w = \#\mathcal{O}_K^{\times}$ be the number of units in $\mathcal{O}_K$, let $h = \#\mathrm{Cl}_K$ be the class number of $K$ and let $\chi$ be the character associated to $K$. Then it holds that*

$$L(\chi, 1) = \frac{2\pi h}{w\sqrt{D}}.$$

*Proof.* Let $\zeta_K$ denote the zeta function of $K$. Then $\zeta_K(s) = \zeta(s) \cdot L(\chi, s)$, see [BS66, p. 343]. As $\zeta(s)$ has a simple pole with residue 1 at $s = 1$, it follows that

$$L(\chi, 1) = \lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2\pi h}{w\sqrt{D}},$$

as is given by the class number formula, see [BS66, p. 313]. □

We can also give a simplified functional equation.

**Theorem 3.3.** *Let $\chi$ be the Dirichlet character modulo $D$ associated to an imaginary quadratic field of discriminant $-D$. Then*

$$L(\chi, s) = \frac{(2\pi)^s}{\pi} D^{\frac{1}{2}-s} \sin(\tfrac{\pi}{2}(s+1))\Gamma(1-s)L(\chi, 1-s).$$

*Proof.* By [Miy06, Lemma 4.8.1], we know that $\tau(\chi) = i\sqrt{D}$. As $\chi$ is real, we have $\bar{\chi} = \chi$. Lastly, note that $\chi(-1) = -1$. Now use Theorem 3.1. □

Using the functional equation, we can deduce that the only zeros of $L(\chi, s)$ in the region $\Re(s) < 0$ are at the negative odd integers.

**Proposition 3.4.** *Let $\chi$ be the Dirichlet character modulo $D$ associated to an imaginary quadratic field of discriminant $-D$. Then the only zeros of $L(\chi, s)$ for $\Re(s) < 0$ are the points $s = -(2n+1)$ with $n \in \mathbf{Z}_{\geq 0}$.*

*Proof.* Consider the functional equation in Theorem 3.3. By (3.1), we know that $L(1-s, \chi)$ does not vanish for $\Re(s) < 0$. Moreover, $\Gamma(1-s)$ does not have any zeros or poles for $\Re(s) < 0$. The proposition now follows from the fact that the only zeros of $\sin(\frac{\pi}{2}(s+1))$ are at the odd integers. □

## 3.2 L-series of modular forms

In this section, we will define the twisted $L$-series $L_g(s, \frac{a}{c})$ associated to a modular form $g \in M_\ell(\Gamma_0(D), \chi)$. In Proposition 3.5, we will show that $L_g(s, \frac{a}{c})$ can be extended meromorphically to the complex plane (holomorphically if $g$ is a cusp form) and that it satisfies a functional equation relating $L_g(s, \frac{a}{c})$ to the twisted $L$-series of an Atkin–Lehner translate of $g$. We conclude the section with Proposition 3.8 by giving a bound on $L_g(s, \frac{a}{c})$ on certain vertical strips of the complex plane that is polynomial in $\Im(s)$.

### Definitions

Let $g \in M_\ell(\Gamma_0(D), \chi)$ be a modular form with $D$ a square-free integer. Consider its Fourier expansion given by

$$g(z) = b(0) + \sum_{n=1}^{\infty} b(n) n^{\frac{\ell-1}{2}} e^{2\pi i n z}.$$

Then we can associate an $L$-series to $g$ given by

$$L_g(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}. \tag{3.7}$$

If we have a bound of the form $b(n) = O(n^\gamma)$ as $n \to \infty$ for some $\gamma > 0$, then the sum converges absolutely for $\Re(s) > 1 + \gamma$ and defines an analytic function there. By Lemma 2.2, we can always take any $\gamma > \frac{\ell-1}{2}$. More generally, given integers $a$ and $c$ with $\gcd(a, c) = 1$, we can consider the twisted $L$-series

$$L_g(s, \tfrac{a}{c}) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s} e^{2\pi i n \frac{a}{c}}. \tag{3.8}$$

This also defines an analytic function on $\Re(s) > 1 + \gamma$ and only depends on the class of $a \pmod c$.

### Continuation and functional equation

In this subsection, we will give the functional equation for $L_g(s, \frac{a}{c})$ and show that $L_g(s, \frac{a}{c})$ can be continued meromorphically to the complex plane. The functional equation involves the Atkin–Lehner operator, which is defined for a decomposition $D = \delta \cdot \delta'$ by (2.5) as

$$g^\delta = g|_\ell \omega_\delta \qquad \text{with} \qquad \omega_\delta = \begin{pmatrix} x & y \\ \delta' & \delta \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix},$$

where $x$ and $y$ are integers such that $x\delta - y\delta' = 1$. We follow the approach in [GZ99, Prop. 4.2].

**Proposition 3.5.** *Let $g \in M_\ell(\Gamma_0(D), \chi)$ with $D$ square-free. Let $a, c \in \mathbf{Z}$ with $\gcd(a,c) = 1$. Decompose $D$ as $D = \delta \cdot \delta'$, where $\delta' = \gcd(D, c)$. Then:*

*(1) The function $L_g(s, \frac{a}{c})$ can be extended meromorphically to the complex plane and its only possible pole is a simple pole at $s = \frac{\ell+1}{2}$ with residue*

$$\mathrm{Res}_{\frac{\ell+1}{2}} \left( L_g \left( s, \frac{a}{c} \right) \right) = \frac{(2\pi)^\ell}{\Gamma(\ell)} \chi_\delta \left( \frac{c}{\delta'} \right) \chi_{\delta'}^{-1}(a\delta) \delta^{-\frac{\ell}{2}} i^\ell c^{-\ell} b^\delta(0).$$

*(2) The function $L_g(s, \frac{a}{c})$ satisfies the functional equation*

$$L_g \left( s, \frac{a}{c} \right) = i^\ell \chi_\delta \left( \frac{c}{\delta'} \right) \chi_{\delta'}^{-1}(a\delta) \left( \frac{\delta c^2}{4\pi^2} \right)^{\frac{1}{2} - s} \frac{\Gamma(\frac{\ell+1}{2} - s)}{\Gamma(\frac{\ell-1}{2} + s)} L_{g^\delta} \left( 1 - s, -\frac{\overline{a\delta}}{c} \right),$$

*where we let $\overline{a\delta}$ denote the inverse of $a\delta \pmod c$.*

The proof of the proposition involves the Mellin transform

$$\int_0^\infty \left[ g\left( \frac{a}{c} + iy \right) - b(0) \right] y^{\frac{\ell-1}{2}+s} \frac{dy}{y}.$$

In order to manipulate this integral, we will need the following lemma.

**Lemma 3.6.** *Let $g \in M_\ell(\Gamma_0(D), \chi)$ with $D$ square-free. Let $a, c \in \mathbf{Z}$ with $\gcd(a,c) = 1$ and decompose $D = \delta \cdot \delta'$ where $\gcd(c, D) = \delta'$. Then for all real $y > 0$, it holds that*

$$g\left( \frac{a}{c} + iy \right) = i^\ell (cy)^{-\ell} \chi_\delta \left( \frac{c}{\delta'} \right) \chi_{\delta'}^{-1}(a\delta) \delta^{-\frac{\ell}{2}} g^\delta \left( \frac{-\overline{a\delta}}{c} + \frac{i}{c^2 \delta y} \right),$$

*where we let $\overline{a\delta}$ denote the inverse of $a\delta \pmod c$.*

*Proof.* Let $x, y \in \mathbf{Z}$ be integers such that $x\delta - y\delta' = 1$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ with $\delta | d$ and set

$$\gamma' = \gamma \begin{pmatrix} x & y \\ \delta' & \delta \end{pmatrix}^{-1} = \begin{pmatrix} a\delta - b\delta' & -ay + bx \\ c\delta - d\delta' & -cy + dx \end{pmatrix} \in \Gamma_0(D).$$

Now observe that

$$\gamma = \gamma' \begin{pmatrix} x & y \\ \delta' & \delta \end{pmatrix} = \gamma' \omega_\delta \begin{pmatrix} \delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

If we use $\delta | d$, $\delta' | c$, $-y\delta' \equiv 1 \pmod \delta$, $x\delta \equiv 1 \pmod{\delta'}$ and $ad \equiv 1 \pmod c$, we find that

$$\chi(-cy + dx) = \chi_\delta(-cy + dx) \chi_{\delta'}(-cy + dx) = \chi_\delta(-cy) \chi_{\delta'}(dx) = \chi_\delta \left( \frac{c}{\delta'} \right) \chi_{\delta'}^{-1}(a\delta),$$

and thus

$$\begin{aligned}(g|_\ell \gamma)(z) = (g|_\ell \gamma' \omega_\delta \left(\begin{smallmatrix} \delta^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right))(z) &= \delta^{-\frac{\ell}{2}} \chi(-cy + dx) \cdot (g|_\ell \omega_\delta) \left( \frac{z}{\delta} \right) \\ &= \chi(-cy + dx) \delta^{-\frac{\ell}{2}} g^\delta \left( \frac{z}{\delta} \right) = \chi_\delta \left( \frac{c}{\delta'} \right) \chi_{\delta'}^{-1}(a\delta) \delta^{-\frac{\ell}{2}} g^\delta \left( \frac{z}{\delta} \right). \quad (3.9)\end{aligned}$$

Next, note that $\gamma z = \frac{a}{c} - \frac{1}{c(cz+d)}$. Combined with (3.9), this yields the following equality:

$$(cz+d)^{-\ell} g\left(\frac{a}{c} - \frac{1}{c(cz+d)}\right) = \chi_\delta\left(\frac{c}{\delta'}\right)\chi_{\delta'}^{-1}(a\delta)\delta^{-\frac{\ell}{2}} g^\delta\left(\frac{z}{\delta}\right).$$

We substitute $z = -\frac{1}{c^2 w} - \frac{d}{c}$, and thus $w = -\frac{1}{c(cz+d)}$, to obtain

$$(-cw)^\ell g\left(\frac{a}{c} + w\right) = \chi_\delta\left(\frac{c}{\delta'}\right)\chi_{\delta'}^{-1}(a\delta)\delta^{-\frac{\ell}{2}} g^\delta\left(-\frac{1}{c^2\delta w} - \frac{d}{c\delta}\right).$$

Taking $w = iy$ and noting that $\frac{d}{\delta} \equiv (a\delta)^{-1} \pmod{c}$ completes the proof. □

We can now prove Proposition 3.5.

*Proof of Proposition 3.5.* Consider the integral

$$L_g^*\left(s, \frac{a}{c}\right) = \int_0^\infty \left[g\left(\frac{a}{c} + iy\right) - b(0)\right] y^{\frac{\ell-1}{2}+s}\frac{dy}{y}.$$

This is the Mellin transform of the function $y \mapsto (g(\frac{a}{c} + iy) - b(0))y^{\frac{\ell-1}{2}}$. Using Lemma 3.6 and the bound $|g(\frac{a}{c} + iy) - b(0)| = O(\exp(-2\pi y))$ as $y \to \infty$, it follows that the integral converges absolutely and uniformly on compact subsets of $\{s \in \mathbf{C} \mid \Re(s) > \frac{\ell+1}{2}\}$. For such $s$, we have

$$\begin{aligned} L_g^*\left(s, \frac{a}{c}\right) &= \sum_{n=1}^\infty b(n) n^{\frac{\ell-1}{2}} e^{2\pi i n\frac{a}{c}} \int_0^\infty e^{-2\pi n y} y^{\frac{\ell-1}{2}+s}\frac{dy}{y} \\ &= \sum_{n=1}^\infty \frac{b(n)}{(2\pi)^{\frac{\ell-1}{2}+s}\cdot n^s} e^{2\pi i n\frac{a}{c}} \int_0^\infty e^{-y} y^{\frac{\ell-1}{2}+s}\frac{dy}{y} \\ &= \frac{\Gamma(\frac{\ell-1}{2}+s)}{(2\pi)^{\frac{\ell-1}{2}+s}} L_g\left(s, \frac{a}{c}\right). \end{aligned} \tag{3.10}$$

Therefore, if we can show that $L_g^*(s, \frac{a}{c})$ can be extended meromorphically and has a functional equation, the corresponding properties for $L_g(s, \frac{a}{c})$ will follow. We can rewrite the integral defining $L_g^*\left(s, \frac{a}{c}\right)$ as

$$\int_0^{\frac{1}{c\sqrt{\delta}}} \left[g\left(\frac{a}{c} + iy\right) - b(0)\right] y^{\frac{\ell-1}{2}+s}\frac{dy}{y} + \int_{\frac{1}{c\sqrt{\delta}}}^\infty \left[g\left(\frac{a}{c} + iy\right) - b(0)\right] y^{\frac{\ell-1}{2}+s}\frac{dy}{y}.$$

We compute the left integral:

$$\int_0^{\frac{1}{c\sqrt{\delta}}} \left[g\left(\frac{a}{c} + iy\right) - b(0)\right] y^{\frac{\ell-1}{2}+s}\frac{dy}{y} = \int_0^{\frac{1}{c\sqrt{\delta}}} g\left(\frac{a}{c} + iy\right) y^{\frac{\ell-1}{2}+s}\frac{dy}{y} - b(0)(c\sqrt{\delta})^{-s-\frac{\ell-1}{2}}\frac{1}{s+\frac{\ell-1}{2}},$$

and using Lemma 3.6, and abbreviating $a' = -\overline{a\delta}$ and $A = \chi_\delta\left(\frac{c}{\delta'}\right)\chi_{\delta'}^{-1}(a\delta)\delta^{-\frac{\ell}{2}}$, we obtain

$$\begin{aligned} \int_0^{\frac{1}{c\sqrt{\delta}}} g\left(\frac{a}{c} + iy\right) y^{\frac{\ell-1}{2}+s}\frac{dy}{y} &= \int_0^{\frac{1}{c\sqrt{\delta}}} \left[i^\ell (cy)^{-\ell} A g^\delta\left(\frac{a'}{c} + \frac{i}{c^2\delta y}\right)\right] y^{s+\frac{\ell-1}{2}}\frac{dy}{y} \\ &= A i^\ell c^{-\ell} \int_{\frac{1}{c\sqrt{\delta}}}^\infty \left[g^\delta\left(\frac{a'}{c} + iy\right) - b^\delta(0)\right](c^2\delta y)^{\frac{\ell+1}{2}-s}\frac{dt}{y} + A i^\ell c^{-\ell} b^\delta(0)(c\sqrt{\delta})^{-s+\frac{\ell+1}{2}}\frac{1}{s-\frac{\ell+1}{2}}. \end{aligned}$$

Here we used the substitution $y \to \frac{1}{c^2\delta y}$ for the last equality. We conclude that

$$\begin{aligned} L_g^*\left(s, \frac{a}{c}\right) = &\int_{\frac{1}{c\sqrt{\delta}}}^{\infty} \left[g\left(\frac{a}{c} + iy\right) - b(0)\right] y^{\frac{\ell-1}{2}+s}\frac{dy}{y} \\ &+ Ai^{\ell}c^{1-2s}\delta^{\frac{\ell+1}{2}-s}\int_{\frac{1}{c\sqrt{\delta}}}^{\infty} \left[g^{\delta}\left(\frac{a'}{c} + iy\right) - b^{\delta}(0)\right] y^{\frac{\ell+1}{2}-s}\frac{dy}{y} \\ &+ Ai^{\ell}c^{-\ell}b^{\delta}(0)(c\sqrt{\delta})^{-s+\frac{\ell+1}{2}}\frac{1}{s-\frac{\ell+1}{2}} - b(0)(c\sqrt{\delta})^{-s-\frac{\ell-1}{2}}\frac{1}{s+\frac{\ell-1}{2}}. \end{aligned} \tag{3.11}$$

These integrals converge absolutely and uniformly on compact subsets of $\mathbf{C}$, as we have bounds of the form $\left|g\left(\frac{a}{c}+iy\right) - b(0)\right| = O\left(\exp(-2\pi y)\right)$ as $y \to \infty$. It follows immediately that $L_g^*\left(s, \frac{a}{c}\right)$ defines a meromorphic function on $\mathbf{C}$, with simple poles at $s = \frac{\ell+1}{2}$ and $s = \frac{1-\ell}{2}$ if $g$ is not a cusp form. As $\Gamma(\frac{\ell-1}{2}+s)$ has a simple pole at $s = \frac{1-\ell}{2}$, we find by (3.10) that $L_g\left(s, \frac{a}{c}\right)$ will not have a pole at $s = \frac{1-\ell}{2}$. This proves (1).

A similar computation for $L_{g^\delta}^*(1-s, \frac{a'}{c})$ shows that

$$\begin{aligned} L_{g^\delta}^*\left(1-s, \tfrac{a'}{c}\right) = &\int_{\frac{1}{c\sqrt{\delta}}}^{\infty} \left[g^{\delta}\left(\frac{a'}{c} + iy\right) - b^{\delta}(0)\right] y^{\frac{\ell+1}{2}-s}\frac{dy}{y} \\ &+ A^{-1}i^{-\ell}c^{2s-1}\delta^{s-\frac{\ell+1}{2}}\int_{\frac{1}{c\sqrt{d}}}^{\infty} \left[g\left(\frac{a}{c} + iy\right) - b(0)\right] y^{s+\frac{\ell-1}{2}}\frac{dy}{y} \\ &- A^{-1}i^{-\ell}c^{\ell}b(0)(c\sqrt{\delta})^{s-\frac{3\ell+1}{2}} \cdot \frac{1}{s+\frac{\ell-1}{2}} + b^{\delta}(0)(c\sqrt{\delta})^{s-\frac{\ell+1}{2}} \cdot \frac{1}{s-\frac{\ell+1}{2}}. \end{aligned}$$

We conclude that

$$L_g^*\left(s, \frac{a}{c}\right) = i^{\ell}\chi_\delta\left(\frac{c}{\delta'}\right)\chi_{\delta'}^{-1}(a\delta)(\delta c^2)^{\frac{1}{2}-s}L_{g^\delta}^*\left(1-s, \frac{a'}{c}\right),$$

and from this the functional equation for $L_g(s, \frac{a}{c})$ follows. □

### Polynomial bounds

Using the functional equation, we can now prove that the $L$-function attached to $g$ is polynomially bounded on certain vertical strips of the complex plane. We will make use of the Phragmen–Lindelöf principle, see [IK04, Thm. 5.53] or [Tit39, Section 5.65].

**Theorem 3.7** (Phragmen–Lindelöf Principle)**.** *Let $a < b$ be real numbers and $f$ be a function that is holomorphic on an open neighborhood of a strip $a \le \sigma \le b$. Suppose that $|f(\sigma+it)| \ll \exp(|\sigma+it|^A)$ for some $A \ge 0$ and all $a \le \sigma \le b$, and that there are real constants $M_a, M_b, \alpha$ and $\beta$ such that*

$$|f(a+it)| \le M_a(1+|t|)^{\alpha},$$
$$|f(b+it)| \le M_b(1+|t|)^{\beta},$$

*for all $t \in \mathbf{R}$. Then*

$$|f(\sigma+it)| \le M_a^{\lambda(\sigma)}M_b^{1-\lambda(\sigma)}(1+|t|)^{\alpha\lambda(\sigma)+\beta(1-\lambda(\sigma))},$$

*for all $s = \sigma+it$ in the strip, where $\lambda$ is the linear function such that $\lambda(a) = 1$ and $\lambda(b) = 0$.* □

In other words, given a mild bound on $f$ on the strip and a polynomial bound on the boundaries of the strip, a polynomial bound on the entire strip is obtained. We will apply this theorem to $L_g(s, \frac{r}{c})$, but we have to deal with the fact that the $L$-function might have a pole at $s = \frac{\ell+1}{2}$.

**Proposition 3.8.** *Let $g \in M_\ell(\Gamma_0(D), \chi)$ and let $r, c \in \mathbf{Z}$ with $\gcd(r, c) = 1$. Decompose $D = \delta \cdot \delta'$ with $\delta' = \gcd(c, D)$. Let $a, b \in \mathbf{R}$ such that $L_{g^\delta}(1 - s)$ converges absolutely for $\Re(s) < a$ and such that $L_g(s)$ converges absolutely for $\Re(s) > b$. Let $q$ denote the residue, which may be zero, of $L_g(s, \frac{r}{c})$ at $s = \frac{\ell+1}{2}$. Then there exists some constant $M$ such that for all $s = \sigma + it \in \mathbf{C}$ with $a \leq \sigma \leq b$, we have the bound*

$$\left| L_g\left(s, \frac{r}{c}\right) - \frac{q}{s - \frac{\ell+1}{2}} \right| \leq M \cdot (1 + |t|)^{(1-2a)\cdot\lambda(\sigma)},$$

*where $\lambda$ is the linear function with $\lambda(a) = 1$ and $\lambda(b) = 0$.*

*Proof.* Write

$$f(s) = L_g\left(s, \frac{r}{c}\right) - \frac{q}{s - \frac{\ell+1}{2}}.$$

This is a holomorphic function on the entire complex plane. We will apply the Phragmen–Lindelöf principle (Theorem 3.7) to $f$. We first note that $|L_g(b + it, \frac{r}{c})| \ll 1$ as $|t| \to \infty$, because the $L$-series converges absolutely there by assumption. As the quotient $q/(s - \frac{\ell+1}{2})$ is bounded away from its pole, it follows that $|f(b + it)| \ll 1$ as $|t| \to \infty$. We similarly have that $|L_{g^\delta}(1 - a - it, -\frac{\overline{r\delta}}{c})| \ll 1$ as $|t| \to \infty$, and so by the functional equation (Proposition 3.5) and Sterling's approximation (2.8), we see that

$$|L_g(a + it, \tfrac{r}{c})| \ll \left(\frac{\delta c^2}{4\pi^2}\right)^{\frac{1}{2}-a} \frac{\sqrt{2\pi} e^{-|t|\pi/2} |t|^{\frac{\ell}{2}-a}(1 + o(1))}{\sqrt{2\pi} e^{-|t|\pi/2} |t|^{\frac{\ell}{2}+a-1}(1 + o(1))} \ll |t|^{1-2a} \qquad \text{as} \quad |t| \to \infty.$$

It follows that $|f(a + it)| \ll |t|^{1-2a}$ as $|t| \to \infty$. We can thus find a constant $M$ such that for all $t \in \mathbf{R}$:

$$|f(a + it)| \leq M(1 + |t|)^{1-2a},$$
$$|f(b + it)| \leq M.$$

Lastly, we need to show that $|f(\sigma + it)| \ll \exp(|\sigma + it|^A)$ for all $a \leq \sigma \leq b$ for some constant $A$. As $q/(s - \frac{\ell+1}{2})$ is bounded for $|s|$ large, it suffices to give such a bound for $L_g(s, \frac{r}{c})$ for $|s|$ large. Recall that

$$L_g\left(s, \frac{r}{c}\right) = \frac{(2\pi)^{\frac{\ell-1}{2}+s}}{\Gamma(\frac{\ell-1}{2} + s)} L_g^*\left(s, \frac{r}{c}\right).$$

By the integral representation (3.11) for $L_g^*(s, \frac{r}{c})$ and Sterling's approximation (2.8) for $\Gamma(s)$, we conclude that $L_g(\sigma + it, \frac{r}{c}) \ll \exp(|\sigma + it|^A)$ for a suitable constant $A$. We can now apply the Phragmen–Lindelöf principle to conclude that $|f(\sigma + it)| \leq M(1 + |t|)^{(1-2a)\lambda(\sigma)}$ for all $a \leq \sigma \leq b$ and $t \in \mathbf{R}$. ☐

**Remark.** In general, one may take any $a < \frac{1-\ell}{2}$ and $b > \frac{\ell+1}{2}$ by Lemma 2.2. In the specific case that $g$ is a theta series, we can use Proposition 2.15 (6) and take any $a < 0$ and $b > 1$.

## 3.3 Rankin–Selberg L-series

In this section, we define the Rankin–Selberg $L$-series $L(f \otimes g, s)$ associated to two modular forms. By varying $f$ while keeping $g$ fixed, we obtain a linear map $S_k(\Gamma_0(N)) \to \mathbf{C}$. Via the Petersson inner product, this linear map is induced by a holomorphic kernel $\Phi_{s,g} \in S_k(\Gamma_0(N))$, which is the central object of interest in this thesis. We illustrate Rankin's method, which was used by Gross and Zagier to obtain their formulas for the Fourier coefficients of the holomorphic kernel [GZ86].

### Definition

Let $k, \ell, D$ and $N$ be positive integers and let $\chi$ be a Dirichlet character modulo $D$. Consider two modular forms $f \in S_k(\Gamma_0(N))$ and $g \in M_\ell(\Gamma_0(D), \chi)$, with Fourier expansions

$$f(z) = \sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} e^{2\pi i n z},$$

and

$$g(z) = b(0) + \sum_{n=1}^{\infty} b(n) n^{\frac{\ell-1}{2}} e^{2\pi i n z}.$$

We then define the Rankin–Selberg $L$-function for $\Re(s) > \frac{\ell+1}{2}$ by

$$L(f \otimes g, s) = \sum_{n=1}^{\infty} \frac{a(n)b(n)}{n^s}.$$

By Lemma 2.2, we have bounds $a(n) = O(n^{\varepsilon})$ and $b(n) = O(n^{\frac{\ell-1}{2}+\varepsilon})$ for any $\varepsilon > 0$, and so it follows that $L(f \otimes g, s)$ converges absolutely and uniformly on compact subsets of $\{s \in \mathbf{C} \mid \Re(s) > \frac{\ell+1}{2}\}$ and defines a holomorphic function there. This $L$-function has a meromorphic continuation to the entire complex plane and satisfies a functional equation [Li79].

### Holomorphic kernel

Fix some modular form $g \in M_\ell(\Gamma_0(D), \chi)$ and a point $s \in \mathbf{C}$ that is not a pole of $L(f \otimes g, s)$ for any $f \in S_k(\Gamma_0(N))$. Then we have a linear map

$$\begin{aligned} S_k(\Gamma_0(N)) &\to \mathbf{C} \\ f &\mapsto L(f \otimes g, s). \end{aligned}$$

As the Petersson inner product is non-degenerate, we can find some modular form $\Phi_{\bar{s},g} \in S_k(\Gamma_0(N))$ such that

$$L(f \otimes g, s) = \langle f, \Phi_{\bar{s},g} \rangle \qquad \text{for all } f \in S_k(\Gamma_0(N)).$$

We refer to $\Phi_{\bar{s},g}$ as the holomorphic kernel (associated to $g$). By understanding $\Phi_{s,g}$, it is possible to prove properties about $L(f \otimes g, s)$. For example, a functional equation for $\Phi_{s,g}$ in terms of $s$ yields a functional equation for $L(f \otimes g, s)$, as we will show in Section 4.6. Naturally, we are interested in formulas for the Fourier coefficients of the holomorphic kernel. In the next chapter, we will use the Goldfeld–Zhang method for this. In this section, we will illustrate the alternative Rankin's method, which was used by Gross and Zagier to obtain their famous formulas [GZ86].

### Rankin's method

For now, fix some $f \in S_k(\Gamma_0(N))$. Let $g$ be as above, and assume for simplicity that $g$ has real coefficients, that $\ell < k$ and that $\chi$ is a real character. We will make use of the following equality:

$$\sum_{n=1}^{\infty} a(n)b(n)e^{-4\pi ny} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a(n)e^{-2\pi ny}b(m)e^{-2\pi my}\int_0^1 e^{2\pi i(n-m)x}dx. \tag{3.12}$$

Using the integral representation for the gamma function, we find for $\Re(s) > \frac{\ell+1}{2}$ that

$$\begin{aligned}
\frac{\Gamma(s+\frac{k+\ell}{2}-1)}{(4\pi)^{s+\frac{k+\ell}{2}-1}}L(f\otimes g,s) &= \sum_{n=1}^{\infty} a(n)b(n)n^{\frac{k+\ell}{2}-1}\int_0^{\infty} y^{s+\frac{k+\ell}{2}-2}e^{-4\pi ny}dy \\
&\overset{(3.12)}{=} \int_0^{\infty}\int_0^1 \sum_{n,m=0}^{\infty} a(n)n^{\frac{k-1}{2}}b(m)m^{\frac{\ell-1}{2}}e^{2\pi in(x+iy)}e^{2\pi im(iy-x)}y^{s+\frac{k+\ell}{2}}\frac{dxdy}{y^2} \\
&= \int_{\Gamma_\infty\backslash\mathcal{H}} f(z)\overline{g(z)}y^{s+\frac{k+\ell}{2}}\frac{dxdy}{y^2}.
\end{aligned}$$

We now introduce an Eisenstein series by rewriting the quotient $\Gamma_\infty\backslash\mathcal{H}$. Let $\mathcal{D}$ denote a fundamental domain for the action of $\Gamma_0(ND)$ on $\mathcal{H}$. For a matrix $\gamma \in \Gamma_0(ND)$, we will write $d_\gamma$ for the lower right entry of $\gamma$. Using (2.2), we see that

$$\begin{aligned}
\int_{\Gamma_\infty\backslash\mathcal{H}} f(z)\overline{g(z)}y^{s+\frac{k+\ell}{2}}\frac{dxdy}{y^2} &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(ND)}\int_{\mathcal{D}} f(\gamma z)\overline{g(\gamma z)}(\Im(\gamma z))^{s+\frac{k+\ell}{2}}\frac{dxdy}{y^2} \\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(ND)}\int_{\mathcal{D}} f(z)\overline{g(z)}\frac{\chi(d_\gamma)}{j(\gamma,\bar{z})^{k-\ell}}\frac{y^{s+\frac{\ell-k}{2}}}{|j(\gamma,z)|^{2s-k+\ell}}y^k\frac{dxdy}{y^2} \\
&= \langle f, gE_{\bar{s}+\frac{\ell-k}{2}}\rangle_{\Gamma_0(ND)},
\end{aligned}$$

with $\langle\cdot,\cdot\rangle_{\Gamma_0(ND)}$ being the Petersson inner product for $\Gamma_0(ND)$ and

$$E_s(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(ND)}\frac{\chi(d_\gamma)}{j(\gamma,z)^{k-\ell}}\frac{y^s}{|j(\gamma,z)|^{2s}} \in \tilde{M}_{k-\ell}(\Gamma_0(ND),\chi),$$

a non-holomorphic Eisenstein series of weight $k-\ell$ and level $ND$. Note that $gE_s \in \tilde{M}_k(\Gamma_0(ND))$, as $\chi$ is a real character. Now one can convert the inner product for $\Gamma_0(ND)$ to an inner product for $\Gamma_0(N)$ by using the trace map

$$\begin{aligned}
\mathrm{Tr}_N^{ND} : \tilde{M}_k(\Gamma_0(ND)) &\to \tilde{M}_k(\Gamma_0(N)), \\
g &\mapsto \sum_{\gamma\in\Gamma_0(DN)\backslash\Gamma_0(N)} g|_k\gamma.
\end{aligned}$$

We obtain

$$\langle f, gE_{\bar{s}}\rangle_{\Gamma_0(ND)} = \langle f, \mathrm{Tr}_N^{ND}(gE_{\bar{s}})\rangle_{\Gamma_0(N)}.$$

We have proved the following proposition, which is related to [GZ86, Prop. IV.1.2].

**Proposition 3.9.** *Let $g \in M_\ell(\Gamma_0(D), \chi)$ be a modular form with real Fourier coefficients and a real quadratic nebentypus $\chi$. Fix weights $k > \ell$ and a level $N \geq 1$. Define the non-holomorphic function*

$$\Phi_{s,g}^{\mathrm{nh}} = \mathrm{Tr}_N^{ND}(gE_{s+\frac{\ell-k}{2}}) \in \tilde{M}_k(\Gamma_0(N)).$$

*Then for any modular form $f \in S_k(\Gamma_0(N))$ and $\Re(s) > \frac{\ell+1}{2}$, we have*

$$L(f \otimes g, s) = \frac{(4\pi)^{s+\frac{k+\ell}{2}-1}}{\Gamma(s+\frac{k+\ell}{2}-1)} \langle f, \Phi_{\bar{s},g}^{\mathrm{nh}} \rangle.$$

□

The holomorphic kernel $\Phi_{s,g}$ can now be obtained from $\Phi_{s,g}^{\mathrm{nh}}$ by taking its holomorphic projection. Broadly speaking, this means that the Fourier coefficients of $\Phi_{s,g}$ are expressed as integrals of the Fourier coefficients of $\Phi_{s,g}^{\mathrm{nh}}$. We will not discuss holomorphic projection in detail in this thesis and instead refer to the paper by Gross and Zagier [GZ86, Lemma IV.5.1]. In the next chapter, we will use the Goldfeld–Zhang method to find an expression for $\Phi_{s,g}$ without making use of traces and holomorphic projections.

# 4 Holomorphic kernel

In this chapter we use the Goldfeld–Zhang method to derive a formula for the Fourier coefficients of the holomorphic kernel $\Phi_{s,g}$ of the Rankin–Selberg $L$-function [GZ99]. In Section 4.1, we detail the main idea behind the method. In Section 4.2, we give a formula for the Fourier coefficients of $\Phi_{s,g}$ for complex $s$ lying in a certain vertical strip. In Section 4.3 and Section 4.4, we give explicit expressions for the functions that occur in this formula. In Section 4.5, we modify the holomorphic kernel by adding oldforms and obtain a modular form with nicer Fourier coefficients. Finally, in Section 4.6, we derive an expression for the Fourier coefficients of this modified holomorphic kernel in the case that the associated modular form $g$ is a theta series. These expressions show that the modified kernel satisfies a functional equation. As a consequence, we obtain a functional equation for the Rankin–Selberg function $L(f \otimes \theta_{\mathcal{A},\psi}, s)$.

## 4.1 Goldfeld–Zhang method

Let $k, \ell, D$ and $N$ be positive integers with $D$ square-free and let $\chi$ be a Dirichlet character modulo $D$. We will assume that $k \geq 4$. Let $g \in M_\ell(\Gamma_0(D), \chi)$ be a modular form with Fourier expansion

$$g(z) = b(0) + \sum_{n=1}^{\infty} b(n) n^{\frac{\ell-1}{2}} e^{2\pi i n z}.$$

Fix a point $s \in \mathbf{C}$ such that the Rankin–Selberg $L$-function $L(f \otimes g, s)$, as defined in Section 3.3, does not have a pole at $s$ for all $f \in S_k(\Gamma_0(N))$. As $S_k(\Gamma_0(N))$ is finite-dimensional and the set of poles of each $L$-function is discrete, we only exclude a discrete set of points in this way. Throughout the remainder of this chapter, we will implicitly exclude such "bad" values. Recall from Section 3.3 the holomorphic kernel $\Phi_{\bar{s},g} \in S_k(\Gamma_0(N))$ associated to $g$ that satisfies

$$L(f \otimes g, s) = \langle f, \Phi_{\bar{s},g} \rangle \qquad \text{for all } f \in S_k(\Gamma_0(N)). \tag{4.1}$$

We denote by $\phi_{\bar{s},g}$ the normalized Fourier coefficients of $\Phi_{\bar{s},g}$, so that

$$\Phi_{\bar{s},g}(z) = \sum_{n=1}^{\infty} \phi_{\bar{s},g}(n) n^{\frac{k-1}{2}} e^{2\pi i n z}.$$

Now, in the case that $f = P_m$, we find by the Petersson formula (Proposition 2.9) that

$$L(P_m \otimes g, s) = \langle P_m, \Phi_{\bar{s},g} \rangle = \overline{\langle \Phi_{\bar{s},g}, P_m \rangle} = \frac{(k-1)!}{(4\pi)^{k-1}} \overline{\phi_{\bar{s},g}(m)},$$

and thus

$$\phi_{s,g}(m) = \frac{(4\pi)^{k-1}}{(k-2)!} \overline{L(P_m \otimes g, \bar{s})}. \tag{4.2}$$

As $L(P_m \otimes g, s)$ can be extended meromorphically to the complex plane, we see that $\phi_{s,g}(m)$ is meromorphic in $s$. As we have an explicit formula for the Fourier coefficients of the Poincaré series $P_m$, given by Proposition 2.8, we find for $\Re(s) > \frac{\ell+1}{2}$ that

$$L(P_m \otimes g, s) = \frac{b(m)}{m^s} + 2\pi i^k \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \sum_{\substack{c=1 \\ N|c}}^{\infty} \frac{K(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Here $K(m,n;c)$ is a Kloosterman sum (Definition 2.5) and $J_{k-1}(x)$ is a Bessel function of the first kind (Definition 2.7). We will write $T_m(s)$ for this double sum:

$$T_m(s) := \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \sum_{\substack{c=1 \\ N|c}}^{\infty} \frac{K(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

As $L(P_m \otimes g, s)$ has a meromorphic extension to the complex plane, it follows that $T_m(s)$ can be extended meromorphically as well. In the next section, we will manipulate this expression for $T_m(s)$ and thereby deriving a formula for the Fourier coefficients of the holomorphic kernel.

## 4.2 Fourier coefficients of the holomorphic kernel

In this section, we will obtain a formula for the Fourier coefficients of the holomorphic kernel $\Phi_{s,g}$ that is valid for $s$ lying in a vertical strip to the right of $\Re(s) = 1$. The main result is given by Theorem 4.4. In order to prove the theorem, we will first need a different representation of $T_m(s)$.

For every $\delta|D$, we have the modular form $g^\delta$ defined by the Atkin–Lehner operator (2.5) with Fourier coefficients $b^\delta(n)$, see (2.6). For every $\delta|D$, let $\gamma_\delta > 0$ such that we have a bound of the form

$$b^\delta(n) = O(n^{\gamma_\delta}) \qquad \text{as } n \to \infty,$$

and let

$$\gamma = \max_{\delta|D} \gamma_\delta. \tag{4.3}$$

In particular, for any divisor $\delta|D$, it holds that $L_{g^\delta}(s)$ converges absolutely for $\Re(s) > 1 + \gamma$. For a general modular form $g$, we will be able to take $\gamma_\delta$, and hence $\gamma$, to be any real number greater than $\frac{\ell-1}{2}$ by Lemma 2.2. In the case that $g$ is a theta series, we have a sharper bound and can take $\gamma_\delta > 0$ arbitrarily small by Proposition 2.15 (6). In the following lemma, we will need to fix an $\epsilon$ satisfying $0 < \epsilon < \frac{k}{2} - \gamma - 1$. As a result, we cannot apply the lemma when $k = 2$, or when $\gamma > \frac{\ell-1}{2}$ and $\ell \geq k - 1$.

**Lemma 4.1.** *Fix some $0 < \epsilon < \frac{k}{2} - \gamma - 1$. Then for all $1 < \Re(s) < \frac{k+\ell}{2} - \epsilon$, it holds that*

$$T_m(s) = \sum_{\substack{c=1 \\ N|c}}^{\infty} \sum_{r \in (\mathbf{Z}/c\mathbf{Z})^\times} e^{2\pi i m \frac{r}{c}} \left( \frac{1}{2\pi i} \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \frac{(2\pi\sqrt{m})^{-2w}}{c^{1-2w}} L_g(s + w, \tfrac{\bar{r}}{c}) dw \right.$$
$$\left. + \frac{\Gamma(\frac{k+\ell}{2} - s)}{\Gamma(\frac{k-\ell}{2} + s)} \frac{(2\pi\sqrt{m})^{2s-\ell-1}}{c^{2s-\ell}} \cdot \operatorname{Res}_{\frac{\ell+1}{2}}(L_g(s, \tfrac{\bar{r}}{c})) \right).$$

*Here $L_g(s, \frac{\bar{r}}{c})$ is the twisted $L$-series associated to $g$, defined by (3.8).*

*Proof.* By the definition of $J_{k-1}(x)$ (Definition 2.7), we have that

$$T_m(s) = \sum_{n=1}^{\infty} \sum_{\substack{c=1 \\ N|c}}^{\infty} \frac{b(n)}{n^s} \frac{K(m,n;c)}{c} \frac{1}{2\pi i} \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \left(\frac{2\pi\sqrt{mn}}{c}\right)^{-2w} dw.$$

We will first show that this representation of $T_m(s)$ converges absolutely and uniformly on closed compact subsets of $\{s \in \mathbf{C} \mid \Re(s) > \frac{k+1}{2} + \gamma - \epsilon\}$, so that it defines an analytic function there. Let $U$ be such a compact subset and let $\sigma > \frac{k+1}{2} + \gamma - \epsilon$ such that $\Re(s) \geq \sigma$ for all $s \in U$. By Lemma 2.6, we know for fixed $m$ that $K(m,n;c) = O(c^{\frac{1}{2}+\alpha})$ as $n, c \to \infty$, for all $\alpha > 0$. In this case, we choose some $\alpha$ with $0 < \alpha < k - 2\epsilon - 2$. It now follows that $T_m(s)$ converges absolutely and uniformly on $U$, as it is absolutely bounded by (for some constant $C$)

$$|T_m(s)| \leq C \cdot \sum_{n=1}^{\infty} n^{\gamma + \frac{k-1}{2} - \epsilon - \sigma} \cdot \sum_{\substack{c=1 \\ N|c}}^{\infty} c^{\alpha + \frac{1}{2} + 2\epsilon - k} \cdot \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \left| \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \right| dw.$$

These sums converge as the exponents of $n$ and $c$ are smaller than $-1$, and because the integral defining $J_{k-1}(2)$ converges absolutely. The fact that $T_m(s)$ converges absolutely allows us to change the order of the sums and integral. Using the definition of a Kloosterman sum (Definition 2.5), we find for $\Re(s) > \frac{k+1}{2} + \gamma - \epsilon$, that

$$\begin{aligned} T_m(s) &= \sum_{\substack{c=1 \\ N|c}}^{\infty} \sum_{r \in (\mathbf{Z}/c\mathbf{Z})^\times} e^{2\pi i m \frac{r}{c}} \frac{1}{2\pi i c} \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \sum_{n=1}^{\infty} e^{\frac{2\pi i n \bar{r}}{c}} \frac{b(n)}{n^s} \left( \frac{2\pi\sqrt{mn}}{c} \right)^{-2w} dw \\ &= \sum_{\substack{c=1 \\ N|c}}^{\infty} \sum_{r \in (\mathbf{Z}/c\mathbf{Z})^\times} e^{2\pi i m \frac{r}{c}} \frac{1}{2\pi i} \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \frac{(2\pi\sqrt{m})^{-2w}}{c^{1-2w}} L_g(s+w, \tfrac{\bar{r}}{c}) dw. \end{aligned} \tag{4.4}$$

This expression converges absolutely and uniformly on compact subsets of $\{s \in \mathbf{C} \mid \frac{1}{2} < \Re(s) < \frac{k+\ell}{2} - \epsilon\}$, but that does not mean that we can take this expression as a representation of $T_m(s)$ there. In fact, if $g$ is not a cusp form, the expression converges to a different meromorphic function there, and a correct representation of $T_m(s)$ is obtained by adding the difference. This difference appears because of the pole of $L_g(s+w, \frac{\bar{r}}{c})$ at $w = \frac{\ell+1}{2} - s$, as we shall show. We first split the integral into two integrals:

$$\begin{aligned} &\int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \frac{(2\pi\sqrt{m})^{-2w}}{c^{1-2w}} \left( L_g(s+w, \tfrac{\bar{r}}{c}) - \frac{\operatorname{Res}_{\frac{\ell+1}{2}}(L_g(s, \frac{\bar{r}}{c}))}{s + w - \frac{\ell+1}{2}} \right) dw \\ &\qquad + \operatorname{Res}_{\frac{\ell+1}{2}}(L_g(s, \tfrac{\bar{r}}{c})) \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \frac{(2\pi\sqrt{m})^{-2w}}{c^{1-2w}} \frac{1}{s + w - \frac{\ell+1}{2}} dw. \end{aligned}$$

The first integral converges absolutely and uniformly on compact subsets of $\{s \in \mathbf{C} \mid \Re(s) > \frac{1}{2}\}$. This can be proven using Proposition 3.8 as follows. Given a compact subset $U$ of $\{s \in \mathbf{C} \mid \Re(s) > \frac{1}{2}\}$, we can choose an $\frac{1}{2} < a < \frac{k-1}{2} - \epsilon - \gamma$ and an $b > \frac{k+1}{2} - \epsilon + \gamma$ such that $a \leq \Re(s) \leq b$ for all $s \in U$. Using Proposition 3.8 with the bounds $a + \epsilon - \frac{k-1}{2}$ and $b + \epsilon - \frac{k-1}{2}$, we obtain a bound

$$\left| L_g\left(s + w, \frac{\bar{r}}{c}\right) - \frac{\operatorname{Res}_{\frac{\ell+1}{2}}(L_g(s, \frac{\bar{r}}{c}))}{s + w - \frac{\ell+1}{2}} \right| \leq M \cdot (1 + |\Im(s) + \Im(w)|)^{k - 2a - 2\epsilon}.$$

Here we used that $k - 2a - 2\epsilon > 1 + \gamma > 0$ and hence we omitted the linear function $\lambda(\sigma)$ in the exponent. Together with Sterling's approximation (2.8), it follows that the integrand of the first integral is $O(|t|^{-2a})$ as $|t| \to \infty$ where $t = \Im(w)$, and so the integral converges absolutely.

The second integral converges absolutely on both $\{s \in \mathbf{C} \mid \Re(s) > \frac{k+\ell}{2} - \epsilon\}$ and $\{s \in \mathbf{C} \mid \Re(s) < \frac{k+\ell}{2} - \epsilon\}$. However, it converges to different meromorphic functions on these regions. To see this, we can compute the integral using the residue theorem. We first consider $\Re(s) > \frac{k+\ell}{2} - \epsilon$. The poles of the integrand that lie to the left of the line $\Re(w) = \epsilon - \frac{k-1}{2}$ are the poles of $\Gamma(\frac{k-1}{2} + w)$ at $w = -n - \frac{k-1}{2}$ for $n \in \mathbf{Z}_{\geq 0}$, and the pole of $(s + w - \frac{\ell+1}{2})^{-1}$ at $w = \frac{\ell+1}{2} - s$. Similarly to how one can express $J_{k-1}(x)$ as a series, see (2.9), we obtain for $\Re(s) > \frac{k+\ell}{2} - \epsilon$:

$$\frac{1}{2\pi i} \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \frac{(2\pi\sqrt{m})^{-2w}}{c^{1-2w}} \frac{1}{s + w - \frac{\ell+1}{2}} dw$$
$$= \frac{\Gamma(\frac{k+\ell}{2} - s)}{\Gamma(\frac{k-\ell}{2} + s)} \frac{(2\pi\sqrt{m})^{2s-\ell-1}}{c^{2s-\ell}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(2\pi\sqrt{m})^{2n+k-1}}{\Gamma(k+n)c^{2n+k}} \frac{1}{s - n - \frac{k+\ell}{2}}.$$

Here we assume for simplicity that $s$ is not of the form $n + \frac{k+\ell}{2}$. It is clear that this infinite sum converges absolutely and locally uniformly for all $s \in \mathbf{C}$ (excluding obvious poles) and thus defines a meromorphic function on the entire complex plane. When we consider $\Re(s) < \frac{k+\ell}{2} - \epsilon$, the pole of $(s + w - \frac{\ell+1}{2})^{-1}$ at $w = \frac{\ell+1}{2} - s$ has been moved to the right of the line $\Re(w) = \epsilon - \frac{k-1}{2}$. It follows thus for $\Re(s) < \frac{k+\ell}{2} - \epsilon$ that

$$\frac{1}{2\pi i} \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \frac{(2\pi\sqrt{m})^{-2w}}{c^{1-2w}} \frac{1}{s + w - \frac{\ell+1}{2}} dw = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(2\pi\sqrt{m})^{2n+k-1}}{\Gamma(k+n)c^{2n+k}} \frac{1}{s - n - \frac{k+\ell}{2}}.$$

We see that the two integrals both define meromorphic functions, and that they differ by one term. As a result, in order to obtain a correct representation of $T_m(s)$ on $1 < \Re(s) < \frac{k+\ell}{2} - \epsilon$, we will have to add the difference to (4.4):

$$T_m(s) = \sum_{\substack{c=1 \\ N|c}}^{\infty} \sum_{r \in (\mathbf{Z}/c\mathbf{Z})^\times} e^{2\pi i m \frac{r}{c}} \left( \frac{1}{2\pi i} \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \frac{(2\pi\sqrt{m})^{-2w}}{c^{1-2w}} L_g(s + w, \tfrac{\bar{r}}{c}) dw \right.$$
$$\left. + \frac{\Gamma(\frac{k+\ell}{2} - s)}{\Gamma(\frac{k-\ell}{2} + s)} \frac{(2\pi\sqrt{m})^{2s-\ell-1}}{c^{2s-\ell}} \cdot \mathrm{Res}_{\frac{\ell+1}{2}}(L_g(s, \tfrac{\bar{r}}{c})) \right).$$

Using the formula for the residue of $L_g(s, \frac{\bar{r}}{c})$ at $\frac{\ell+1}{2}$ given by Proposition 3.5, it follows that this representation converges absolutely and uniformly on compact subsets of the vertical strip $\{s \in \mathbf{C} \mid 1 < \Re(s) < \frac{k+\ell}{2} - \epsilon\}$ and thus defines a holomorphic function there. □

**Remark.** It seems that the presence of this additional term corresponding to the residues of $L_g(s, \frac{\bar{r}}{c})$ was overlooked in [GZ99].

We will use this lemma to obtain a formula for the Fourier coefficients of the holomorphic kernel. Two functions that will appear are the following series and Mellin–Barnes type integral.

**Definition 4.2.** Let $D$ and $N$ be two levels with a decomposition $D = \delta \cdot \delta'$. Then for an $s \in \mathbf{C}$ with $\Re(s) > 1$ and an integer $B$, we define

$$S^{\delta}(s,B) = \sum_{\substack{c=1 \\ N|c \\ (c,D)=\delta'}}^{\infty} \frac{\chi_\delta\left(\frac{c}{\delta'}\right)}{c^{2s}} \sum_{r \in (\mathbf{Z}/c\mathbf{Z})^\times} \chi_{\delta'}(r) e^{2\pi i \frac{Br}{c}}.$$

**Definition 4.3.** Let $s \in \mathbf{C}$ with $\Re(s) > \frac{1}{2}$ and fix some $0 < \epsilon < \frac{k-1}{2}$. Then for $x > 0$, we define

$$I_s(x) = \frac{1}{2\pi i} \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \frac{\Gamma(\frac{\ell+1}{2} - s - w)}{\Gamma(\frac{\ell-1}{2} + s + w)} x^{-w} dw.$$

This integral converges absolutely, and the value of $I_s(x)$ does not depend on the choice of $\epsilon$.

We can now give an explicit expression for the Fourier coefficients of $\Phi_{s,g}(z)$ on some vertical strip to the right of $\Re(s) = 1$.

**Theorem 4.4.** *Let $k, \ell, N$ and $D$ be positive integers with $D$ square-free and $k \geq 4$. Let $\chi$ be a Dirichlet character modulo $D$ and $g \in M_\ell(\Gamma_0(D), \chi)$ a modular form with normalized $q$-expansion:*

$$g(z) = b(0) + \sum_{n=1}^{\infty} b(n) n^{\frac{\ell-1}{2}} e^{2\pi i n z}.$$

*Let $\gamma$ be a bound on the coefficients of $g$ and its Atkin–Lehner translates as in (4.3). Then:*

*(a) The Fourier expansion of the holomorphic kernel $\Phi_{s,g}(z)$ is given by*

$$\Phi_{s,g}(z) = \frac{(4\pi)^{k-1}}{(k-2)!} \sum_{m=1}^{\infty} \overline{L(P_m \otimes g, \bar{s})} m^{\frac{k-1}{2}} e^{2\pi i m z}.$$

*(b) For $1 < \Re(s) < \frac{k-1}{2} - \gamma$, the Rankin–Selberg L-function $L(P_m \otimes g, s)$ is given by*

$$L(P_m \otimes g, s) = \frac{b(m)}{m^s} + 2\pi i^k \sum_{\delta | D} T_m^{\delta}(s),$$

*where*

$$T_m^{\delta}(s) = i^{\ell} \left(\frac{\delta}{4\pi^2}\right)^{\frac{1}{2}-s} \sum_{n=0}^{\infty} b^{\delta}(n) S^{\delta}(s, m\delta - n) V_s(n, m\delta),$$

*with*

$$V_s(n,m) = \begin{cases} \frac{\Gamma(\frac{k+\ell}{2} - s)}{\Gamma(\frac{k-\ell}{2} + s)\Gamma(\ell)} m^{s - \frac{\ell+1}{2}} & \text{if } n = 0, \\ I_s\left(\frac{m}{n}\right) n^{s-1} & \text{if } n > 0. \end{cases} \tag{4.5}$$

*Here $S^{\delta}$ is given by Definition 4.2, and $I_s$ is given by Definition 4.3.*

*Proof.* Part (a) follows from (4.2). For part (b), consider some $s \in \mathbf{C}$ with $1 < \Re(s) < \frac{k-1}{2} - \gamma$. It suffices to show that

$$T_m(s) = \sum_{\delta | D} T_m^\delta(s).$$

Fix an $\epsilon > 0$ such that $\Re(s) < \frac{k-1}{2} - \gamma - \epsilon$ and recall the representation for $T_m(s)$ given by Lemma 4.1:

$$T_m(s) = \sum_{\substack{c=1 \\ N|c}}^\infty \sum_{r \in (\mathbf{Z}/c\mathbf{Z})^\times} e^{2\pi i m \frac{r}{c}} \left( \frac{1}{2\pi i} \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \frac{(2\pi\sqrt{m})^{-2w}}{c^{1-2w}} L_g(s+w, \tfrac{\bar{r}}{c}) dw \right.$$
$$\left. + \frac{\Gamma(\frac{k+\ell}{2} - s)}{\Gamma(\frac{k-\ell}{2} + s)} \frac{(2\pi\sqrt{m})^{2s-\ell-1}}{c^{2s-\ell}} \cdot \mathrm{Res}_{\frac{\ell+1}{2}}(L_g(s, \tfrac{\bar{r}}{c})) \right).$$

We will use the functional equation of $L_g(s, \frac{\bar{r}}{c})$ to rewrite this expression. Fix an integer $c \geq 1$ with $N|c$ and a residue class $r \in (\mathbf{Z}/c\mathbf{Z})^\times$. As shown in Proposition 3.5, the functional equation of $L_g(s, \frac{\bar{r}}{c})$ depends on $c$, and in particular on $\delta' = \gcd(c, D)$ and $\delta = \frac{D}{\delta'}$ In that case, we find that

$$L_g\left(s+w, \frac{\bar{r}}{c}\right) = i^\ell \chi_\delta\left(\frac{c}{\delta'}\right) \chi_{\delta'}^{-1}(\bar{r}\delta) \left(\frac{\delta c^2}{4\pi^2}\right)^{\frac{1}{2}-s-w} \frac{\Gamma(\frac{\ell+1}{2} - s - w)}{\Gamma(\frac{\ell-1}{2} + s + w)} L_{g^\delta}\left(1-s-w, -\frac{r\bar{\delta}}{c}\right).$$

Moreover, we also know by Proposition 3.5 that

$$\mathrm{Res}_{\frac{\ell+1}{2}}\left(L_g\left(s, \frac{\bar{r}}{c}\right)\right) = \frac{(2\pi)^\ell}{\Gamma(\ell)} \chi_\delta\left(\frac{c}{\delta'}\right) \chi_{\delta'}^{-1}(\bar{r}\delta) \delta^{-\frac{\ell}{2}} i^\ell c^{-\ell} b^\delta(0).$$

Given the bound $b^\delta(n) = O(n^{\gamma_\delta})$ as $n \to \infty$, we may replace $L_{g^\delta}(s, \frac{\bar{r}}{c})$ by its series for $\Re(s) > 1 + \gamma_\delta$. So, for $1 < \Re(s) < \frac{k-1}{2} - \gamma_\delta - \epsilon$, we can use the functional equation and unfold the $L$-series $L_{g^\delta}(1-s-w, -\frac{r\bar{\delta}}{c})$ that appears. If we now sum over the different values for $\delta$, $c$ and $r$, it follows that for $1 < \Re(s) < \frac{k-1}{2} - \gamma - \epsilon$:

$$T_m(s) = \sum_{\delta | D} \sum_{\substack{c=1 \\ N|c \\ (c,D)=\delta'}}^\infty \sum_{r \in (\mathbf{Z}/c\mathbf{Z})^\times} i^\ell \chi_\delta\left(\frac{c}{\delta'}\right) \chi_{\delta'}^{-1}(\bar{r}\delta) \left(\frac{\delta}{4\pi^2}\right)^{\frac{1}{2}-s} c^{-2s} e^{2\pi i m \frac{r}{c}}$$
$$\cdot \left( \frac{1}{2\pi i} \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \frac{\Gamma(\frac{\ell+1}{2} - s - w)}{\Gamma(\frac{\ell-1}{2} + s + w)} \sum_{n=1}^\infty \frac{b^\delta(n)}{n^{1-s}} e^{-2\pi i n \frac{r\bar{\delta}}{c}} \left(\frac{\delta m}{n}\right)^{-w} dw \right.$$
$$\left. + b^\delta(0) \frac{\Gamma(\frac{k+\ell}{2} - s)}{\Gamma(\ell)\Gamma(\frac{k-\ell}{2} + s)} (\delta m)^{s - \frac{\ell+1}{2}} \right).$$

Using a similar argument as before, it can be shown that this expression converges absolutely and uniformly on compact subsets of $1 < \Re(s) < \frac{k-1}{2} - \gamma - \epsilon$, and so we may exchange the order of summation there. We conclude that

$$\begin{aligned}
T_m(s) &= \sum_{\delta|D} i^\ell \left(\frac{\delta}{4\pi^2}\right)^{\frac{1}{2}-s} \sum_{n=0}^{\infty} b^\delta(n) V_s(n, m\delta) \sum_{\substack{c=1\\ N|c\\ (c,D)=\delta'}}^{\infty} \frac{\chi_\delta\left(\frac{c}{\delta'}\right)}{c^{2s}} \sum_{r\in(\mathbf{Z}/c\mathbf{Z})^\times} \chi_{\delta'}^{-1}(\bar{r}\delta) e^{2\pi i \frac{r(m-n\bar{\delta})}{c}} \\
&= \sum_{\delta|D} i^\ell \left(\frac{\delta}{4\pi^2}\right)^{\frac{1}{2}-s} \sum_{n=1}^{\infty} b^\delta(n) V_s(n, m\delta) \sum_{\substack{c=1\\ N|c\\ (c,D)=\delta'}}^{\infty} \frac{\chi_\delta\left(\frac{c}{\delta'}\right)}{c^{2s}} \sum_{r\in(\mathbf{Z}/c\mathbf{Z})^\times} \chi_{\delta'}(r) e^{2\pi i \frac{r(\delta m-n)}{c}} \\
&= \sum_{\delta|D} i^\ell \left(\frac{\delta}{4\pi^2}\right)^{\frac{1}{2}-s} \sum_{n=1}^{\infty} b^\delta(n) S^\delta(s, m\delta - n) V_s(n, m\delta).
\end{aligned}$$

Here we used the substitution $r \mapsto r\delta$ to simplify the summation. □

### 4.3 Ramanujan sums

In the previous section, we derived a formula for the Fourier coefficients of $\Phi_{s,g}$ in terms of the functions $S^\delta(s,B)$ and $I_s(x)$. In the present section, we will take a closer look at $S^\delta(s,B)$, which is given by Definition 4.2. It is a series whose terms have a factor of the form

$$\sum_{r\in(\mathbf{Z}/c\mathbf{Z})^\times} \chi_{\delta'}(r) e^{2\pi i \frac{Br}{c}},$$

which is a twisted Ramanujan sum. Given two integers $n$ and $D$ with $n$ nonzero, we will write $n_{[D]}$ for the largest positive divisor of $n$, whose prime factors also divide $D$. We will write $n^{[D]}$ for the largest divisor of $n$ that is coprime to $D$ and has the same sign as $n$. Note that $n_{[D]} \cdot n^{[D]} = n$. For $n = 0$, we define $n_{[D]} = 0$ and $n^{[D]} = 1$. Recall from (3.5) that the Gauss sum associated to a Dirichlet character with modulus $D$ is defined as

$$\tau(\chi) = \sum_{r\in(\mathbf{Z}/D\mathbf{Z})^\times} \chi(r) e^{2\pi i \frac{r}{D}}.$$

The following lemma is a restatement of [GZ99, Lemma 5.3].

**Lemma 4.5.** *Let $D$ be a square-free integer and $\chi$ a primitive character modulo $D$. Let $c$ and $B$ be non-zero integers with $c > 0$ and $D|c$. Then*

$$\sum_{r\in(\mathbf{Z}/c\mathbf{Z})^\times} \chi(r) e^{2\pi i \frac{Br}{c}} = \begin{cases} \tau(\chi) B_{[D]} \chi\left(\frac{c^{[D]}}{B^{[D]}}\right) \sum_{d|(c^{[D]},B^{[D]})} \mu\left(\frac{c^{[D]}}{d}\right) d & \text{if } c_{[D]} = B_{[D]} \cdot D, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* See the proof of [GZ99, Lemma 5.3] (or alternatively [Nel13, Lemma 2.7]). The function $G(\delta)$ that Goldfeld and Zhang use coincides with with Gauss sum $\tau(\chi_\delta)$. □

This lemma allows us to derive a different expression for $S^\delta(s,B)$, which is given by Definition 4.2 as

$$S^\delta(s,B) = \sum_{\substack{c=1\\ N|c\\ (c,D)=\delta'}}^{\infty} \frac{\chi_\delta\left(\frac{c}{\delta'}\right)}{c^{2s}} \sum_{r\in(\mathbf{Z}/c\mathbf{Z})^\times} \chi_{\delta'}(r) e^{2\pi i \frac{Br}{c}}.$$

Recall from (3.2), that for an integer $e$, we define the Dirichlet $L$-function $L^{(e)}(\chi, s)$ by

$$L^{(e)}(\chi, s) = \sum_{\substack{n=1 \\ (n,e)=1}}^{\infty} \frac{\chi(n)}{n^s}.$$

**Proposition 4.6.** *Let $D = \delta \cdot \delta'$ a decomposition of a square-free positive integer. Let $\chi$ be a primitive Dirichlet character modulo $D$. Let $B \in \mathbf{Z}$ be an integer and define*

$$S_e^\delta(s, B) = \begin{cases} \frac{\tau(\chi_{\delta'})\chi_\delta(B_{[\delta']})}{(\delta')^{2s} B_{[\delta']}^{2s-1} \chi_{\delta'}(B^{[\delta']})} \sum_{\substack{d \mid \frac{B}{e} \\ (d,D)=1}} \chi(d) d^{1-2s} & \text{if } B \neq 0,\ N_{[\delta']} | B_{[\delta']}\delta',\ e|B,\ (N, \delta) = 1, \\ L(\chi, 2s-1) & \text{if } B = 0,\ \delta = D,\ (N, D) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

*Then for $\Re(s) > 1$, the series $S^\delta(s, B)$ as in Definition 4.2 satisfies*

$$S^\delta(s, B) = \frac{\chi(N^{[D]})(N^{[D]})^{1-2s}}{L^{(N)}(\chi, 2s)} \sum_{e | N^{[D]}} \mu\left(\frac{N^{[D]}}{e}\right) \frac{e}{N^{[D]}} S_e^\delta(s, B).$$

*Proof.* We follow the structure of the proof of [GZ99, Prop. 7.1], but note that our definition of $S^\delta(s, B)$ differs from theirs. We first note that we may assume that $\gcd(N, \delta) = 1$. Otherwise, we know that $S_e^\delta(s, B) = 0$ for any integer $e$ by definition and that $S^\delta(s, B) = 0$ due to the sum being empty (there are no integers $c$ with $N|c$ and $\gcd(c, D) = \delta'$), and so the lemma follows. As a result, we may also assume that $\gcd(N^{[\delta']}, D) = 1$ and that $N^{[D]} = N^{[\delta']}$.

We now consider the case $B \neq 0$. We find by Lemma 4.5 that

$$S^\delta(s, B) = \tau(\chi_{\delta'}) B_{[\delta']} \sum_{\substack{c=1 \\ N|c \\ (c,D)=\delta' \\ c_{[\delta']}=B_{[\delta']}\delta'}}^{\infty} \frac{\chi_\delta(\frac{c}{\delta'})}{c^{2s}} \chi_{\delta'}\left(\frac{c^{[\delta']}}{B^{[\delta']}}\right) \sum_{d | (c^{[\delta']}, B^{[\delta']})} \mu\left(\frac{c^{[\delta']}}{d}\right) d.$$

As $N|c$ implies that $N_{[\delta']}|c_{[\delta']}$, we may assume that $N_{[\delta']}|B_{[\delta']}\delta'$ – otherwise the sum would be empty, in which case the lemma follows trivially. As any integer $c$ can be decomposed as $c = c_{[\delta']} \cdot c^{[\delta']}$, we can replace $c$ by $B_{[\delta']}\delta' c$ in the sum and interchange the summations to find that

$$S^\delta(s, B) = \frac{\tau(\chi_{\delta'})\chi_\delta(B_{[\delta']})}{(\delta')^{2s} B_{[\delta']}^{2s-1} \chi_{\delta'}(B^{[\delta']})} \sum_{\substack{d | B^{[\delta']} \\ (d,D)=1}} d \sum_{\substack{c=1 \\ N^{[\delta']}|c \\ d|c \\ (c,D)=1}}^{\infty} \frac{\chi(c)}{c^{2s}} \mu\left(\frac{c}{d}\right).$$

Fix some divisor $d|B^{[\delta']}$ with $\gcd(d, D) = 1$. We now claim that

$$\sum_{\substack{c=1 \\ N^{[\delta']}|c \\ d|c \\ (c,D)=1}}^{\infty} \frac{\chi(c)}{c^{2s}} \mu\left(\frac{c}{d}\right) = \sum_{e | (N^{[\delta']}, d)} \sum_{\substack{k=1 \\ (k,De)=1}}^{\infty} \frac{\chi(d\frac{N^{[\delta']}}{e}k)}{(d\frac{N^{[\delta']}}{e}k)^{2s}} \mu\left(\frac{N^{[\delta']}}{e}k\right). \tag{4.6}$$

To see this, let $c \geq 1$ be given with $N^{[\delta']}|c$, $d|c$ and $\gcd(c,D) = 1$. We can then define the integers $e = dN^{[\delta']}/\gcd(c, dN^{[\delta']})$ and $k = c/\gcd(c, dN^{[\delta']})$. From the assumptions on $c$, it follows that $e|\gcd(N^{[\delta']}, d)$ and $\gcd(k, De) = 1$. By construction, it holds that $c = dN^{[\delta']}k/e$.

Conversely, given $e|\gcd(N^{[\delta']}, d)$ and $k \geq 1$ with $\gcd(k, De) = 1$, we can construct $c = dN^{[\delta']}k/e$. Then $N^{[\delta']}|c$, $d|c$ and $\gcd(c,D) = 1$.

This correspondence between integers $c$ and pairs $(e,k)$ is one-to-one, and as the sum on the left in (4.6) converges absolutely, our claim follows. Next, note that

$$\sum_{e|(N^{[\delta']},d)} \sum_{\substack{k=1 \\ (k,De)=1}}^{\infty} \frac{\chi(d\frac{N^{[\delta']}}{e}k)}{(d\frac{N^{[\delta']}}{e}k)^{2s}} \mu\left(\frac{N^{[\delta']}}{e}k\right) = \sum_{e|(N^{[\delta']},d)} \frac{\chi(d\frac{N^{[\delta']}}{e})}{(d\frac{N^{[\delta']}}{e})^{2s}} \mu\left(\frac{N^{[\delta']}}{e}\right) \cdot \sum_{\substack{k=1 \\ (k,DN)=1}}^{\infty} \frac{\chi(k)}{k^{2s}} \mu(k),$$

Now, using (3.4) and the fact that $N^{[\delta']} = N^{[D]}$, we obtain

$$\begin{aligned} S^{\delta}(s,B) &= \frac{\tau(\chi_{\delta'})\chi_{\delta}(B_{[\delta']})}{(\delta')^{2s} B_{[\delta']}^{2s-1} \chi_{\delta'}(B^{[\delta']})} \sum_{e|N^{[D]}} \sum_{\substack{d|B^{[\delta']} \\ e|d \\ (d,D)=1}} d \frac{\chi(\frac{d}{e}N^{[D]})}{(\frac{d}{e}N^{[D]})^{2s}} \mu\left(\frac{N^{[D]}}{e}\right) \frac{1}{L^{(N)}(\chi, 2s)} \\ &= \frac{\tau(\chi_{\delta'})\chi_{\delta}(B_{[\delta']})}{(\delta')^{2s} B_{[\delta']}^{2s-1} \chi_{\delta'}(B^{[\delta']})} \frac{\chi\left(N^{[D]}\right)(N^{[D]})^{1-2s}}{L^{(N)}(\chi, 2s)} \sum_{e|(N^{[D]},B)} \mu\left(\frac{N^{[D]}}{e}\right) \frac{e}{N^{[D]}} \sum_{\substack{d|\frac{B}{e} \\ (d,D)=1}} \chi(d) d^{1-2s} \\ &= \frac{\chi\left(N^{[D]}\right)(N^{[D]})^{1-2s}}{L^{(N)}(\chi, 2s)} \sum_{e|N^{[D]}} \mu\left(\frac{N^{[D]}}{e}\right) \frac{e}{N^{[D]}} S_e^{\delta}(s,B). \end{aligned}$$

Now we consider the case $B = 0$. Note that:

$$S^{\delta}(s,0) = \sum_{\substack{c=1 \\ N|c \\ (c,D)=\delta'}}^{\infty} \frac{\chi_{\delta}(\frac{c}{\delta'})}{c^{2s}} \sum_{r \in (\mathbf{Z}/c\mathbf{Z})^{\times}} \chi_{\delta'}(r).$$

The second sum vanishes if $\chi_{\delta'}$ is non-principal and thus if $\delta' \neq 1$. We will thus assume that $\delta = D$ and that $\gcd(N,D) = 1$. In this case, the sum $\sum_{r \in (\mathbf{Z}/c\mathbf{Z})^{\times}} \chi_{\delta'}(r)$ reduces to $\varphi(c) = \sum_{d|c} \mu(\frac{c}{d})d$. Using the same method to rewrite a sum over $c$ to a sum over $(k,e)$ and using (3.4), we find:

$$\begin{aligned} S^{D}(s,0) &= \sum_{\substack{d=1 \\ (d,D)=1}}^{\infty} \sum_{\substack{c=1 \\ N|c \\ d|c \\ (c,D)=1}}^{\infty} \frac{\chi(c)}{c^{2s}} \mu\left(\frac{c}{d}\right) d = \frac{\chi(N)N^{1-2s}}{L^{(N)}(\chi, 2s)} \sum_{\substack{d=1 \\ (d,D)=1}}^{\infty} \sum_{\substack{e|N \\ e|d}} \frac{\chi(d/e)}{(d/e)^{2s}} \mu\left(\frac{N}{e}\right) \frac{d}{N} \\ &= \frac{\chi(N)N^{1-2s}}{L^{(N)}(\chi, 2s)} \sum_{e|N} \mu\left(\frac{N}{e}\right) \frac{e}{N} \sum_{\substack{d=1 \\ (d,D)=1}}^{\infty} \frac{\chi(d)}{d^{2s-1}} = \frac{\chi(N)N^{1-2s}}{L^{(N)}(\chi, 2s)} \sum_{e|N} \mu\left(\frac{N}{e}\right) \frac{e}{N} S_e^{D}(s,0). \end{aligned}$$

□

## 4.4 Mellin–Barnes integrals

The second function that occurs in the formula of the coefficients of $\Phi_{s,g}$ is the Mellin–Barnes type integral $I_s(x)$, defined by Definition 4.3. For $x > 0$ and $\Re(s) > \frac{1}{2}$, it is given by

$$I_s(x) = \frac{1}{2\pi i} \int_{\epsilon - \frac{k-1}{2} - i\infty}^{\epsilon - \frac{k-1}{2} + i\infty} \frac{\Gamma(\frac{k-1}{2} + w)}{\Gamma(\frac{k+1}{2} - w)} \frac{\Gamma(\frac{\ell+1}{2} - s - w)}{\Gamma(\frac{\ell-1}{2} + s + w)} x^{-w} dw,$$

where $0 < \epsilon < \frac{k-1}{2}$ is fixed. The value of the integral does not depend on the choice of $\epsilon$. Convergence conditions of these types of integrals are discussed in [Erd+81a, p. 49-50]. We will first define a completed version of $I_s(x)$, which has a nice functional equation.

**Proposition 4.7.** *Let $x > 0$ and $s \in \mathbf{C}$. Define*

$$\tilde{I}_s(x) = \frac{\Gamma(\frac{k-\ell}{2} + s)}{\Gamma(\frac{k+\ell}{2} - s)} I_s(x). \tag{4.7}$$

*Then for $x > 0$ with $x \neq 1$, we have a functional equation*

$$\frac{\tilde{I}_{1-s}(x)}{|x-1|^{1-s}} = \operatorname{sgn}(x-1)^{k-\ell} \frac{\tilde{I}_s(x)}{|x-1|^s}.$$

*Proof.* See [GZ99, Prop. 8.6]. □

We now give an explicit expression for the value of $\tilde{I}_s(x)$ in terms of the Gaussian hypergeometric function $F(a, b; c; z)$ defined by

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}. \tag{4.8}$$

By taking a countour integral and using the residue theorem, Goldfeld and Zhang show that $I_s(x)$ naturally relates to the Gaussian hypergeometric function [GZ99, Prop. 8.3]. For the completed function, we can give the following formula.

**Proposition 4.8.** *Assume that $\Re(s) < \frac{k+\ell}{2}$. Then $\tilde{I}_s(x)$ is given by*

$$\tilde{I}_s(x) = \begin{cases} x^{\frac{k-1}{2}} \dfrac{\Gamma(\frac{k-\ell}{2} + s)}{\Gamma(k)\Gamma(\frac{\ell-k}{2} + s)} (1-x)^{\frac{\ell-k}{2} - 1 + s} F\left(\frac{k-\ell}{2} + s, \frac{k-\ell}{2} + 1 - s; k; \frac{x}{x-1}\right) & \text{if } 0 < x < 1, \\[2ex] \dfrac{\Gamma(2s-1)}{\Gamma(\frac{\ell-k}{2} + s)\Gamma(\frac{\ell+k}{2} + s - 1)} & \text{if } x = 1, \\[2ex] x^{\frac{1-k}{2}} \dfrac{1}{\Gamma(\ell)} (x-1)^{\frac{k-\ell}{2} - 1 + s} F\left(\frac{\ell-k}{2} + 1 - s, \frac{\ell-k}{2} + s; \ell; \frac{1}{1-x}\right) & \text{if } x > 1. \end{cases}$$

*Proof.* This follows from [GZ99, Prop. 8.3]. □

## 4.5 Modified holomorphic kernel

By summing up the holomorphic kernel functions for different levels, we will obtain a new kernel function that satisfies an inner product relation for newforms and has nicer Fourier coefficients. The main result of this section is Proposition 4.11.

For any divisor $e|N^{[D]}$, we define

$$A_s(e) = \frac{\Gamma(\frac{\ell+k}{2}-s)(2\pi)^{2s}\chi(e)e^{1-2s}}{\Gamma(\frac{k-\ell}{2}+s)L^{(e)}(\chi,2s)}.$$

Moreover, for $e|N^{[D]}$, $\delta|D$, and $m>0$, we define

$$T^{\delta}_{m,e}(s) = \frac{\delta^{\frac{1}{2}-s}i^{\ell}}{2\pi}\sum_{n=0}^{\infty} b^{\delta}(n)S^{\delta}_e(s,m\delta-n)\tilde{V}_s(n,\delta m), \tag{4.9}$$

with $S^{\delta}_e(s,m\delta-n)$ as in Proposition 4.6 and

$$\tilde{V}_s(n,m) = \begin{cases} \Gamma(\ell)^{-1}m^{s-\frac{\ell+1}{2}} & \text{if } n=0, \\ \tilde{I}_s\left(\frac{m}{n}\right)n^{s-1} & \text{if } n>0. \end{cases}$$

Note that for $V_s(n,m)$ as defined in (4.5), it holds that

$$\tilde{V}_s(n,m) = \frac{\Gamma(\frac{k-\ell}{2}+s)}{\Gamma(\frac{k+\ell}{2}-s)}V_s(n,m).$$

**Lemma 4.9.** *With notation as above and $T^{\delta}_m(s)$ as in Theorem 4.4, we have for every $\delta|D$ that*

$$T^{\delta}_m(s) = A_s(N^{[D]})\sum_{e|N^{[D]}}\mu\left(\frac{N^{[D]}}{e}\right)\frac{e}{N^{[D]}}T^{\delta}_{m,e}(s),$$

*Proof.* The equality follows immediately from Proposition 4.6, where it was proven that

$$S^{\delta}(s,B) = \frac{\chi(N^{[D]})(N^{[D]})^{1-2s}}{L^{(N)}(\chi,2s)}\sum_{e|N^{[D]}}\mu\left(\frac{N^{[D]}}{e}\right)\frac{e}{N^{[D]}}S^{\delta}_e(s,B).$$

□

We now define

$$T_{m,e}(s) = \sum_{\delta|D}T^{\delta}_{m,e}(s). \tag{4.10}$$

Recall from Theorem 4.4 that

$$T_m(s) = \sum_{\delta|D}T^{\delta}_m(s).$$

An obvious consequence of Lemma 4.9 is the following fact.

**Lemma 4.10.** *With $T_m(s)$ and $T_{m,e}(s)$ as above, we have*

$$T_m(s) = A_s(N^{[D]}) \sum_{e|N^{[D]}} \mu\left(\frac{N^{[D]}}{e}\right) \frac{e}{N^{[D]}} T_{m,e}(s).$$

□

Note that the functions $S_e^\delta(s,B)$ and hence $T_{m,e}(s)$ only (implicitly) depend on $N_{[D]}$, and not on $N^{[D]}$. In this sense, if $e$ is a proper divisor of $N^{[D]}$, the function $T_{m,e}(s)$ comes from holomorphic kernels of a lower level, and thus from oldforms. By choosing the right linear combination of holomorphic kernels, we are able to construct a modified kernel whose coefficients are expressible solely in terms of $T_{m,N^{[D]}}(s)$.

**Proposition 4.11.** *Fix positive integers $k, \ell, N$ and $D$ with $D$ square-free and $k \geq 4$. Let $\chi$ be a primitive Dirichlet character modulo $D$. Let $g \in M_\ell(\Gamma_0(D), \chi)$ be a modular form with Fourier expansion*

$$g(z) = b(0) + \sum_{n=1}^{\infty} b(n) n^{\frac{\ell-1}{2}} e^{2\pi i n z}.$$

*Let $\gamma$ be a bound on the coefficients of $g$ and its Atkin–Lehner translates as in (4.3). For $s \in \mathbf{C}$, define*

$$\tilde{\Phi}_{s,g} = \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{e|N^{[D]}} \frac{e}{N^{[D]}} \overline{A_{\bar{s}}(e)^{-1}} \Phi_{s,g,e},$$

*where $\Phi_{s,g,e}$ is the holomorphic kernel of level $eN_{[D]}$. Then $\tilde{\Phi}_{\bar{s},g}$ has a Fourier expansion*

$$\tilde{\Phi}_{\bar{s},g}(z) = \sum_{m=1}^{\infty} \tilde{\phi}_{\bar{s},g}(m) m^{\frac{k-1}{2}} e^{2\pi i m z},$$

*whose coefficients $\tilde{\phi}_{\bar{s},g}(m)$ for $1 < \Re(s) < \frac{k-1}{2} - \gamma$ are given by*

$$\overline{\tilde{\phi}_{\bar{s},g}(m)} = \frac{b(m)}{m^s} \sum_{e|N^{[D]}} \frac{e}{N^{[D]}} \frac{\Gamma(\frac{k-\ell}{2}+s) L^{(e)}(\chi, 2s)}{\Gamma(\frac{k+\ell}{2}-s)(2\pi)^{2s}\chi(e) e^{1-2s}} + 2\pi i^k T_{m,N^{[D]}}(s),$$

*where $T_{m,N^{[D]}}$ is defined as in (4.9). Moreover, $\tilde{\Phi}_{\bar{s},g} \in S_k(\Gamma_0(N))$, and for any $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$, we have*

$$L(f \otimes g, s) = \frac{(4\pi)^{k-1}}{(k-2)!} \frac{\Gamma(\frac{k+\ell}{2}-s)(2\pi)^{2s}\chi(N^{[D]})(N^{[D]})^{1-2s}}{\Gamma(\frac{k-\ell}{2}+s) L^{(N)}(\chi, 2s)} \langle f, \tilde{\Phi}_{\bar{s},g}\rangle.$$

*Proof.* For a divisor $e|N^{[D]}$, let $P_{m,e}$ denote the $m^{\text{th}}$ Poincaré series for $\Gamma_0(eN_{[D]})$ of weight $k$. Recall that

$$L(P_{m,e} \otimes g, s) = \frac{b(m)}{m^s} + 2\pi i^k T_m(e,s), \tag{4.11}$$

where we write $T_m(e,s)$ instead of $T_m(s)$ to explicitly indicate the dependence on the level $eN_{[D]}$. Now note that $S_e^\delta(s,B)$, and hence $T_{m,e}^\delta(s)$ and $T_{m,e}(s)$, only depend on $N_{[D]}$ and not on $N^{[D]}$.

Together with the observation that $(eN_{[D]})_{[D]} = N_{[D]}$, this shows that the coefficients $T_{m,e}(s)$ are equal for all levels that we consider. We find by Lemma 4.10 that for every divisor $e|N^{[D]}$,

$$T_m(e,s) = A_s(e)\sum_{d|e}\mu\left(\frac{e}{d}\right)\frac{d}{e}T_{m,d}(s).$$

This allows us to compute:

$$\begin{aligned}\sum_{e|N^{[D]}}\frac{e}{N^{[D]}}A_s(e)^{-1}T_m(e,s) &= \sum_{d|N^{[D]}}\frac{d}{N^{[D]}}T_{m,d}(s)\sum_{e|N^{[D]},d|e}\mu\left(\frac{e}{d}\right)\\ &= \sum_{d|N^{[D]}}\frac{d}{N^{[D]}}T_{m,d}(s)\sum_{e|\frac{N^{[D]}}{d}}\mu(e) = T_{m,N^{[D]}}(s),\end{aligned} \tag{4.12}$$

where we use that

$$\sum_{d|n}\mu(d) = \begin{cases}1 & \text{if } n=1,\\ 0 & \text{otherwise.}\end{cases}$$

If we now write $\phi_{s,g,e}(m)$ for the $m^{\text{th}}$ Fourier coefficient of $\Phi_{s,g,e}(z)$, it follows that

$$\begin{aligned}\overline{\tilde{\phi}_{\bar{s},g}(m)} &= \frac{(k-2)!}{(4\pi)^{k-1}}\sum_{e|N^{[D]}}\frac{e}{N^{[D]}}A_s(e)^{-1}\overline{\phi_{\bar{s},g,e}(m)}\\ &\overset{(4.2)}{=}\sum_{e|N^{[D]}}\frac{e}{N^{[D]}}A_s(e)^{-1}L(P_{m,e}\otimes g,s)\\ &\overset{(4.11)}{=}\sum_{e|N^{[D]}}\frac{e}{N^{[D]}}A_s(e)^{-1}\left(\frac{b(m)}{m^s}+2\pi i^kT_m(e,s)\right)\\ &\overset{(4.12)}{=}\frac{b(m)}{m^s}\sum_{e|N^{[D]}}\frac{e}{N^{[D]}}A_s(e)^{-1}+2\pi i^kT_{m,N^{[D]}}(s).\end{aligned}$$

This proves the formula for the coefficients of $\tilde{\Phi}_{\bar{s},g}$. As each of the holomorphic kernels $\Phi_{\bar{s},g,e}$ is a cusp form of weight $k$ for $\Gamma_0(eN_{[D]})$, each of them can be lifted to a cusp form of weight $k$ for $\Gamma_0(N)$. It follows that $\tilde{\Phi}_{\bar{s},g}(z)$ is a cusp form of weight $k$ for $\Gamma_0(N)$. Moreover, for a newform $f \in S_k^{\text{new}}(\Gamma_0(N))$, we have that $\langle f, \Phi_{\bar{s},g,e}\rangle = 0$ for all $e|N^{[D]}$ with $e \neq N^{[D]}$, as newforms are orthogonal to oldforms by definition, see (2.4). We conclude that

$$\begin{aligned}\langle f, \tilde{\Phi}_{\bar{s},g}\rangle &= \frac{(k-2)!}{(4\pi)^{k-1}}\sum_{e|N^{[D]}}\frac{e}{N^{[D]}}A_s(e)^{-1}\langle f, \Phi_{\bar{s},g,e}\rangle\\ &= \frac{(k-2)!}{(4\pi)^{k-1}}A_s(N^{[D]})^{-1}L(f\otimes g,s),\end{aligned}$$

as only $\langle f, \Phi_{\bar{s},g,N^{[D]}}\rangle \overset{(4.1)}{=} L(f\otimes g,s)$ survives in the sum. It follows that

$$L(f\otimes g,s) = \frac{(4\pi)^{k-1}}{(k-2)!}\frac{\Gamma(\frac{k+\ell}{2}-s)(2\pi)^{2s}\chi(N^{[D]})(N^{[D]})^{1-2s}}{\Gamma(\frac{k-\ell}{2}+s)L^{(N)}(\chi,2s)}\langle f, \tilde{\Phi}_{\bar{s},g}\rangle.$$

□

## 4.6 Modified holomorphic kernel for theta series

In this section, we will derive a formula for the Fourier coefficients of the modified holomorphic kernel $\tilde{\Phi}_{s,g}$ in the case that $g$ is a theta series (Theorem 4.13). From that formula, we deduce that $\tilde{\Phi}_{s,g}$ satisfies a functional equation in $s$ centered at $\frac{1}{2}$ (Theorem 4.15). We conclude this section with a functional equation for the Rankin–Selberg $L$-function $L(f \otimes \theta_{\mathcal{A},\psi}, s)$ (Theorem 4.16). In particular, we recover [LS24, Thm. 3.1].

### Fourier coefficients

We will now additionally assume that $D \equiv 3 \pmod 4$ and that $N$ is coprime to $D$. Consider the imaginary quadratic field $K = \mathbf{Q}(\sqrt{-D})$ of discriminant $\Delta_K = -D$. It has an associated Dirichlet character $\chi$ modulo $D$ given by the Kronecker symbol $\chi(\cdot) = \left(\frac{\Delta_K}{\cdot}\right)$. To state the final formula for the coefficients, we need the following arithmetic function.

**Definition 4.12.** Let $D = \delta \cdot \delta'$ be a decomposition. We define the multiplicative arithmetic function $\hat{\chi}_{\delta\cdot\delta'} : \mathbf{Z} \to \mathbf{C}$ by

$$\hat{\chi}_{\delta\cdot\delta'}(n) = \chi_\delta(n_{[\delta']}) \cdot \chi_{\delta'}(n^{[\delta']}).$$

Note that $\hat{\chi}_{\delta\cdot\delta'}(n) = \pm 1$ if $n \neq 0$ and that $\hat{\chi}_{\delta\cdot\delta'}(0) = 0$. For a decomposition $D = \delta_1 \cdot \delta_2 \cdot \delta_3$, it satisfies

$$\hat{\chi}_{\delta_1\delta_2\cdot\delta_3}(n) \cdot \hat{\chi}_{\delta_1\delta_3\cdot\delta_2}(n) \cdot \hat{\chi}_{\delta_1\cdot\delta_2\delta_3}(n) = 1. \tag{4.13}$$

By taking $\delta_1 = 1$, it follows that $\hat{\chi}_{\delta\cdot\delta'}$ depends on the order of the decomposition in the following way:

$$\hat{\chi}_{\delta\cdot\delta'}(n) = \chi(n^{[D]}) \cdot \hat{\chi}_{\delta'\cdot\delta}(n). \tag{4.14}$$

Recall from (2.12) the character on ideals $\chi_{\delta\cdot\delta'}$. Given a non-zero ideal $\mathfrak{a} \subseteq \mathcal{O}_K$, it relates to $\hat{\chi}_{\delta\cdot\delta'}$ via

$$\chi_{\delta\cdot\delta'}(\mathfrak{a}) = \hat{\chi}_{\delta\cdot\delta'}(\mathrm{N}(\mathfrak{a})). \tag{4.15}$$

In fact, if one extends $\hat{\chi}_{\delta\cdot\delta'}$ to $\mathbf{Q}$ via $\hat{\chi}_{\delta\cdot\delta'}(\frac{m}{n}) = \hat{\chi}_{\delta\cdot\delta'}(mn)$, then (4.15) holds for any fractional ideal $\mathfrak{a}$ of $\mathcal{O}_K$.

We can now give a formula for the Fourier coefficients of $\tilde{\Phi}_{\bar{s},g}$ in the case that the modular form $g$ is the theta series associated to an unramified Hecke character, as considered in Section 2.3. As we assume that the Hecke character has infinity type $(\ell-1, 0)$, it follows that the weight of $g$ must be odd.

We will write $\tilde{\Phi}_{s,\theta}$ instead of $\tilde{\Phi}_{s,\theta_{\mathcal{A},\psi}}$ to make the notation more concise.

**Theorem 4.13.** *Let $k, \ell, D$ and $N$ be positive integers, with $k \geq 4$ even, $\ell$ odd, $D$ square-free and $\gcd(D, N) = 1$. Assume furthermore that $D \equiv 3 \pmod 4$. Let $\chi$ be the Dirichlet character modulo $D$ given by the Kronecker symbol corresponding to the imaginary quadratic field $K = \mathbf{Q}(\sqrt{-D})$. Let $\psi$ be an unramified Hecke character of infinity type $(\ell-1, 0)$, and consider for a class $\mathcal{A} \in \mathrm{Cl}_K$ the associated theta series*

$$\theta_{\mathcal{A},\psi}(z) = b(0) + \sum_{n=1}^{\infty} b(n) n^{\frac{\ell-1}{2}} e^{2\pi i n z} \in M_\ell(\Gamma_0(D), \chi).$$

*Then for $\frac{3-k}{2} < \Re(s) < \frac{k-1}{2}$, the holomorphic kernel function $\tilde{\Phi}_{\bar{s},\theta}(z)$ has Fourier coefficients whose complex conjugates are given by:*

$$\begin{aligned}\overline{\tilde{\phi}_{\bar{s},\theta}(m)} = i^{k+\ell-1}\chi(-N)L(\chi,2s)\frac{N^{2s-1}}{(2\pi)^{2s}}\frac{\Gamma(\frac{1}{2}-s)\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{\ell-k}{2}+1-s)\Gamma(\frac{\ell+k}{2}-s)}\frac{b(m)}{m^s}\\ + i^{k+\ell-1}L(\chi,2-2s)\frac{D^{1-2s}}{(2\pi)^{2-2s}}\frac{\Gamma(s-\frac{1}{2})\Gamma(\frac{3}{2}-s)}{\Gamma(\frac{\ell-k}{2}+s)\Gamma(\frac{\ell+k}{2}-1+s)}\frac{b(m)}{m^{1-s}}\\ + i^kD^{\frac{1}{2}-s}\sum_{\substack{n\geq 1, n\neq mD\\ n\equiv mD \pmod N}}\frac{b(n)}{n^{1-s}}M_{s,\mathcal{A}}\left(\frac{mD-n}{N}\right)\tilde{I}_s\left(\frac{mD}{n}\right)\\ + i^{k+\ell-1}\frac{m^{s-\frac{\ell+1}{2}}}{\Gamma(\ell)\sqrt{D}}b(0)M_{s,\mathcal{A}}\left(\frac{mD}{N}\right),\end{aligned}$$

*with*

$$M_{s,\mathcal{A}}(t) = \sum_{\substack{d|t\\(d,D)=1}}\chi(d)d^{1-2s}\cdot\sum_{\delta'|(D,t)}\hat{\chi}_{\delta\cdot\delta'}(-Nt)\chi_{\delta\cdot\delta'}(\mathcal{A})t_{[\delta']}^{1-2s}. \tag{4.16}$$

*Proof.* By the assumption that $N$ and $D$ are coprime, we have that $N$ decomposes as $N_{[D]} = 1$ and $N^{[D]} = N$. Moreover, we have a bound $b^\delta(n) = O(n^\varepsilon)$ as $n\to\infty$, for any $\varepsilon > 0$ and $\delta|D$. By Proposition 4.11, we find for $1 < \Re(s) < \frac{k-1}{2}$ that

$$\overline{\tilde{\phi}_{\bar{s},g}(m)} = \frac{b(m)}{m^s}\sum_{e|N}\frac{e}{N}\frac{\Gamma(\frac{k-\ell}{2}+s)L^{(e)}(\chi,2s)}{\Gamma(\frac{k+\ell}{2}-s)(2\pi)^{2s}\chi(e)e^{1-2s}} + 2\pi i^kT_{m,N}(s).$$

Recall by (3.3) that

$$L^{(e)}(\chi,2s) = L(\chi,2s)\prod_{p|e}\left(1-\frac{\chi(p)}{p^{2s}}\right).$$

Moreover, we have

$$\begin{aligned}\sum_{e|N}\chi(e)e^{2s}\prod_{p|e}\left(1-\frac{\chi(p)}{p^{2s}}\right) &= \sum_{e|N}\chi(e)e^{2s}\sum_{d|e}\mu(d)\chi(d)d^{-2s} = \sum_{d|N}\mu(d)\sum_{e|N, d|e}\chi(\tfrac{e}{d})\left(\frac{e}{d}\right)^{2s}\\ &= \sum_{d|N}\mu(d)\sum_{e|\frac{N}{d}}\chi(e)e^{2s} = \sum_{e|N}\chi(e)e^{2s}\sum_{d|\frac{N}{e}}\mu(d) = \chi(N)N^{2s}.\end{aligned} \tag{4.17}$$

It follows that

$$\sum_{e|N}\frac{e}{N}\frac{L^{(e)}(\chi,2s)}{\chi(e)e^{1-2s}} \overset{(3.3)}{=} \frac{L(\chi,2s)}{N}\sum_{e|N}\chi(e)e^{2s}\prod_{p|e}\left(1-\frac{\chi(p)}{p^{2s}}\right) \overset{(4.17)}{=} \chi(N)L(\chi,2s)N^{2s-1}.$$

This allows us to obtain the first term:

$$\begin{aligned}\frac{b(m)}{m^s}\frac{\Gamma(\frac{k-\ell}{2}+s)}{\Gamma(\frac{k+\ell}{2}-s)(2\pi)^{2s}}\sum_{e|N}\frac{e}{N}\frac{L^{(e)}(\chi,2s)}{\chi(e)e^{1-2s}} &= \frac{b(m)}{m^s}\chi(N)L(\chi,2s)N^{2s-1}\frac{\Gamma(\frac{k-\ell}{2}+s)}{\Gamma(\frac{k+\ell}{2}-s)(2\pi)^{2s}}\\ &= i^{k+\ell-1}\frac{b(m)}{m^s}\chi(-N)L(\chi,2s)\frac{N^{2s-1}}{(2\pi)^{2s}}\frac{\Gamma(\frac{1}{2}-s)\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{\ell-k}{2}+1-s)\Gamma(\frac{k+\ell}{2}-s)}.\end{aligned}$$

Here we used that $\chi(-1) = -1$, and we used the following equality, which follows from the reflection formula for $\Gamma(s)$:

$$\Gamma\left(\frac{k-\ell}{2}+s\right)\Gamma\left(\frac{\ell-k}{2}+1-s\right) = \frac{(-1)^{\frac{k-\ell-1}{2}}\pi}{\cos(\pi s)} = -i^{k+\ell-1}\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(\frac{1}{2}+s\right).$$

Now recall from (4.9) and (4.10) that

$$T_{m,N}(s) = \sum_{\delta|D} \frac{\delta^{\frac{1}{2}-s}i^{\ell}}{2\pi} \sum_{n=0}^{\infty} b^{\delta}(n)S_N^{\delta}(s, m\delta - n)\tilde{V}_s(n, \delta m). \tag{4.18}$$

In order to obtain the second and third term of $\tilde{\phi}_{\bar{s},\theta}(m)$, we consider the summands for $n \geq 1$ in the series above. After multiplying by $2\pi i^k$ and substituting $n \mapsto \frac{n}{\delta'}$, these summands yield

$$i^{k+\ell}\sum_{\delta|D} D^{\frac{1}{2}-s}\sum_{\substack{n=1\\ \delta'|n}}^{\infty} \frac{b^{\delta}(\frac{n}{\delta'})}{n^{1-s}}\sqrt{\delta'}S_N^{\delta}\left(s, \frac{mD-n}{\delta'}\right)\tilde{I}_s\left(\frac{mD}{n}\right).$$

By Proposition 4.6, the term corresponding to some $n \geq 1$ is only non-zero if $N$ divides $\frac{mD-n}{\delta'}$ and thus only if $N$ divides $mD - n$. Moreover, using Proposition 2.15 (with the corresponding property above each equal sign below), we know that

$$b^{\delta}(\tfrac{n}{\delta'}) \overset{(4)}{=} \kappa(\delta)^{-1}\chi_{\delta'}(\delta)\chi_{\delta\cdot\delta'}(\mathcal{A})b(\tfrac{n}{\delta'}\delta) \overset{(2)}{=} \kappa(\delta)^{-1}\chi_{\delta'}(\delta)\chi_{\delta\cdot\delta'}(\mathcal{A})b(nD) \overset{(1)}{=} i^{\ell-1}\kappa(\delta)^{-1}\chi_{\delta'}(\delta)\chi_{\delta\cdot\delta'}(\mathcal{A})b(n),$$

and so we obtain the two terms that will lead to the second and third term:

$$\begin{aligned}
&i^{k+1}\sum_{\delta|D}D^{\frac{1}{2}-s}\sum_{\substack{n=1\\ \delta'|n}}^{\infty}\frac{\kappa(\delta)^{-1}\chi_{\delta'}(\delta)\chi_{\delta\cdot\delta'}(\mathcal{A})b(n)}{n^{1-s}}\sqrt{\delta'}S_N^{\delta}\left(s, \frac{mD-n}{\delta'}\right)\tilde{I}_s\left(\frac{mD}{n}\right)\\
&\qquad = i^k D^{\frac{1}{2}-s}\sum_{\substack{n=1\\ N|mD-n}}^{\infty}\frac{b(n)}{n^{1-s}}\sum_{\delta'|(D,n)}\kappa(\delta')\chi_{\delta'}(\delta)\chi_{\delta\cdot\delta'}(\mathcal{A})\sqrt{\delta'}S_N^{\delta}\left(s, \frac{mD-n}{\delta'}\right)\tilde{I}_s\left(\frac{mD}{n}\right)\\
&\qquad = i^{k+\ell-1}\frac{1}{D^{\frac{1}{2}}}\frac{b(m)}{m^{1-s}}M_{s,\mathcal{A}}^*(mD)\tilde{I}_s(1) + i^k D^{\frac{1}{2}-s}\sum_{\substack{n=1, n\neq mD\\ n\equiv mD \pmod N}}^{\infty}\frac{b(n)}{n^{1-s}}M_{s,\mathcal{A}}^*(n)\tilde{I}_s\left(\frac{mD}{n}\right),
\end{aligned}$$

where we let

$$M_{s,\mathcal{A}}^*(n) = \sum_{\delta'|(D,n)}\kappa(\delta')\chi_{\delta'}(\delta)\chi_{\delta\cdot\delta'}(\mathcal{A})\sqrt{\delta'}S_N^{\delta}\left(s, \frac{mD-n}{\delta'}\right).$$

By Proposition 4.6, we know that $S_N^{\delta}(s, 0) = 0$ for $\delta \neq D$, and that $S_N^D(s, 0) = L(\chi, 2s-1)$. From this, it follows that $M_{s,\mathcal{A}}^*(mD)$ reduces to $L(\chi, 2s-1)$. Using the functional equation for $L(\chi, s)$ as given in Theorem 3.3, we obtain the second term:

$$\begin{aligned}
i^{k+\ell-1}\frac{1}{D^{\frac{1}{2}}}\frac{b(m)}{m^{1-s}}M_{s,\mathcal{A}}^*(mD)\tilde{I}_s(1) &= i^{k+\ell-1}\frac{1}{D^{\frac{1}{2}}}\frac{b(m)}{m^{1-s}}L(\chi, 2s-1)\tilde{I}_s(1)\\
&= i^{k+\ell-1}\frac{1}{D^{\frac{1}{2}}}\frac{b(m)}{m^{1-s}}2^{2s-1}\pi^{2s-2}D^{\frac{3}{2}-2s}\sin(\pi s)\frac{\Gamma(2-2s)\Gamma(2s-1)}{\Gamma(\frac{\ell-k}{2}+s)\Gamma(\frac{\ell+k}{2}-1+s)}L(\chi, 2-2s)\\
&= i^{k+\ell-1}L(\chi, 2-2s)\frac{D^{1-2s}}{(2\pi)^{2-2s}}\frac{b(m)}{m^{1-s}}\frac{\Gamma(s-\frac{1}{2})\Gamma(\frac{3}{2}-s)}{\Gamma(\frac{\ell-k}{2}+s)\Gamma(\frac{\ell+k}{2}-1+s)}.
\end{aligned}$$

Here we used Proposition 4.8 to rewrite $\tilde{I}(1)$, and we used the following equality, which follows from the reflection formula for $\Gamma(s)$:

$$2\sin(\pi s)\Gamma(2-2s)\Gamma(2s-1) = \frac{-2\pi\sin(\pi s)}{\sin(2\pi s)} = \frac{-\pi}{\cos(\pi s)} = \Gamma(s-\tfrac{1}{2})\Gamma(\tfrac{3}{2}-s).$$

We will derive the third term from

$$i^k D^{\frac{1}{2}-s} \sum_{\substack{n=1, n\neq mD \\ n\equiv mD \pmod N}}^{\infty} \frac{b(n)}{n^{1-s}} M^*_{s,\mathcal{A}}(n) \tilde{I}_s\left(\frac{mD}{n}\right),$$

and so it suffices to show that $M^*_{s,\mathcal{A}}(n) = M_{s,\mathcal{A}}(\frac{mD-n}{N})$ for all $n \geq 1$ with $n \equiv mD \pmod N$ and $n \neq mD$. Write $t = \frac{mD-n}{N}$, so that $\frac{mD-n}{\delta'} = \frac{Nt}{\delta'}$. By definition of $S_N^\delta$ (see Proposition 4.6), we then have for $\delta' |\gcd(D,n)$:

$$S_N^\delta\left(s, \tfrac{Nt}{\delta'}\right) = \frac{\tau(\chi_{\delta'})\chi_\delta((Nt/\delta')_{[\delta']})}{(\delta')^{2s}(t_{[\delta']}/\delta')^{2s-1}\chi_{\delta'}((Nt/\delta')^{[\delta']})} \sum_{\substack{d|\frac{t}{\delta'} \\ (d,D)=1}} \chi(d) d^{1-2s}.$$

First, we note that we may replace $\frac{t}{\delta'}$ with $t$ in the subscript of the summation, due to the coprimality condition. Next, we note that

$$\frac{\chi_\delta((Nt/\delta')_{[\delta']})}{\chi_{\delta'}((Nt/\delta')^{[\delta']})} = \chi_\delta((Nt/\delta')_{[\delta']}) \cdot \chi_{\delta'}((Nt/\delta')^{[\delta']}) = \hat{\chi}_{\delta\cdot\delta'}(Nt/\delta') = \hat{\chi}_{\delta\cdot\delta'}(Nt\delta').$$

Lastly, by [Miy06, Lemma 4.8.1], we know that $\tau(\chi_{\delta'}) = \kappa(\delta')\sqrt{\delta'}$. We find that

$$\begin{aligned}
M^*_{s,\mathcal{A}}(n) &= \sum_{\delta'|(D,n)} \kappa(\delta')\chi_{\delta'}(\delta)\chi_{\delta\cdot\delta'}(\mathcal{A})\sqrt{\delta'} S_N^\delta\left(s, \frac{mD-n}{\delta'}\right) \\
&= \sum_{\delta'|(D,n)} \chi_{\delta'}(-1)\chi_{\delta'}(\delta)\chi_{\delta\cdot\delta'}(\mathcal{A})\hat{\chi}_{\delta\cdot\delta'}(Nt\delta') t_{[\delta']}^{1-2s} \sum_{\substack{d|t \\ (d,D)=1}} \chi(d) d^{1-2s} \\
&= \sum_{\delta'|(D,t)} \hat{\chi}_{\delta\cdot\delta'}(-Nt)\chi_{\delta\cdot\delta'}(\mathcal{A}) t_{[\delta']}^{1-2s} \sum_{\substack{d|t \\ (d,D)=1}} \chi(d) d^{1-2s} \\
&= M_{s,\mathcal{A}}(t).
\end{aligned}$$

Here we used that $\chi_{\delta'}(\delta) \cdot \hat{\chi}_{\delta\cdot\delta'}(\delta') = \hat{\chi}_{\delta\cdot\delta'}(D) = 1$.

For the fourth term, we consider the summands for $n = 0$ occurring in (4.18), namely

$$\sum_{\delta|D} \frac{\delta^{\frac{1}{2}-s} i^\ell}{2\pi\Gamma(\ell)} b^\delta(0) S_N^\delta(s, m\delta)(m\delta)^{s-\frac{\ell+1}{2}}.$$

Note that this sum is only non-zero if $\theta_{\mathcal{A},\psi}$ is not a cusp form, i.e., if $\ell = 1$. Moreover, the terms of this sum are only nonzero if $N|m\delta$, or equivalently if $N|m$ or if $N|mD$. In that case, by Proposition 4.6, we know that

$$S_N^\delta(s, m\delta) = \frac{\sqrt{\delta'}\kappa(\delta')\hat{\chi}_{\delta\cdot\delta'}(m\delta)}{\delta'^{2s} m_{[\delta']}^{2s-1}} \sum_{\substack{d \mid \frac{m}{N} \\ (d,D)=1}} \chi(d) d^{1-2s}.$$

Together with Proposition 2.15 (5), we find that the fourth term is given by

$$\begin{aligned} i^{k+\ell} \sum_{\delta \mid D} \frac{\delta^{\frac{1}{2}-s} i^\ell}{\Gamma(\ell)} b(0) \delta^{\frac{\ell-1}{2}} \kappa(\delta)^{-1} \chi_{\delta'}(\delta) \chi_{\delta\cdot\delta'}(\mathcal{A}) (m\delta)^{s-\frac{\ell+1}{2}} \frac{\sqrt{\delta'}\kappa(\delta')\hat{\chi}_{\delta\cdot\delta'}(m\delta)}{\delta'^{2s} m_{[\delta']}^{2s-1}} \sum_{\substack{d \mid \frac{m}{N} \\ (d,D)=1}} \chi(d) d^{1-2s} \\ = \frac{i^{k+\ell-1} m^{s-\frac{\ell+1}{2}}}{\Gamma(\ell)\sqrt{D}} b(0) \sum_{\delta' \mid D} (mD)_{[\delta']}^{1-2s} \hat{\chi}_{\delta\cdot\delta'}(-mD) \chi_{\delta\cdot\delta'}(\mathcal{A}) \sum_{\substack{d \mid \frac{mD}{N} \\ (d,D)=1}} \chi(d) d^{1-2s}. \end{aligned}$$

We have now shown the theorem for $1 < \Re(s) < \frac{k-1}{2}$. For a more general $\frac{3-k}{2} < \Re(s) < \frac{k-1}{2}$, we appeal to the uniqueness of a meromorphic extension. It is clear that the first, second and fourth term have a meromorphic extension to the whole complex plane. For the third term, we first consider $\frac{1}{2} \le \Re(s) < \frac{k-1}{2}$. In that case, we have the bounds $b(n) = O(n^\varepsilon)$, $\tilde{I}_s\left(\frac{mD}{n}\right) = O(n^{\frac{1-k}{2}})$ and $M_{s,\mathcal{A}}(n) = O(n^\varepsilon)$ as $n \to \infty$. It follows that the summand of the third term satisfies the bound $O(n^{s+\epsilon-\frac{k+1}{2}})$ as $n \to \infty$, for any $\varepsilon > 0$. In particular, we see that the third term converges absolutely and uniformly on compact subsets of $\{s \in \mathbf{C} \mid \frac{1}{2} \le \Re(s) < \frac{k-1}{2}\}$ (excluding obvious poles of the summand).

Now consider $\frac{3-k}{2} < \Re(s) \le \frac{1}{2}$. In that case we have the same bounds for $b(n)$ and $\tilde{I}_s\left(\frac{mD}{n}\right)$. For $M_{s,\mathcal{A}}(n)$, we have the bound $M_{s,\mathcal{A}}(n) = O(n^{1-2s+\varepsilon})$ as $n \to \infty$. We see that the summand is bounded by $O(n^{\varepsilon+\frac{1-k}{2}-s})$ as $n \to \infty$, for any $\varepsilon > 0$. It follows that the sum converges absolutely and uniformly on compact subsets of $\{s \in \mathbf{C} \mid \frac{3-k}{2} < \Re(s) \le \frac{1}{2}\}$ (once again, excluding obvious poles of the summand). We see that the third term is meromorphic on both regions and as they overlap on the line $\Re(s) = \frac{1}{2}$, we conclude that the third term defines a meromorphic function on $\frac{3-k}{2} < \Re(s) < \frac{k-1}{2}$. This concludes the proof. □

**Functional equation of $\tilde{\Phi}_{s,\theta}$**

This special version of the modified holomorphic kernel satisfies a functional equation relating $\tilde{\Phi}_{1-s,\theta}$ to $\tilde{\Phi}_{s,\theta}$. We prove this using the formulas we have obtained in Theorem 4.13. As such, we need a functional equation for $M_{s,\mathcal{A}}(t)$.

**Lemma 4.14.** *Let $s \in \mathbf{C}$ and let $t$ be a non-zero integer. Decompose $D = D_1 \cdot D_2$ with $|D_1| = \gcd(t, D)$. Then*

$$M_{1-s,\mathcal{A}}(t) = \operatorname{sgn}(t)\chi(-N)|t|^{2s-1}\hat{\chi}_{D_1\cdot D_2}(-Nt)\chi_{D_1\cdot D_2}(\mathcal{A})M_{s,\mathcal{A}}(t).$$

*Proof.* Let $n$ be the norm of any ideal in $\mathcal{A}$, so that $\chi_{\delta\cdot\delta'}(\mathcal{A}) = \hat{\chi}_{\delta\cdot\delta'}(n)$ by (4.15) for any decomposition $D = \delta \cdot \delta'$. Note that as $d$ ranges over the divisors of $t$ that are coprime to $D$, so does

$|t|/(t_{[D]}d) = \mathrm{sgn}(t)|t|^{[D]}/d$. Using the substitution $d \mapsto |t|/(t_{[D]}d)$ in the formula (4.16) for $M_{1-s,\mathcal{A}}(t)$, we see that

$$\begin{aligned} M_{1-s,\mathcal{A}}(t) &= \sum_{\substack{d|t \\ (d,D)=1}} \chi(|t|/(t_{[D]}d)) \left(\frac{|t|}{t_{[D]}d}\right)^{2s-1} \sum_{\delta'|(D,t)} \hat{\chi}_{\delta\cdot\delta'}(-Nt)\chi_{\delta\cdot\delta'}(\mathcal{A})t_{[\delta']}^{2s-1} \\ &= \mathrm{sgn}(t)\chi(t^{[D]})|t|^{2s-1} \sum_{\substack{d|t \\ (d,D)=1}} \chi(d)d^{1-2s} \sum_{\delta'|(D,t)} \hat{\chi}_{\delta\cdot\delta'}(-Ntn)t_{[\delta]}^{1-2s}. \end{aligned}$$

As $\delta'$ ranges over the divisors of $(D,t)$, we can write $\delta = D^{[t]} \cdot \delta^*$, where $\delta^*$ ranges over the divisors of $(D,t)$. Note that $t_{[\delta]}^{1-2s} = t_{[\delta^*]}^{1-2s}$, as $t$ and $D^{[t]}$ do not have any divisors in common. It follows that

$$\begin{aligned} \sum_{\delta'|(D,t)} \hat{\chi}_{\delta\cdot\delta'}(-Ntn)t_{[\delta]}^{1-2s} &= \sum_{\delta^*|(D,t)} \hat{\chi}_{D^{[t]}\delta^*\cdot\delta'}(-Ntn)t_{[\delta^*]}^{1-2s} \\ &\overset{(4.13)}{=} \hat{\chi}_{D^{[t]}\cdot D_{[t]}}(-Ntn) \sum_{\delta^*|(D,t)} \hat{\chi}_{D^{[t]}\delta'\cdot\delta^*}(-Ntn)t_{[\delta^*]}^{1-2s}. \end{aligned}$$

Now note that $\chi(t^{[D]}) = \chi(-N \cdot (-Nt)^{[D]})$. If we let $D = D_1 \cdot D_2$ with $|D_1| = D_{[t]} = \gcd(D,t)$, we conclude that

$$\begin{aligned} M_{1-s,\mathcal{A}}(t) &= \mathrm{sgn}(t)\chi(t^{[D]})|t|^{2s-1}\hat{\chi}_{D_2\cdot D_1}(-Nt)\chi_{D_2\cdot D_1}(\mathcal{A})M_{s,\mathcal{A}}(t) \\ &\overset{(4.14)}{=} \mathrm{sgn}(t)\chi(-N)|t|^{2s-1}\hat{\chi}_{D_1\cdot D_2}(-Nt)\chi_{D_1\cdot D_2}(\mathcal{A})M_{s,\mathcal{A}}(t). \end{aligned}$$

□

**Theorem 4.15.** *Under the assumptions of Theorem 4.13, $\tilde{\Phi}_{s,\theta}$ satisfies the functional equation given by*

$$\tilde{\Phi}_{1-s,\theta} = \chi(-N)D^{2s-1}N^{1-2s}\tilde{\Phi}_{s,\theta}.$$

*Proof.* It suffices to show for $m \geq 1$ that

$$\overline{\tilde{\phi}_{\overline{1-s},\theta}(m)} = \chi(-N)D^{2s-1}N^{1-2s}\overline{\tilde{\phi}_{\bar{s},\theta}(m)}.$$

It is clear that the first two terms of $\overline{\tilde{\phi}_{\bar{s},\theta}(m)}$ as in Theorem 4.13 are swapped up to multiplication by $\chi(-N)D^{2s-1}N^{1-2s}$ under the transformation $s \mapsto 1-s$. For the third term, we will use the functional equations of $M_{s,\mathcal{A}}(t)$ and $\tilde{I}_s(x)$ to show that every summand satisfies the functional equation. Let $n \geq 1$ with $n \neq mD$ and $n \equiv mD \pmod{N}$. Decompose $D = D_1 \cdot D_2$ with $|D_1| = \gcd(D, \frac{n-mD}{N})$. Note that $\gcd(D, \frac{n-mD}{N}) = \gcd(D,n)$, as $N$ is coprime to $D$. If $b(n) = 0$, then the entire term corresponding to $n$ vanishes and we are done. We will thus assume $b(n) \neq 0$, and may in particular assume that there is some ideal $\mathfrak{a} \in \mathcal{A}$ with $\mathrm{N}(\mathfrak{a}) = n$. This tells us that

$$\hat{\chi}_{D_1\cdot D_2}(n-mD) \cdot \chi_{D_1\cdot D_2}(\mathcal{A}) = \chi_{D_2}(n-mD) \cdot \chi_{D_2}(n) = \chi_{D_2}(n)^2 = 1,$$

as $D_2 = D^{[n-mD]} = D^{[n]}$, which is coprime to both $n$ and $n-mD$. It follows by Lemma 4.14 that

$$M_{1-s,\mathcal{A}}\left(\frac{mD-n}{N}\right) = \mathrm{sgn}\left(mD-n\right)\chi(-N)\left|\frac{mD-n}{N}\right|^{2s-1} M_{s,\mathcal{A}}\left(\frac{mD-n}{N}\right).$$

By the functional equation of $\tilde{I}_s(x)$ (Proposition 4.7), we have

$$\tilde{I}_{1-s}\left(\frac{mD}{n}\right) = \left|\frac{mD-n}{n}\right|^{1-2s} \operatorname{sgn}(mD-n)\, \tilde{I}_s\left(\frac{mD}{n}\right).$$

We conclude for every $n \geq 1$ that

$$i^k D^{s-\frac{1}{2}} \frac{b(n)}{n^s} M_{1-s,\mathcal{A}}\left(\frac{mD-n}{N}\right) \tilde{I}_{1-s}\left(\frac{mD}{n}\right) \\ = \chi(-N) D^{2s-1} N^{1-2s} \cdot i^k D^{\frac{1}{2}-s} \frac{b(n)}{n^{1-s}} M_{s,\mathcal{A}}\left(\frac{mD-n}{N}\right) \tilde{I}_s\left(\frac{mD}{n}\right).$$

For the fourth term, we use the functional equation of $M_{s,\mathcal{A}}\left(\frac{mD}{N}\right)$. As $\gcd(D, \frac{mD}{N}) = D$, we find that

$$M_{1-s,\mathcal{A}}\left(\frac{mD}{N}\right) = \hat{\chi}_{D\cdot 1}(-mD)\chi_{D\cdot 1}(\mathcal{A}) \left(\frac{mD}{N}\right)^{2s-1} \chi(-N) M_{s,\mathcal{A}}\left(\frac{mD}{N}\right).$$

As $\hat{\chi}_{D\cdot 1}(-mD) = 1$ and $\chi_{D\cdot 1}(\mathcal{A}) = 1$, we deduce that

$$i^{k+\ell-1} \frac{m^{\frac{\ell-1}{2}-s}}{\Gamma(\ell)\sqrt{D}} b(0) M_{1-s,\mathcal{A}}\left(\frac{mD}{N}\right) = \chi(-N) D^{2s-1} N^{1-2s} i^{k+\ell-1} \frac{m^{s-\frac{\ell+1}{2}}}{\Gamma(\ell)\sqrt{D}} b(0) M_{s,\mathcal{A}}\left(\frac{mD}{N}\right).$$

□

**Functional equation of $L(f \otimes \theta_{\mathcal{A},\psi}, s)$**

Using Proposition 4.11 and Theorem 4.15, we can now deduce a functional equation for $L(f \otimes \theta_{\mathcal{A},\psi}, s)$ for a newform $f \in S_k^{\text{new}}(\Gamma_0(N))$. Define the completed $L$-function

$$\Lambda(f \otimes \theta_{\mathcal{A},\psi}, s) = L^{(N)}(\chi, 2s) D^s N^s (2\pi)^{-2s} \Gamma(\tfrac{k+\ell}{2} + s - 1) \Gamma(\tfrac{|k-\ell|}{2} + s) L(f \otimes \theta_{\mathcal{A},\psi}, s). \tag{4.19}$$

**Theorem 4.16.** *Let $k, \ell, D$ and $N$ be positive integers, with $D$ square-free and $\gcd(D, N) = 1$. Assume furthermore that $k \geq 4$ is even, that $\ell$ is odd, and that $D \equiv 3 \pmod 4$. Let $\chi$ be the Dirichlet character modulo $D$ given by the Kronecker symbol corresponding to the imaginary quadratic field $K = \mathbf{Q}(\sqrt{-D})$. Let $\psi$ be an unramified Hecke character of infinity type $(\ell - 1, 0)$, and consider for a class $\mathcal{A} \in \mathrm{Cl}_K$ the theta series $\theta_{\mathcal{A},\psi}$. Then for a newform $f \in S_k^{\text{new}}(\Gamma_0(N))$, the completed Rankin–Selberg L-function $\Lambda(f \otimes \theta_{\mathcal{A},\psi}, s)$ satisfies the functional equation*

$$\Lambda(f \otimes \theta_{\mathcal{A},\psi}, 1-s) = \chi(\epsilon N) \Lambda(f \otimes \theta_{\mathcal{A},\psi}, s),$$

*with*

$$\epsilon = \begin{cases} -1 & \text{if } k > \ell, \\ 1 & \text{if } \ell > k. \end{cases}$$

*Proof.* We first consider the case $k > \ell$. By Proposition 4.11, we know that

$$L(f \otimes \theta_{\mathcal{A},\psi}, s) = \frac{(4\pi)^{k-1}}{(k-2)!} \frac{\Gamma(\frac{k+\ell}{2} - s)(2\pi)^{2s} \chi(N) N^{1-2s}}{\Gamma(\frac{k-\ell}{2} + s) L^{(N)}(\chi, 2s)} \langle f, \tilde{\Phi}_{\bar{s},\theta} \rangle,$$

and so

$$\Lambda(f\otimes\theta_{\mathcal{A},\psi},s)=\frac{(4\pi)^{k-1}}{(k-2)!}\chi(N)D^sN^{1-s}\Gamma(\tfrac{k+\ell}{2}-s)\Gamma(\tfrac{k+\ell}{2}+s-1)\langle f,\tilde{\Phi}_{\bar{s},\theta}\rangle.$$

Next, note that by Theorem 4.15, we have

$$\langle f,\tilde{\Phi}_{1-\bar{s},\theta}\rangle=\overline{\chi(-N)D^{2\bar{s}-1}N^{1-2\bar{s}}}\langle f,\tilde{\Phi}_{\bar{s},\theta}\rangle=\chi(-N)D^{2s-1}N^{1-2s}\langle f,\tilde{\Phi}_{\bar{s},\theta}\rangle.$$

We conclude that

$$\begin{aligned}\Lambda(f\otimes\theta_{\mathcal{A},\psi},1-s)&=\frac{(4\pi)^{k-1}}{(k-2)!}\chi(N)D^{1-s}N^s\Gamma(\tfrac{k+\ell}{2}+s-1)\Gamma(\tfrac{k+\ell}{2}-s)\langle f,\tilde{\Phi}_{1-\bar{s},\theta}\rangle\\&=-\frac{(4\pi)^{k-1}}{(k-2)!}D^sN^{1-s}(2\pi)^{2-2s}\Gamma(\tfrac{k+\ell}{2}s-1)\Gamma(\tfrac{k+\ell}{2}-s)\langle f,\tilde{\Phi}_{\bar{s},\theta}\rangle\\&=\chi(-N)\Lambda(f\otimes\theta_{\mathcal{A},\psi},s).\end{aligned}$$

In the case that $\ell>k$, the gamma functions do not cancel and following the same steps yields

$$\Lambda(f\otimes\theta_{\mathcal{A},\psi},1-s)=\chi(-N)\frac{\Gamma(\frac{\ell-k}{2}+1-s)\Gamma(\frac{k-\ell}{2}+s)}{\Gamma(\frac{k-\ell}{2}+1-s)\Gamma(\frac{\ell-k}{2}+s)}\Lambda(f\otimes\theta_{\mathcal{A},\psi},s).$$

This quotient of gamma functions is equal to $-1$, as can be shown using the reflection formula. □

**Remark.** This recovers [LS24, Thm. 3.1] by the Goldfeld–Zhang method.

# 5 Special values and derivatives

In this chapter, we will derive formulas for the values and derivatives of Rankin–Selberg $L$-series at specific points. In this way, we recover the results of Gross–Zagier [GZ86] and Lilienfeldt–Shnidman [LS24] via the Goldfeld–Zhang method, thereby verifying the claim made in the final remark of [GZ99]. To facilitate the comparison with these papers, we will introduce their notation and normalizations and relate it to ours in Section 5.1. Then, in the remainder of this chapter, we deduce a formula for the special values (Section 5.2), the central value (Section 5.3) and the central derivative (Section 5.4) of the Rankin–Selberg $L$-function. We remark here that both Gross–Zagier and Lilienfeldt–Shnidman write $2k$ for the weight of the newforms, whereas we use $k$.

## 5.1 Preliminaries

Fix positive coprime integers $D$ and $N$ with $D$ square-free. We will assume that $D \equiv 3 \pmod 4$. Let $k$ and $\ell$ be positive integers with $k$ even, $\ell$ odd and $\ell < k$. Consider the imaginary quadratic field $K = \mathbf{Q}(\sqrt{-D})$ of discriminant $\Delta_K = -D$ with the corresponding Dirichlet character $\chi$ modulo $D$. Finally, let $\psi : I_K \to \mathbf{C}^\times$ be an unramified Hecke character of infinity type $(\ell - 1, 0)$ and fix a class $\mathcal{A} \in \mathrm{Cl}_K$.

Given a cusp form $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$, we will write its Fourier expansion as

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z} = \sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} e^{2\pi i n z}.$$

So far, we have used the second normalization, whereas Gross–Zagier and Lilienfeldt–Shnidman use the first normalization. We also have a theta series $\theta_{\mathcal{A},\psi} \in M_\ell(\Gamma_0(N), \chi)$ given by

$$\theta_{\mathcal{A},\psi}(z) = r_{\mathcal{A},\psi}(0) + \sum_{n=1}^{\infty} r_{\mathcal{A},\psi}(n) e^{2\pi i n z} = b(0) + \sum_{n=1}^{\infty} b(n) n^{\frac{\ell-1}{2}} e^{2\pi i n z}.$$

Recall that we defined the Dirichlet $L$-function $L^{(N)}(\chi, s)$ by

$$L^{(N)}(\chi, s) = \sum_{\substack{n=1 \\ (n,DN)=1}}^{\infty} \frac{\chi(n)}{n^s}.$$

We can then define the $L$-function

$$\begin{aligned} L_{\mathcal{A}}(f, \psi, s) &= L^{(N)}(\chi, 2s - k - \ell + 2) \cdot \sum_{n=1}^{\infty} \frac{a_f(n) r_{\mathcal{A},\psi}(n)}{n^s} \\ &= L^{(N)}(\chi, 2s - k - \ell + 2) \cdot L(f \otimes \theta_{\mathcal{A},\psi}, s - \tfrac{k-1}{2} - \tfrac{\ell-1}{2}). \end{aligned} \tag{5.1}$$

Gross–Zagier and Lilienfeldt–Shnidman obtain a formula for $L_{\mathcal{A}}(f, \psi, s)$, respectively, when $\ell = 1$ and $\ell > 1$, while we have given a formula for $L(f \otimes \theta_{\mathcal{A},\psi}, s)$ for any $\ell \geq 1$ (see Proposition 4.11). Note that Gross–Zagier use the notation $L_{\mathcal{A}}(f, s)$, which coincides with $L_{\mathcal{A}}(f, \psi, s)$ if $\psi$ is the trivial unramified finite order Hecke character.

**Evaluating $M_{s,\mathcal{A}}(n)$**

Given a non-zero integer $n \in \mathbf{Z}$ and a divisor $d|n$, we define

$$\chi_{\mathcal{A}}(n,d) = \begin{cases} 0 & \text{if } \gcd(d, \frac{n}{d}, D) \neq 1, \\ \chi_{D_1}(d)\chi_{D_2}(-N\frac{n}{d})\chi_{D_1 \cdot D_2}(\mathcal{A}) & \text{if } \gcd(d, \frac{n}{d}, D) = 1, \end{cases}$$

where $\Delta_K = D_1 \cdot D_2$ is a decomposition of discriminants with $|D_2| = \gcd(d, D)$. For some $s \in \mathbf{C}$, we now define for $n \in \mathbf{Z}$,

$$\sigma_{s,\mathcal{A}}(n) = \begin{cases} (-D)^{\frac{\ell-1}{2}} \frac{\chi(-N)}{2} L(\chi, -s) & \text{if } n = 0, \\ \sum_{d|n} \chi_{\mathcal{A}}(n,d)(n/d)^s & \text{if } n \neq 0, \end{cases} \tag{5.2}$$

and for $n > 1$,

$$\tilde{\sigma}_{\mathcal{A}}(n) = \sum_{d|n} \chi_{\mathcal{A}}(n,d) \log\left(\frac{n}{d^2}\right).$$

These functions appear in the analytic formulas in [GZ86] and [LS24]. Recall that our final formula for the coefficients of $\tilde{\Phi}_{s,\theta_{\mathcal{A},\psi}}$, given in Theorem 4.13, used the function

$$M_{s,\mathcal{A}}(t) = \sum_{\substack{d|t \\ (d,D)=1}} \chi(d) d^{1-2s} \sum_{\delta'|(D,t)} \hat{\chi}_{\delta\cdot\delta'}(-Nt)\chi_{\delta\cdot\delta'}(\mathcal{A}) t_{[\delta']}^{1-2s}.$$

**Lemma 5.1.** *Let $n$ be a non-zero integer. Then $M_{s,\mathcal{A}}(n) = n^{1-2s}\sigma_{2s-1,\mathcal{A}}(n)$ for all $s \in \mathbf{C}$.*

*Proof.* Any divisor $d|n$ that satisfies $\gcd(d, \frac{n}{d}, D) = 1$ can be uniquely written as $d = d' \cdot n_{[\delta']}$ where $d'|n^{[D]}$ and $\delta'|(D,n)$. Decompose $\Delta_K = D_1 \cdot D_2$ with $|D_1| = \delta$ and $|D_2| = \delta'$, so that

$$\chi_{\mathcal{A}}(n, d'n_{[\delta']}) = \chi_{D_1}(d'n_{[\delta']})\chi_{D_2}(-N\tfrac{n}{d'n_{[\delta']}})\chi_{D_1\cdot D_2}(\mathcal{A}) = \chi(d')\hat{\chi}_{D_1\cdot D_2}(-Nn)\chi_{D_1\cdot D_2}(\mathcal{A}).$$

Here we used that

$$\chi_{D_1}(d'n_{[\delta']}) = \hat{\chi}_{D_2\cdot D_1}(d'n_{[\delta']}) \overset{(4.14)}{=} \chi(d')\hat{\chi}_{D_1\cdot D_2}(d'n_{[\delta']}).$$

We conclude that

$$\begin{aligned} M_{s,\mathcal{A}}(n) &= \sum_{\substack{d'|n \\ (d',D)=1}} \chi(d') d'^{1-2s} \sum_{\delta'|(D,n)} \hat{\chi}_{\delta\cdot\delta'}(-nN)\chi_{\delta\cdot\delta'}(\mathcal{A}) n_{[\delta']}^{1-2s} \\ &= \sum_{\substack{d'|n \\ (d',D)=1}} \sum_{\delta'|(D,n)} \chi(d')\hat{\chi}_{\delta\cdot\delta'}(-Nn)\chi_{\delta\cdot\delta'}(\mathcal{A})(d'n_{[\delta']})^{1-2s} \\ &= n^{1-2s} \sum_{d|n} \chi_{\mathcal{A}}(n,d)(n/d)^{2s-1} \\ &= n^{1-2s}\sigma_{2s-1,\mathcal{A}}(n). \end{aligned}$$

□

By taking the derivative with respect to $s$, we obtain the following.

**Lemma 5.2.** *Let $n \geq 1$ be an integer. Then* $\frac{\partial}{\partial s} M_{s,\mathcal{A}}(n)\big|_{s=\frac{1}{2}} = \tilde{\sigma}_{\mathcal{A}}(n) - \log(n)\sigma_{0,\mathcal{A}}(n)$.

*Proof.* We first note that

$$\frac{\partial}{\partial s}\sigma_{2s-1,\mathcal{A}}(n) = 2\sum_{d|n}\chi_{\mathcal{A}}(n,d)\log\left(\frac{n}{d}\right)\left(\frac{n}{d}\right)^{2s-1} = \sum_{d|n}\chi_{\mathcal{A}}(n,d)\left(\frac{n}{d}\right)^{2s-1}\left(\log\left(\frac{n}{d^2}\right)+\log(n)\right),$$

and so

$$\frac{\partial}{\partial s}\sigma_{2s-1,\mathcal{A}}(n)\bigg|_{s=\frac{1}{2}} = \tilde{\sigma}_{\mathcal{A}}(n) + \log(n)\sigma_{0,\mathcal{A}}(n).$$

Differentiating both sides of $M_{s,\mathcal{A}}(n) = n^{1-2s}\sigma_{2s-1,\mathcal{A}}(n)$ yields

$$\frac{\partial}{\partial s}M_{s,\mathcal{A}} = n^{1-2s}\left(\frac{\partial}{\partial s}\sigma_{2s-1,\mathcal{A}}(n) - 2\log(n)\sigma_{2s-1,\mathcal{A}}(n)\right),$$

and evaluating at $s = \frac{1}{2}$ proves the lemma. □

**Evaluating $I_s(x)$**

Next, we take a look at the function $I_s(x)$ given by Definition 4.3 that occurs in our formulas. By evaluating it or its derivative at special points, we can relate it to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and to the Jacobi functions of the second kind $Q_n^{(\alpha,\beta)}(x)$.

**Definition 5.3** ([Erd+81b, 10.8 (12)])**.** Let $\alpha, \beta \in \mathbf{R}$ and $n \in \mathbf{Z}_{\geq 0}$. Then we define the Jacobi polynomial $P_n^{(\alpha,\beta)}$ by

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n}\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^m.$$

It is immediate that

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}, \tag{5.3}$$

and that

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n\binom{n+\beta}{n}. \tag{5.4}$$

Moreover, as stated in [Erd+81b, 10.8 (16)], one can relate $P_n^{(\alpha,\beta)}$ to the Gaussian hypergeometric function (see (4.8)) via

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(\beta+1)}\left(\frac{x-1}{2}\right)^n F\left(-n-\alpha, -n; \beta+1; \frac{x+1}{x-1}\right). \tag{5.5}$$

**Definition 5.4.** Let $\alpha, \beta \in \mathbf{R}$ and $n \in \mathbf{Z}_{\geq 0}$. Then we define the Jacobi function of the second kind $Q_n^{(\alpha,\beta)}$ for $x \in \mathbf{C}$ outside of the real segment $[-1, 1]$ by

$$Q_n^{(\alpha,\beta)}(x) = \frac{2^{-n-1}}{(x-1)^\alpha(x+1)^\beta}\int_{-1}^{1}\frac{(1-u)^{n+\alpha}(1+u)^{n+\beta}}{(x-u)^{n+1}}du.$$

As proven in [Sze75, Thm. 4.61.2], we have the relation

$$Q_n^{(\alpha,\beta)}(x) = 2^{n+\alpha+\beta}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)}(x-1)^{-n-\alpha-1}(x+1)^{-\beta} \cdot F\left(n+\alpha+1, n+1; 2n+\alpha+\beta+2; \frac{2}{1-x}\right). \quad (5.6)$$

**Lemma 5.5.** *Let $0<\ell<k$ be integers with $k$ even and $\ell$ odd. Let $0 \le r \le \frac{k-\ell-1}{2}$ an integer. Let $x \in \mathbf{R}$ with $x \neq 1$. If $0<x<1$, then $\tilde{I}_{\frac{k-\ell}{2}-r}(x) = 0$. If $x>1$, then*

$$\tilde{I}_{\frac{k-\ell}{2}-r}(x) = (-1)^r x^{\frac{1-k}{2}+r}(x-1)^{k-\ell-1-2r}\frac{\Gamma(r+1)}{\Gamma(r+\ell)}P_r^{(k-\ell-1-2r,\ell-1)}(\tfrac{2}{x}-1).$$

*Here $P_r^{(k-\ell-1-2r,\ell-1)}$ is a Jacobi polynomial as in Definition 5.3.*

*Proof.* The case $0<x<1$ follows from Proposition 4.8 and the fact that $\Gamma(\frac{\ell-k}{2}+s)$ has a pole at $s=\frac{k-\ell}{2}-r$. In the case that $x>1$, we find by Proposition 4.8 that

$$\tilde{I}_{\frac{k-\ell}{2}-r}(x) = x^{\frac{1-k}{2}}\frac{1}{\Gamma(\ell)}(x-1)^{k-\ell-1-r}F\left(\ell-k+1+r, -r; \ell; \frac{1}{1-x}\right).$$

Using (5.5) with $\alpha = k-\ell-1-2r$, $\beta=\ell-1$ and $n=-r$, and then substituting $x \mapsto \frac{2}{x}-1$, we know that

$$P_r^{(k-\ell-1-2r,\ell-1)}(\tfrac{2}{x}-1) = \frac{\Gamma(r+\ell)}{\Gamma(r+1)\Gamma(\ell)}\left(\frac{1-x}{x}\right)^r F\left(\ell-k+1+r, -r; \ell; \frac{1}{1-x}\right).$$

We conclude that

$$\tilde{I}_{\frac{k-\ell}{2}-r}(x) = (-1)^r x^{\frac{1-k}{2}+r}(x-1)^{k-\ell-1-2r}\frac{\Gamma(r+1)}{\Gamma(r+\ell)}P_r^{(k-\ell-1-2r,\ell-1)}(\tfrac{2}{x}-1).$$

□

For the derivative at $s=\frac{1}{2}$, we have the following formulas.

**Lemma 5.6.** *Let $0<\ell<k$ be integers with $k$ even and $\ell$ odd. Let $x \in \mathbf{R}$ with $x \neq 1$.*
*If $0<x<1$, then*

$$\frac{\partial}{\partial s}\tilde{I}_s(x)\bigg|_{s=\frac{1}{2}} = (-1)^{\frac{k-\ell-1}{2}}x^{-\frac{\ell}{2}}\frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})}\cdot 2Q_{\frac{k-\ell-1}{2}}^{(0,\ell-1)}\left(\frac{2}{x}-1\right),$$

*with $Q_{\frac{k-\ell-1}{2}}^{(0,\ell-1)}$ a Jacobi function of the second kind as in Definition 5.4.*
*If $x>1$, then*

$$\frac{\partial}{\partial s}\tilde{I}_s(x)\bigg|_{s=\frac{1}{2}} = \log(x-1)x^{-\frac{\ell}{2}}\frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})}P_{\frac{k-\ell-1}{2}}^{(0,\ell-1)}(1-\tfrac{2}{x}).$$

*with $P_{\frac{k-\ell-1}{2}}^{(0,\ell-1)}$ a Jacobi polynomial as in Definition 5.3.*

*Proof.* For $0 < x < 1$, recall the formula for $I_s(x)$ as given in Proposition 4.8

$$\tilde{I}_s(x) = x^{\frac{k-1}{2}} \frac{\Gamma(\frac{k-\ell}{2}+s)}{\Gamma(k)\Gamma(\frac{\ell-k}{2}+s)}(1-x)^{\frac{\ell-k}{2}-1+s} F\left(\frac{k-\ell}{2}+s, \frac{k-\ell}{2}+1-s; k; \frac{x}{x-1}\right).$$

Note that $\Gamma(\frac{\ell-k}{2}+s)$ has a simple pole at $s=\frac{1}{2}$ and that

$$\frac{d}{ds}\Gamma(\tfrac{\ell-k}{2}+s)^{-1}\big|_{s=\frac{1}{2}} = \left(\mathrm{Res}_{\frac{1}{2}}(\Gamma(\tfrac{\ell-k}{2}+s))\right)^{-1} = (-1)^{\frac{k-\ell-1}{2}}\Gamma(\tfrac{k-\ell+1}{2}).$$

It follows that

$$\tilde{I}'_{\frac{1}{2}}(x) = (-1)^{\frac{k-\ell-1}{2}} x^{\frac{k-1}{2}} \frac{\Gamma(\frac{k-\ell+1}{2})^2}{\Gamma(k)}(1-x)^{\frac{\ell-k-1}{2}} F\left(\frac{k-\ell+1}{2}, \frac{k-\ell+1}{2}; k; \frac{x}{x-1}\right).$$

Using (5.6) with $\alpha=0$, $\beta=\ell-1$ and $n=\frac{k-\ell-1}{2}$, and substituting $x \mapsto \frac{2}{x}-1$ yields

$$Q^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}\left(\frac{2}{x}-1\right) = \frac{2^{\frac{k+\ell-3}{2}}\Gamma(\frac{k-\ell+1}{2})\Gamma(\frac{k+\ell-1}{2})}{\Gamma(k)(\frac{2}{x}-2)^{\frac{k-\ell+1}{2}}(\frac{2}{x})^{\ell-1}} F\left(\frac{k-\ell+1}{2}, \frac{k-\ell+1}{2}; k; \frac{x}{x-1}\right).$$

We conclude for $0 < x < 1$, that

$$\begin{aligned}
\frac{\partial}{\partial s}\tilde{I}_s(x)\bigg|_{s=\frac{1}{2}} &= (-1)^{\frac{k-\ell-1}{2}} x^{\frac{k-1}{2}}(1-x)^{\frac{\ell-k-1}{2}} 2^{\frac{-k-\ell+3}{2}}(\tfrac{2}{x}-2)^{\frac{k-\ell+1}{2}}\left(\tfrac{2}{x}\right)^{\ell-1}\frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})} Q^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}\left(\frac{2}{x}-1\right)\\
&= (-1)^{\frac{k-\ell-1}{2}} x^{-\frac{\ell}{2}} \cdot \frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})} \cdot 2Q^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}\left(\frac{2}{x}-1\right).
\end{aligned}$$

For $x > 1$, we know by Proposition 4.8 that

$$\tilde{I}_s(x) = x^{\frac{1-k}{2}}(x-1)^{\frac{k-\ell}{2}-1+s} F\left(\frac{\ell-k}{2}+1-s, \frac{\ell-k}{2}+s; \ell; \frac{1}{1-x}\right).$$

As $F(\frac{\ell-k}{2}+1-s, \frac{\ell-k}{2}+s; \ell; \frac{1}{1-x})$ is invariant under $s \leftrightarrow 1-s$, its derivative with relation to $s$ vanishes at $s=\frac{1}{2}$. Using Lemma 5.5 with $r=\frac{k-\ell-1}{2}$, we deduce that

$$\frac{\partial}{\partial s}\tilde{I}_s(x)\bigg|_{s=\frac{1}{2}} = \log(x-1)\cdot \tilde{I}_{\frac{1}{2}}(x) = \log(x-1)x^{-\frac{\ell}{2}}\frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})} P^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}(1-\tfrac{2}{x}).$$

□

**Final preparations**

The final result we need is the following lemma.

**Lemma 5.7.** *For $1 \le \ell < k$ with $\ell$ odd and $k$ even, define*

$$f(s) = \frac{\Gamma(\frac{1}{2}-s)\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{\ell-k}{2}+1-s)\Gamma(\frac{\ell+k}{2}-s)}.$$

*Then, for any integer* $0 \le r \le \frac{k-\ell-1}{2}$,

$$f(\tfrac{k-\ell}{2} - r) = (-1)^{\frac{k-\ell-1}{2}} \frac{\Gamma(k-\ell-r)}{\Gamma(\ell+r)},$$

*and*

$$f'(\tfrac{1}{2}) = (-1)^{\frac{k-\ell-1}{2}} \frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})} \left( \frac{\Gamma'}{\Gamma}(\tfrac{k+\ell-1}{2}) + \frac{\Gamma'}{\Gamma}(\tfrac{k-\ell+1}{2}) \right).$$

*Sketch of proof.* Given an integer $0 \le r \le \frac{k-\ell-1}{2}$, both $\Gamma(\frac{1}{2}-s)$ and $\Gamma(\frac{\ell-k}{2}+1-s)$ have a pole at $s = \frac{k-\ell}{2} - r$. Computing the residues and evaluating yields the first part of this lemma.

The second part can be proven using the Laurent expanion of the four gamma functions around $s = \frac{1}{2}$. As this computation is not very insightful, we have omitted it. □

We now have all the results that we need to derive the analytic results in [GZ86] and in [LS24]. To make the notation more concise, we will write $\tilde{\Phi}_{s,\theta}$ instead of $\tilde{\Phi}_{s,\theta_{\mathcal{A},\psi}}$ and use Theorem 4.13 to write

$$\overline{\tilde{\phi}_{\bar{s},\theta}} = t_{m,1}(s) + t_{m,2}(s) + t_{m,3}(s) + t_{m,4}(s),$$

with

$$\begin{aligned} t_{m,1}(s) &= i^{k+\ell-1}\chi(-N)L(\chi,2s)\frac{N^{2s-1}}{(2\pi)^{2s}}\frac{\Gamma(\frac{1}{2}-s)\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{\ell-k}{2}+1-s)\Gamma(\frac{\ell+k}{2}-s)}\frac{b(m)}{m^s}, \\ t_{m,2}(s) &= i^{k+\ell-1}L(\chi,2-2s)\frac{D^{1-2s}}{(2\pi)^{2-2s}}\frac{\Gamma(s-\frac{1}{2})\Gamma(\frac{3}{2}-s)}{\Gamma(\frac{\ell-k}{2}+s)\Gamma(\frac{\ell+k}{2}-1+s)}\frac{b(m)}{m^{1-s}}, \\ t_{m,3}(s) &= i^k D^{\frac{1}{2}-s}\sum_{\substack{n\ge1, n\neq mD \\ mD\equiv n \pmod N}} \frac{b(n)}{n^{1-s}} M_{s,\mathcal{A}}\left(\frac{mD-n}{N}\right)\tilde{I}_s\left(\frac{mD}{n}\right), \\ t_{m,4}(s) &= i^{k+\ell-1}\frac{m^{s-\frac{\ell+1}{2}}}{\Gamma(\ell)\sqrt{D}} b(0) M_{s,\mathcal{A}}\left(\frac{mD}{N}\right). \end{aligned} \tag{5.7}$$

## 5.2 Special values

In [GZ86, Thm. IV.5.5], the value of $L_{\mathcal{A}}(f,\psi,k-r-1)$ is derived for integers $0 \le r < \frac{k-2}{2}$ in the case that $\ell = 1$. We generalize this theorem by allowing $\ell > 1$ in the following way.

**Theorem 5.8.** *Let* $K = \mathbf{Q}(\sqrt{-D})$ *be an imaginary quadratic field with* $D \equiv 3 \pmod 4$. *Fix a level* $N$ *coprime to* $D$ *and fix weights* $1 \le \ell < k$ *with* $k \ge 4$ *even and* $\ell$ *odd. Let* $\psi$ *be an unramified Hecke character of infinity type* $(\ell-1,0)$. *Finally, let* $0 \le r < \frac{k-\ell-1}{2}$ *be an integer. For* $m \ge 1$, *define*

$$b_{m,r} = (mD)^r \sum_{0\le t\le \frac{mD}{N}} r_{\bar{\mathcal{A}},\psi}(mD-tN)\sigma_{k-\ell-2r-1,\mathcal{A}}(t)\, P_r^{(k-\ell-1-2r,\ell-1)}\left(1-\frac{2tN}{mD}\right),$$

*with* $P_r^{(k-\ell-1-2r,\ell-1)}$ *a Jacobi polynomial as in Definition 5.3 and* $\sigma_{s,\mathcal{A}}(n)$ *as in (5.2). Then* $\sum_{m\ge1} b_{m,r}e^{2\pi imz} \in S_k(\Gamma_0(N))$ *and for any* $f \in S_k^{\text{new}}(\Gamma_0(N))$, *we have*

$$L_{\mathcal{A}}(f,\psi,k-r-1) = \frac{(-1)^{\frac{k}{2}-r}(2\pi)^{2k-\ell-2r-1}}{(k-\ell-r-1)!}\frac{2^{k-1}}{(k-2)!}\frac{\chi(N)r!}{D^{k-\frac{\ell}{2}-r-1}}\left\langle f, \sum_{m\ge1} b_{m,r}e^{2\pi imz}\right\rangle,$$

*Proof.* By (5.1), we know that

$$L_{\mathcal{A}}(f,\psi,k-r-1) = L^{(N)}(\chi,k-\ell-2r)\cdot L(f\otimes\theta_{\mathcal{A},\psi},\tfrac{k-\ell}{2}-r).$$

Now Proposition 4.11 tells us that

$$L(f\otimes\theta_{\mathcal{A},\psi},\tfrac{k-\ell}{2}-r) = \frac{(4\pi)^{k-1}}{(k-2)!}\frac{\Gamma(\ell+r)(2\pi)^{k-\ell-2r}\chi(N)N^{2r+\ell+1-k}}{\Gamma(k-\ell-r)L^{(N)}(\chi,k-\ell-2r)}\langle f,\tilde{\Phi}_{\frac{k-\ell}{2}-r,\theta}\rangle.$$

It follows that

$$L_{\mathcal{A}}(f,\psi,k-r-1) = \frac{(-1)^{\frac{k}{2}-r}(2\pi)^{2k-\ell-2r-1}}{(k-\ell-r-1)!}\frac{2^{k-1}}{(k-2)!}\frac{\chi(N)r!}{D^{k-\frac{\ell}{2}-r-1}}\langle f,C_r\cdot\tilde{\Phi}_{\frac{k-\ell}{2}-r,\theta}\rangle,$$

with

$$C_r = (-1)^{\frac{k}{2}-r}N^{2r+\ell+1-k}D^{k-\frac{\ell}{2}-r-1}\frac{(r+\ell-1)!}{r!}.$$

We will now calculate the Fourier coefficients of $C_r\cdot\tilde{\Phi}_{\frac{k-\ell}{2}-r,\theta}$. Fix some $m\geq 1$, and consider the complex conjugate of the $m^{\text{th}}$ coefficient, given by (5.7) as

$$\overline{\tilde{\phi}_{\frac{k-\ell}{2}-r,\theta}(m)} = t_{m,1}(\tfrac{k-\ell}{2}-r)+t_{m,2}(\tfrac{k-\ell}{2}-r)+t_{m,3}(\tfrac{k-\ell}{2}-r)+t_{m,4}(\tfrac{k-\ell}{2}-r).$$

For the first term, we need the functional equation of $L(\chi,s)$ given by Theorem 3.3 as

$$L(\chi,s) = \frac{(2\pi)^s}{\pi}D^{\frac{1}{2}-s}\sin\left(\frac{\pi}{2}(s+1)\right)\Gamma(1-s)L(\chi,1-s).$$

A simple computation with Laurent series reveals that

$$\lim_{s\to k-\ell-2r}\sin\left(\frac{\pi}{2}(s+1)\right)\Gamma(1-s) = (-1)^{\frac{k-\ell-1}{2}-r}\cdot\frac{\pi}{2}\cdot\frac{1}{(k-\ell-2r-1)!},$$

and so we find that

$$L(\chi,k-\ell-2r) = (-1)^{\frac{k-\ell-1}{2}-r}\frac{(2\pi)^{k-\ell-2r-1}D^{\frac{1}{2}+\ell+2r-k}}{(k-\ell-2r-1)!}L(\chi,2r+\ell-k+1).$$

Using Lemma 5.7, we deduce that

$$\begin{aligned} t_{m,1}(\tfrac{k-\ell}{2}-r) &= (-1)^{k-1}\chi(-N)L(\chi,k-\ell-2r)\frac{N^{k-\ell-2r-1}}{(2\pi)^{k-\ell-2r}}\frac{\Gamma(k-\ell-r)}{\Gamma(\ell+r)}\frac{b(m)}{m^{\frac{k-\ell}{2}-r}} \\ &= (-1)^{\frac{k+\ell-1}{2}-r}\frac{\chi(-N)}{2}L(\chi,2r+\ell-k+1)N^{k-\ell-2r-1}D^{\frac{1}{2}+\ell+2r-k} \\ &\qquad\cdot\frac{(k-\ell-r-1)!}{(k-\ell-2r-1)!(\ell+r-1)!}\frac{r_{\mathcal{A},\psi}(m)}{m^{\frac{k-1}{2}-r}}, \end{aligned}$$

and hence, by (5.3), we conclude that

$$\begin{aligned} C_r t_{m,1}(\tfrac{k-\ell}{2}-r)m^{\frac{k-1}{2}} &= (mD)^r r_{\mathcal{A},\psi}(m)(-D)^{\frac{\ell-1}{2}}\frac{\chi(-N)}{2}L(\chi,2r+\ell-k+1)\frac{(k-\ell-r-1)!}{(k-\ell-2r-1)!r!} \\ &= (mD)^r r_{\mathcal{A},\psi}(m)\sigma_{k-2r-\ell-1,\mathcal{A}}(0)P_r^{(k-\ell-1-2r,\ell-1)}(1). \end{aligned}$$

For the second term, we note that $t_{m,2}(\frac{k-\ell}{2} - r) = 0$, as $2 + 2r + \ell - k$ is a negative odd integer, and so $L(\chi, 2-2s)$ vanishes at $s = \frac{k-\ell}{2} - r$ by Proposition 3.4.

For the third term, we first note by Lemma 5.5 that the terms of the summation for $n > mD$ vanish. We can then make the substitution $n \mapsto mD - tN$ where $1 \leq t < \frac{mD}{N}$, to obtain

$$t_{m,3}(\tfrac{k-\ell}{2} - r) = i^k D^{\frac{\ell-k+1}{2}+r} \sum_{1 \leq t < \frac{mD}{N}} \frac{r_{\mathcal{A},\psi}(mD-tN)}{(mD-tN)^{1+r+\ell-\frac{1+k}{2}}} M_{\frac{k-\ell}{2}-r,\mathcal{A}}(t)\, \tilde{I}_{\frac{k-\ell}{2}-r}\left(\frac{mD}{mD-tN}\right).$$

As a consequence of Lemma 5.5, it holds that

$$\tilde{I}_{\frac{k-\ell}{2}-r}\left(\frac{mD}{mD-tN}\right) = (-1)^r (mD-tN)^{1+r+\ell-\frac{1+k}{2}} (mD)^{\frac{1-k}{2}+r} (tN)^{k-\ell-1-2r} \frac{r!}{(r+\ell-1)!} P_r^{(k-\ell-1-2r,\ell-1)}\left(1 - \frac{2tN}{mD}\right).$$

Together with Lemma 5.1, which shows that

$$M_{\frac{k-\ell}{2}-r}(t) = t^{1+\ell+2r-k} \sigma_{k-\ell-2r-1,\mathcal{A}}(t),$$

we conclude that

$$C_r t_{m,3}(\tfrac{k-\ell}{2} - r) m^{\frac{k-1}{2}} = (mD)^r \sum_{1 \leq t < \frac{mD}{N}} r_{\mathcal{A},\psi}(mD-tN) \sigma_{k-\ell-2r-1,\mathcal{A}}(t)\, P_r^{(k-\ell-1-2r,\ell-1)}\left(1 - \frac{2tN}{mD}\right).$$

For the fourth and final term, we calculate

$$\begin{aligned} t_{m,4}(\tfrac{k-\ell}{2} - r) &= i^{k+\ell-1} \frac{m^{\frac{k-1}{2}-r-\ell}}{\Gamma(\ell)\sqrt{D}} b(0) M_{\frac{k-\ell}{2}-r,\mathcal{A}}\left(\frac{mD}{N}\right) \\ &= i^{k+\ell-1} \frac{m^{\frac{1-k}{2}+r}}{\Gamma(\ell)} r_{\mathcal{A},\psi}(0) N^{k-\ell-2r+1} D^{\frac{1}{2}+\ell+2r-k} \sigma_{k-\ell-2r-1,\mathcal{A}}\left(\frac{mD}{N}\right), \end{aligned}$$

and hence, by (5.4), we find

$$\begin{aligned} C_r t_{m,4}(\tfrac{k-\ell}{2} - r) m^{\frac{k-1}{2}} &= (-1)^r (-D)^{\frac{\ell-1}{2}} \frac{(r+l-1)!}{(l-1)!r!} r_{\mathcal{A},\psi}(0) (mD)^r \sigma_{k-\ell-2r-1,\mathcal{A}}\left(\frac{mD}{N}\right) \\ &= (-D)^{\frac{\ell-1}{2}} r_{\mathcal{A},\psi}(0) \sigma_{k-\ell-2r-1,\mathcal{A}}\left(\frac{mD}{N}\right) (mD)^r P_r^{(k-\ell-1-2r,\ell-1)}(-1). \end{aligned}$$

As the fourth term is only non-zero if $\ell = 1$, we may drop the factor $(-D)^{\frac{\ell-1}{2}}$ from the formula above.

Summing up all the terms yields

$$C_r \overline{\tilde{\phi}_{\frac{k-\ell}{2}-r}(m)} m^{\frac{k-1}{2}} = (mD)^r \sum_{0 \leq t \leq \frac{mD}{N}} r_{\mathcal{A},\psi}(mD-tN) \sigma_{k-\ell-2r-1,\mathcal{A}}(t)\, P_r^{(k-\ell-1-2r,\ell-1)}\left(1 - \frac{2tN}{mD}\right).$$

To obtain a formula for the coefficients, we can now take the complex conjugate of the right hand side, as $C_r$ and $m^{\frac{k-1}{2}}$ are both real. The only factors that may not be real are the factors $r_{\mathcal{A},\psi}(n)$. We can conjugate these as follows:

$$\overline{r_{\mathcal{A},\psi}(n)} = \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_K \\ [\mathfrak{a}]=\mathcal{A} \\ \mathrm{N}(\mathfrak{a})=n}} \overline{\psi(\mathfrak{a})} = \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_K \\ [\mathfrak{a}]=\mathcal{A} \\ \mathrm{N}(\mathfrak{a})=n}} \psi(\bar{\mathfrak{a}}) = \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_K \\ [\mathfrak{a}]=\bar{\mathcal{A}} \\ \mathrm{N}(\mathfrak{a})=n}} \psi(\mathfrak{a}) = r_{\bar{\mathcal{A}},\psi}(n).$$

The theorem immediately follows. □

**Remark.** In the case that $\ell = 1$ and $\psi$ is the trivial finite order Hecke character, we recover [GZ86, Thm. IV.5.5]. Note that the polynomial $P_{\frac{k}{2},r}(Nn, mD)$ that they use relates to a Jacobi polynomial via

$$P_{\frac{k}{2},r}(Nn, mD) = (mD)^r P_r^{(k-2-2r,0)}\left(1 - \frac{2tN}{mD}\right).$$

Moreover, for $\psi$ trivial, it holds that $r_{\mathcal{A},\psi}(n) = r_{\bar{\mathcal{A}},\psi}(n)$, as the norm of an ideal is invariant under conjugation. We note that from the result for the trivial character, one can recover the result for any class group character $\psi$, as one has the relation

$$L(f \otimes \theta_{\mathcal{A},\psi}, s) = \psi(\mathcal{A}) \cdot L(f \otimes \theta_{\mathcal{A}}, s),$$

where $\theta_{\mathcal{A}}$ is the theta series associated with the trivial class group character.

## 5.3 Central value

We shall now derive an expression for the $L$-function $L_{\mathcal{A}}(f, \psi, s)$ at $s = \frac{k+\ell-1}{2}$. This point corresponds to the central point $s = \frac{1}{2}$ of $L(f \otimes \theta_{\mathcal{A},\psi}, s)$, and by the functional equation (Theorem 4.16), we know that $L(f \otimes \theta_{\mathcal{A},\psi}, \frac{1}{2})$ vanishes for $\chi(N) = 1$. As such, we will assume that $\chi(N) = -1$. In the case that $\ell = 1$ and $\psi$ is trivial, we recover [GZ86, Thm. IV.5.6] for $k \geq 4$.

**Theorem 5.9.** *Let $K = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field with $D \equiv 3 \pmod 4$. Fix a level $N$ coprime to $D$ and fix weights $1 \leq \ell < k$ with $k \geq 4$ even and $\ell$ odd. Assume that $\chi(N) = -1$. Let $\psi$ be an unramified Hecke character of infinity type $(\ell - 1, 0)$. For $m \geq 1$, define*

$$b_m = m^{\frac{k-\ell-1}{2}} \left( (-D)^{\frac{\ell-1}{2}} r_{\bar{\mathcal{A}},\psi}(m) \frac{h}{u} + \sum_{1 \leq t \leq \frac{mD}{N}} \sigma_{\mathcal{A}}(t) r_{\bar{\mathcal{A}},\psi}(mD - tN) P^{(0,\ell-1)}_{\frac{k-\ell-1}{2}} \left(1 - \frac{2tN}{mD}\right) \right),$$

*with $P^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}$ a Jacobi polynomial as in Definition 5.3, and*

$$\sigma_{\mathcal{A}}(n) = \sum_{d|n} \chi_{\mathcal{A}}(n, d).$$

*Then $\sum_{m\geq 1} b_m e^{2\pi i m z} \in S_k(\Gamma_0(N))$, and for any $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$, we have*

$$L_{\mathcal{A}}(f, \psi, \tfrac{k+\ell-1}{2}) = \frac{(2\pi)^k 2^{k-1}}{(k-2)!\sqrt{D}(-D)^{\frac{\ell-1}{2}}} \left\langle f, \sum_{m\geq 1} b_m e^{2\pi i m z} \right\rangle.$$

*Proof.* By (5.1), we know that

$$L_{\mathcal{A}}(f,\psi,\tfrac{k+\ell-1}{2}) = L^{(N)}(\chi,1)\cdot L(f\otimes\theta_{\mathcal{A},\psi},\tfrac{1}{2}).$$

As Proposition 4.11 gives us that

$$L(f\otimes\theta_{\mathcal{A},\psi},\tfrac{1}{2}) = \frac{(2\pi)^k 2^{k-1}}{(k-2)!}\frac{\Gamma(\frac{k+\ell-1}{2})\chi(N)}{\Gamma(\frac{k-\ell+1}{2})L^{(N)}(\chi,1)}\langle f,\tilde{\Phi}_{\frac{1}{2},\theta}\rangle,$$

it follows that

$$L_{\mathcal{A}}(f,\psi,\tfrac{k+\ell-1}{2}) = \frac{(2\pi)^k 2^{k-1}}{(k-2)!\sqrt{D}(-D)^{\frac{\ell-1}{2}}}\langle f, C_\psi\cdot\tilde{\Phi}_{\frac{1}{2},\theta}\rangle,$$

with

$$C_\psi = (-1)^{\frac{\ell+1}{2}} D^{\frac{\ell}{2}}\frac{\Gamma(\frac{k+\ell-1}{2})}{\Gamma(\frac{k-\ell+1}{2})}.$$

Fix some $m\geq 1$, and consider the complex conjugate of the $m^{\text{th}}$ coefficient, given by (5.7) as

$$\overline{\tilde{\phi}_{\frac{1}{2},\theta}(m)} = t_{m,1}(\tfrac{1}{2}) + t_{m,2}(\tfrac{1}{2}) + t_{m,3}(\tfrac{1}{2}) + t_{m,4}(\tfrac{1}{2}).$$

Using Proposition 3.2 and Lemma 5.7, we find that

$$t_{m,1}(\tfrac{1}{2}) = -\frac{h}{w\sqrt{D}}\frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})}\frac{b(m)}{m^{\frac{1}{2}}} = -\frac{h}{w\sqrt{D}}\frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})}\frac{r_{\mathcal{A},\psi}(m)}{m^{\frac{\ell}{2}}}.$$

We deduce that

$$C_\psi t_{m,1}(\tfrac{1}{2})m^{\frac{k-1}{2}} = m^{\frac{k-\ell-1}{2}}(-D)^{\frac{\ell-1}{2}}\frac{h}{w}r_{\mathcal{A},\psi}(m).$$

As $t_{m,1}(1-s) = \chi(-N)D^{2s-1}N^{1-2s}t_{m,2}(s)$ for all $s\in\mathbf{C}$, it follows that $t_{m,2}(\frac{1}{2}) = t_{m,1}(\frac{1}{2})$, and thus

$$C_\psi(t_{m,1}(\tfrac{1}{2}) + t_{m,1}(\tfrac{1}{2}))m^{\frac{k-1}{2}} = m^{\frac{k-\ell-1}{2}}(-D)^{\frac{\ell-1}{2}}\frac{h}{u}r_{\mathcal{A},\psi}(m).$$

For the third term, it follows by Lemma 5.5 that the terms in the summation for $n > mD$ vanish. Using the substitution $n\mapsto mD-tN$ for $1\leq t<\frac{mD}{N}$, we obtain

$$t_{m,3}(\tfrac{1}{2}) = i^k\sum_{1\leq t<\frac{mD}{N}}\frac{b(mD-tN)}{(mD-tN)^{\frac{1}{2}}}M_{\frac{1}{2},\mathcal{A}}(t)\tilde{I}_{\frac{1}{2}}\left(\frac{mD}{mD-tN}\right).$$

By Lemma 5.5 with $r=\frac{k-\ell-1}{2}$, we know that

$$\tilde{I}_{\frac{1}{2}}\left(\frac{mD}{mD-tN}\right) = (-1)^{\frac{k-\ell-1}{2}}(mD-tN)^{\frac{\ell}{2}}(mD)^{-\frac{\ell}{2}}\frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})}P^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}\left(1-\frac{2tN}{mD}\right).$$

Moreover, by Lemma 5.1, we see that $M_{\frac{1}{2},\mathcal{A}}(t) = \sigma_{\mathcal{A}}(t)$, and so

$$C_\psi t_{m,3}(\tfrac{1}{2})m^{\frac{k-1}{2}} = m^{\frac{k-\ell-1}{2}}\sum_{1\leq t<\frac{mD}{N}} r_{\mathcal{A},\psi}(mD-tN)\sigma_{\mathcal{A}}(t)P^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}\left(1-\frac{2tN}{mD}\right).$$

For the final term, we see that

$$t_{m,4}(\tfrac{1}{2}) = i^{k+\ell-1}\frac{m^{-\frac{\ell}{2}}}{\Gamma(\ell)\sqrt{D}} r_{\mathcal{A},\psi}(0)\sigma_{\mathcal{A}}\left(\frac{mD}{N}\right),$$

and so by using (5.4), we obtain

$$\begin{aligned} C_\psi t_{m,4}(\tfrac{1}{2})m^{\frac{k-1}{2}} &= m^{\frac{k-\ell-1}{2}}(-1)^{\frac{k}{2}+\ell}\frac{\Gamma(\frac{k+\ell-1}{2})}{\Gamma(\frac{k-\ell+1}{2})\Gamma(\ell)}D^{\frac{\ell-1}{2}} r_{\mathcal{A},\psi}(0)\sigma_{\mathcal{A}}\left(\frac{mD}{N}\right) \\ &= m^{\frac{k-\ell-1}{2}}(-D)^{\frac{\ell-1}{2}} r_{\mathcal{A},\psi}(0)\sigma_{0,\mathcal{A}}\left(\frac{mD}{N}\right) P^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}(-1). \end{aligned}$$

As this term is only present for $\ell = 1$, we may drop the factor $(-D)^{\frac{\ell-1}{2}}$. As a final step, we conjugate all of the terms, which means that we replace $r_{\mathcal{A},\psi}(n)$ by $r_{\bar{\mathcal{A}},\psi}(n)$, just as in the proof of Theorem 5.8. This proves the theorem. □

**Remark.** In the case that $\psi$ is the trivial finite order Hecke character with $\ell = 1$, one obtains [GZ86, Thm. IV.5.6] by using [GZ86, Prop. IV.4.6] to replace $\sigma_{\mathcal{A}}(t)$ by $\delta(t)R_{\{\mathcal{A}\mathfrak{n}\}}(t)$ (defined in [GZ86, p. 285]). Note that $r_{\mathcal{A}}(mD) = r_{\mathcal{A}}(m)$, as there is a unique ideal of norm $D$, which is principal. We believe that the factor $(k-1)!$ in the numerator of the equation in [GZ86, Thm. IV.5.6] should be omitted. As our results are restricted to weight $k \geq 4$, we do not recover the case $k = 2$.

## 5.4 Central derivative

In this section, we will calculate the derivative of $L_{\mathcal{A}}(f,\psi,s)$ a $s = \frac{k-\ell-1}{2}$ in the case that $\chi(N) = 1$. For $\ell = 1$ and $\psi$ trivial, we obtain [GZ86, Thm. IV.5.8]. For $\ell > 1$, we obtain [LS24, Thm. 3.6].

**Theorem 5.10.** *Let $K = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field with $D \equiv 3 \pmod 4$. Fix a level $N$ coprime to $D$ and fix weights $1 \leq \ell < k$ with $k \geq 4$ even and $\ell$ odd. Assume that $\chi(N) = 1$. Let $\psi$ be an unramified Hecke character of infinity type $(\ell-1, 0)$. For $m \geq 1$, define*

$$\begin{aligned} a_m = m^{\frac{k-\ell-1}{2}}\Bigg[&-\sum_{1\leq t\leq \frac{mD}{N}} r_{\bar{\mathcal{A}},\psi}(mD-tN)\tilde{\sigma}_{\mathcal{A}}(t)P^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}\left(1-\frac{2tN}{mD}\right) \\ &+(-D)^{\frac{\ell-1}{2}} r_{\bar{\mathcal{A}},\psi}(m)\frac{h}{u}\cdot\left(\frac{\Gamma'}{\Gamma}(\tfrac{k+\ell-1}{2})+\frac{\Gamma'}{\Gamma}(\tfrac{k-\ell+1}{2})+\log\left(\frac{DN}{4\pi^2 m}\right)+2\frac{L'(\chi,1)}{L(\chi,1)}\right) \\ &-\sum_{t=1}^{\infty} r_{\bar{\mathcal{A}},\psi}(mD+tN)\sigma_{\mathcal{A}}(-t)\, 2Q^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}\left(1+\frac{2tN}{mD}\right)\Bigg], \end{aligned}$$

*with $P^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}$ a Jacobi polynomial as in Definition 5.3, $Q^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}$ a Jacobi function of the second kind as in Definition 5.4 and*

$$\sigma_{\mathcal{A}}(n) = \sum_{d|n}\chi_{\mathcal{A}}(n,d) \qquad \text{and} \qquad \tilde{\sigma}_{\mathcal{A}}(n) = \sum_{d|n}\chi_{\mathcal{A}}(n,d)\log\left(\frac{n}{d^2}\right).$$

*Then* $\sum_{m\geq 1} a_m e^{2\pi imz} \in S_k(\Gamma_0(N))$, *and for any* $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$, *we have* $L_{\mathcal{A}}(f,\psi,\frac{k+\ell-1}{2}) = 0$ *and*

$$L'_{\mathcal{A}}(f,\psi,\tfrac{k+\ell-1}{2}) = \frac{(2\pi)^k 2^{k-1}}{(k-2)!\sqrt{D}(-D)^{\frac{\ell-1}{2}}} \left\langle f, \sum_{m\geq 1} a_m e^{2\pi imz} \right\rangle.$$

*Proof.* By (5.1), we know that

$$L_{\mathcal{A}}(f,\psi,s+\tfrac{k+\ell-2}{2}) = L^{(N)}(\chi,2s)\cdot L(f\otimes\theta_{\mathcal{A},\psi},s).$$

As Proposition 4.11 shows that

$$L(f\otimes\theta_{\mathcal{A},\psi},s) = \frac{(4\pi)^{k-1}}{(k-2)!}\frac{\Gamma(\frac{k+\ell}{2}-s)(2\pi)^{2s}\chi(N)N^{1-2s}}{\Gamma(\frac{k-\ell}{2}+s)L^{(N)}(\chi,2s)}\langle f,\tilde{\Phi}_{\bar{s},\theta}\rangle,$$

it follows that

$$L_{\mathcal{A}}(f,\psi,s+\tfrac{k+\ell-2}{2}) = \frac{(4\pi)^{k-1}}{(k-2)!}\frac{\Gamma(\frac{k+\ell}{2}-s)(2\pi)^{2s}N^{1-2s}}{\Gamma(\frac{k-\ell}{2}+s)}\langle f,\tilde{\Phi}_{\bar{s},\theta}\rangle.$$

As $\tilde{\Phi}_{s,\theta}$ vanishes at $s=\frac{1}{2}$, we obtain

$$L'_{\mathcal{A}}(f,\psi,\tfrac{k+\ell-1}{2}) = \frac{(2\pi)^k 2^{k-1}}{(k-2)!\sqrt{D}(-D)^{\frac{\ell-1}{2}}} \left\langle f, C_\psi\cdot\frac{\partial}{\partial s}\tilde{\Phi}_{s,\theta}\bigg|_{s=\frac{1}{2}} \right\rangle,$$

where

$$C_\psi = (-1)^{\frac{\ell-1}{2}}D^{\frac{\ell}{2}}\frac{\Gamma(\frac{k+\ell-1}{2})}{\Gamma(\frac{k-\ell+1}{2})}.$$

The normalized Fourier coefficients of $\frac{\partial}{\partial s}\tilde{\Phi}_{s,\theta}\big|_{s=\frac{1}{2}}$ are given by (5.7) as

$$\frac{\partial}{\partial s}\tilde{\phi}_{s,\theta}(m)\bigg|_{s=\frac{1}{2}} = \overline{t'_{m,1}(\tfrac{1}{2})} + \overline{t'_{m,2}(\tfrac{1}{2})} + \overline{t'_{m,3}(\tfrac{1}{2})} + \overline{t'_{m,4}(\tfrac{1}{2})}.$$

Using Proposition 3.2 and Lemma 5.7, we first calculate

$$t'_{m,1}(\tfrac{1}{2}) = (mD)^{-\frac{1}{2}}\frac{h}{w}b(m)\frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})}\cdot\left(2\frac{L'(\chi,1)}{L(\chi,1)} + 2\log\left(\frac{N}{2\pi\sqrt{m}}\right) + \frac{\Gamma'}{\Gamma}(\tfrac{k+\ell-1}{2}) + \frac{\Gamma'}{\Gamma}(\tfrac{k-\ell+1}{2})\right).$$

From $t_{m,2}(1-s) = \chi(-N)D^{2s-1}N^{1-2s}t_{m,1}(s)$, it follows that $t'_{m,2}(\frac{1}{2}) = t'_{m,1}(\frac{1}{2}) + 2\log(DN^{-1})t_{m,1}(\frac{1}{2})$. As in the case of the central value (Theorem 5.9), one calculates

$$t_{m,1}(\tfrac{1}{2}) = \frac{h}{w\sqrt{D}}\frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})}\frac{b(m)}{m^{\frac{1}{2}}},$$

and so

$$\begin{aligned} &t'_{m,1}(\tfrac{1}{2}) + t'_{m,2}(\tfrac{1}{2}) \\ &\qquad = (mD)^{-\frac{1}{2}}\frac{h}{u}b(m)\frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})}\cdot\left(2\frac{L'(\chi,1)}{L(\chi,1)} + \log\left(\frac{ND}{4\pi^2 m}\right) + \frac{\Gamma'}{\Gamma}(\tfrac{k+\ell-1}{2}) + \frac{\Gamma'}{\Gamma}(\tfrac{k-\ell+1}{2})\right). \end{aligned}$$

We deduce that

$$C_\psi\left(t'_{m,1}(\tfrac12)+t'_{m,2}(\tfrac12)\right)m^{\frac{k-1}{2}}$$
$$= m^{\frac{k-\ell-1}{2}}(-D)^{\frac{\ell-1}{2}}\frac{h}{u}r_{\mathcal{A},\psi}(m)\cdot\left(\frac{\Gamma'}{\Gamma}(\tfrac{k+\ell-1}{2})+\frac{\Gamma'}{\Gamma}(\tfrac{k-\ell+1}{2})+\log\left(\frac{DN}{4\pi^2 m}\right)+2\frac{L'(\chi,1)}{L(\chi,1)}\right).$$

For the third term, we split the sum into two parts: a finite sum for $n<mD$ and an infinite sum for $n>mD$. After a substitution, we obtain $t_{m,3}(s)=S_1(s)+S_2(s)$, with

$$S_1(s)=i^kD^{\frac12-s}\sum_{1\le t<\frac{mD}{N}}\frac{b(mD-tN)}{(mD-tN)^{1-s}}M_{s,\mathcal{A}}(t)\,\tilde{I}_s\left(\frac{mD}{mD-tN}\right),$$
$$S_2(s)=i^kD^{\frac12-s}\sum_{t=1}^{\infty}\frac{b(mD+tN)}{(mD+tN)^{1-s}}M_{s,\mathcal{A}}(-t)\,\tilde{I}_s\left(\frac{mD}{mD+tN}\right).$$

We first calculate $S_1'(\frac12)$. By Lemma 5.5 and Lemma 5.6, we know that

$$\tilde{I}_{\frac12}\left(\frac{mD}{mD-tN}\right)=(-1)^{\frac{k-\ell-1}{2}}\left(\frac{mD-tN}{mD}\right)^{\frac{\ell}{2}}\frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})}P^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}\left(1-\frac{2tN}{mD}\right),$$

and

$$\frac{\partial}{\partial s}\tilde{I}_s\left(\frac{mD}{mD-tN}\right)\bigg|_{s=\frac12}=\log\left(\frac{tN}{mD-tN}\right)\tilde{I}_{\frac12}\left(\frac{mD}{mD-tN}\right).$$

Using Lemma 5.2 and keeping track of all the logarithms that appear yields

$$S_1'(\tfrac12)=i^k\log\left(\frac{N}{D}\right)\sum_{1\le t<\frac{mD}{N}}\frac{b(mD-tN)}{(mD-tN)^{\frac12}}M_{\frac12,\mathcal{A}}(t)\tilde{I}_{\frac12}\left(\frac{mD}{mD+tN}\right)$$
$$+(-1)^{\frac{\ell+1}{2}}(mD)^{-\frac{\ell}{2}}\frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})}\sum_{1\le t<\frac{mD}{N}}r_{\mathcal{A}}(mD-tN)\tilde{\sigma}_{\mathcal{A}}(t)P^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}\left(1-\frac{2tN}{mD}\right).$$

Note that all terms of the first sum vanish. This follows from (the proof of) the functional equation of $\tilde{\Phi}_{s,\theta}$, see Theorem 4.15. For the computation of $S_2'(\frac12)$, we note that $\tilde{I}_s(\frac{mD}{mD+tN})$ vanishes at $s=\frac12$ by Lemma 5.5. Using Lemma 5.1 and Lemma 5.6, it follows that

$$S_2'(\tfrac12)=(-1)^{\frac{\ell+1}{2}}(mD)^{-\frac{\ell}{2}}\frac{\Gamma(\frac{k-\ell+1}{2})}{\Gamma(\frac{k+\ell-1}{2})}\sum_{t=1}^{\infty}r_{\mathcal{A}}(mD+tN)\sigma_{\mathcal{A}}(-t)\,2Q^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}\left(1+\frac{2tN}{mD}\right).$$

We find that

$$C_\psi t'_{m,3}(\tfrac12)m^{\frac{k-1}{2}}=-m^{\frac{k-\ell-1}{2}}\cdot\left[\sum_{1\le t<\frac{mD}{N}}r_{\mathcal{A}}(mD-tN)\tilde{\sigma}_{\mathcal{A}}(t)P^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}\left(1-\frac{2tN}{mD}\right)\right.$$
$$\left.+\sum_{t=1}^{\infty}r_{\mathcal{A}}(mD+tN)J_{\frac12}(-t)\,2Q^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}\left(1+\frac{2tN}{mD}\right)\right].$$

For the fourth term, we calculate using Lemma 5.2:

$$\begin{aligned} t'_{m,4}(\tfrac{1}{2}) &= i^{k+\ell-1}\frac{m^{-\frac{\ell}{2}}}{\Gamma(\ell)\sqrt{D}} r_{\mathcal{A},\psi}(0)(\tilde{\sigma}_{\mathcal{A}}\left(\tfrac{mD}{N}\right) - \log\left(\tfrac{mD}{N}\right)\sigma_{0,\mathcal{A}}(\tfrac{mD}{N})) \\ &= i^{k+\ell-1}\frac{m^{-\frac{\ell}{2}}}{\Gamma(\ell)\sqrt{D}} r_{\mathcal{A},\psi}(0)\tilde{\sigma}_{\mathcal{A}}\left(\tfrac{mD}{N}\right), \end{aligned}$$

where we used that $t_{m,4}(s)$ vanishes at $s = \frac{1}{2}$. It follows that

$$C_\psi t'_{m,4}(\tfrac{1}{2}) m^{\frac{k-1}{2}} = m^{\frac{k-\ell-1}{2}}(-D)^{\frac{\ell-1}{2}} r_{\mathcal{A},\psi}(0)\tilde{\sigma}_{\mathcal{A}}\left(\frac{mD}{N}\right) P^{(0,\ell-1)}_{\frac{k-\ell-1}{2}}(-1).$$

As in the previous two theorems, we may drop the factor $(-D)^{\frac{\ell-1}{2}}$, as this term is only present for $\ell = 1$.

Summing up the four terms and taking the complex conjugate finishes the proof. $\square$

# 6 Conclusion and further research

In this thesis, we corrected and solidified the method introduced by Goldfeld and Zhang in [GZ99]. We have recovered their main results, but with an additional term coming from the residues of the twisted $L$-functions $L_g(s, \frac{a}{c})$. By evaluating the obtained expressions at special points, we were able to recover [GZ86, Thms. IV.5.5, IV.5.6, IV.5.8] and [LS24, Thm. 3.6]. In particular, for $k > 2$, we have validated the final claim in [GZ99]. This substantiates the correctness of our results.

However, the Gross–Zagier formula for the derivative when $k = 2$ [GZ86, Thm. IV.6.9] is the one that is needed to obtain their results on $L$-series of elliptic curves. With the current approach, we are unable to recover this formula, as interchanging the summations in the proof of Theorem 4.4 is not allowed when $k = 2$. Therefore, it might prove interesting to find a way to adapt the approach and recover a formula for the case $k = 2$ as well. One way this might be done is by replacing the Poincaré series $P_m$ by a non-holomorphic Poincaré series $P_{m,s}$ (as in (2.10)) and taking a limit.

One way the method has already been extended is by Nelson [Nel13], as they consider $f \in S_k(\Gamma_0(N), \varepsilon)$ with nebentypus $\varepsilon$. In particular, this allows the weight $k$ of $f$ to be odd. This also essentially covers the case where $f \in S_k(\Gamma_1(N))$, as any such $f$ decomposes as a linear combination of modular forms with nebentypus. They prove this generalization by using a formula for the coefficients of the Poincaré series for $S_k(\Gamma_0(N), \epsilon)$, which is similar to the one for the coefficients of the Poincaré series $P_m$ that we consider, but involves a Kloosterman sum that is twisted by the character $\epsilon$. Perhaps it is possible to look at even more general congruence subgroups $\Gamma$ instead of $\Gamma_0(N)$ and $\Gamma_1(N)$.

One assumption we make is that the level $D \geq 1$ is square-free, which forces us to assume $D \equiv 3 \pmod 4$ in the case of theta series. The reason for this is that we need to understand the behavior of $g$ at the various cusps of $\Gamma_1(D)$, and for that we use the Atkin–Lehner operator. This operator only allows for a decomposition of $D$ into two coprime factors. As such, if $D$ is divisible by any square, then there is not an Atkin–Lehner operator for all decompositions of $D$. It would be interesting to find a way to overcome this restriction. Note that in [GZ86] a similar assumption is used.

When talking about theta series, we only consider unramified Hecke characters of infinity type $(t-1, 0)$. We need this assumption on the ramification to prove that the associated theta series behaves nicely when slashed with any matrix in $\mathrm{SL}_2(\mathbf{Z})$, as in Proposition 2.14. If this proposition generalizes to ramified Hecke characters, then all results in Chapter 5 could be generalized to any Hecke character of infinity type $(t-1, 0)$, as long as the norm of its conductor is square-free and coprime to $ND$. We have not found a generalization of Proposition 2.14 in the literature.

Finally, when deriving the special values, the central value and the central derivative in Chapter 5, we only consider the case where $\ell < k$. Using the Goldfeld–Zhang method, it should be possible to also obtain a formula for $\ell > k$. In that case, it is perhaps more natural to replace the modified holomorphic kernel $\tilde{\Phi}_{s,\theta}$ by $\frac{\Gamma(\frac{\ell-k}{2}+s)}{\Gamma(\frac{k-\ell}{2}+s)}\tilde{\Phi}_{s,\theta}$. More generally, in the construction of the modified holomorphic kernel in Proposition 4.11, a factor $\Gamma(\frac{k-\ell}{2}+s)$ is introduced. As suggested by the definition of the completed $L$-function $\Lambda(f \otimes \theta, s)$ in (4.19), it might be more natural to use a factor $\Gamma(\frac{|k-\ell|}{2}+s)$ instead. For $\ell > k$, this changes the sign of the functional equation of $\tilde{\Phi}_{s,\theta}$, and ensures that $\tilde{\Phi}_{s,\theta}$ does not have a pole at $s = \frac{1}{2}$. Our final claim is that it should not be hard to derive a formula for the value and the derivative of $L(f \otimes \theta, s)$ at the center in this case.

# A Geometric context

In this appendix, we give some additional context and motivation behind the main results of this thesis. Most of the results mentioned can be found in [Sil09] and in [Dar04].

## A.1 Modular curves

In this section, we list some important properties and constructions related to elliptic curves. We describe two examples of moduli spaces: the modular curve $X_0(1)$ and its generalization $X_0(N)$.

### Elliptic curves

An elliptic curve $E$ over a field $K$ is a smooth, projective curve of genus 1, together with a distinguished point $O \in E(K)$. It can always be given by a non-singular affine Weierstrass equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with $a_i \in K$ and $O$ being the point at infinity. The set of $K$-rational points $E(K)$ has a natural abelian group structure, with the point $O$ being the identity element. In the case that $K$ is a number field, this abelian group is finitely generated.

**Theorem A.1** (Mordell–Weil, [Sil09, Thm. VIII.6.7])**.** *Let $E$ be an elliptic curve over a number field $K$. Then $E(K)$ is finitely generated. In particular, there exists some integer $r \geq 0$ such that*

$$E(K) \cong \mathbf{Z}^r \times E(K)_{\text{tors}},$$

*where $E(K)_{\text{tors}}$ is the finite torsion group.*

By the work of Mazur [Maz78] and Merel [Mer96], we understand $E(K)_{\text{tors}}$ well. We call $r$ the (algebraic) rank of an elliptic curve. The proof of Theorem A.1 uses the Néron–Tate height of a point. This height can be used to define a positive definite quadratic form on the $\mathbf{R}$-vector space $E(K) \otimes \mathbf{R}$. In particular, the height of a point is zero if and only if the point is torsion.

### L-series

Let $E$ be an elliptic curve over a number field $K$. For a prime $\mathfrak{p}$ of $K$, one can reduce $E$ modulo $\mathfrak{p}$ and look at the curve $\tilde{E}_{\mathfrak{p}}$ over $k_{\mathfrak{p}}$, the residue field of $\mathfrak{p}$. For only finitely many primes $\mathfrak{p}$ this curve is singular, in which case we say that $E$ has bad reduction modulo $\mathfrak{p}$. One can associate an ideal $N \subseteq \mathcal{O}_K$ to $E$ such that $\mathfrak{p}$ divides $N$ if and only if $E$ has bad reduction modulo $\mathfrak{p}$. We call $N$ the conductor of $E$ and two isogenous elliptic curves have the same conductor. As $k_{\mathfrak{p}}$ is a finite field, $\tilde{E}_{\mathfrak{p}}$ contains finitely many points and so we define for $\mathfrak{p} \nmid N$ the coefficient $a_{\mathfrak{p}} = \mathrm{N}(\mathfrak{p}) + 1 - \#\tilde{E}_{\mathfrak{p}}$. For each prime $\mathfrak{p}$ with bad reduction, we can choose some $a_{\mathfrak{p}} \in \{-1, 0, 1\}$ depending on the type of bad reduction. We now define the $L$-series $L(E/K, s)$ for $\Re(s) > \frac{3}{2}$ by

$$L(E/K, s) = \prod_{\mathfrak{p} \nmid N}(1 - a_{\mathfrak{p}}\mathrm{N}(\mathfrak{p})^{-s} + \mathrm{N}(\mathfrak{p})^{1-2s})^{-1} \prod_{\mathfrak{p} \mid N}(1 - a_{\mathfrak{p}}\mathrm{N}(\mathfrak{p})^{-s})^{-1}. \tag{A.1}$$

We write $L(E, s)$ for $L(E/\mathbf{Q}, s)$ if $E$ is defined over $\mathbf{Q}$. In the latter case, this $L$-function can be continued analytically to the complex plane, but this is a very deep result. The key ingredient here is the modularity theorem, as we will see in the next section. But first, we will consider a method for classifying elliptic curves over $\mathbf{C}$.

### Complex elliptic curves

Over the complex numbers, one can classify elliptic curves using lattices. These are discrete subgroups of the complex plane and can be written as $\Lambda = \mathbf{Z}\omega_1 \oplus \mathbf{Z}\omega_2$ with $\omega_1, \omega_2 \in \mathbf{C}$ linearly independent over $\mathbf{R}$. It turns out that any elliptic curve $E$ over $\mathbf{C}$ is isomorphic (as a complex Lie group) to the quotient $\mathbf{C}/\Lambda$ for some lattice $\Lambda$. Moreover, the following correspondence holds.

**Proposition A.2** ([Sil09, Corollary VI.4.1.1])**.** *Let $E_1$ and $E_2$ be two elliptic curves over $\mathbf{C}$ that correspond to lattices $\Lambda_1$ and $\Lambda_2$. Then $E_1$ and $E_2$ are isomorphic over $\mathbf{C}$ if and only if $\Lambda_1$ and $\Lambda_2$ are homothetic, i.e., if there is some $\alpha \in \mathbf{C}^\times$ such that $\Lambda_1 = \alpha\Lambda_2$.*

Given a lattice $\Lambda \subseteq \mathbf{C}$, it is homothetic to a lattice of the form $\Lambda_\tau = \mathbf{Z} \oplus \tau\mathbf{Z}$ with $\tau \in \mathcal{H}$, the complex upper half-plane. We write $E_\tau$ for the complex elliptic curve associated to $\Lambda_\tau$. One can define an action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathcal{H}$ by the Möbius transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

Now, two lattices $\Lambda_\tau$ and $\Lambda_{\tau'}$ with $\tau, \tau' \in \mathcal{H}$ are homothetic if and only if there exists some $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ with $\tau' = \gamma\tau$. In this way, the quotient $\mathrm{SL}_2(\mathbf{Z})\backslash\mathcal{H}$ naturally classifies all complex elliptic curves up to isomorphism.

### Modular curves

In a more general way, we can create a moduli space for complex elliptic curves with a cyclic subgroup of a fixed size. Let $N \geq 1$ be an integer and consider the subgroup of $\mathrm{SL}_2(\mathbf{Z})$ given by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}.$$

**Definition A.3.** An *enhanced elliptic curve for* $\Gamma_0(N)$ is a pair $(E, C)$ consisting of a complex elliptic curve $E$ and a cyclic subgroup $C \subseteq E(\mathbf{C})$ of order $N$.

Two enhanced elliptic curves $(E, C)$ and $(E', C')$ for $\Gamma_0(N)$ are said to be equivalent if there is an isomorphism $\phi : E \to E'$ such that $\phi(C) = C'$. In that case we write $(E, C) \sim (E', C')$. Now define the moduli space

$$S_0(N) = \big\{\text{enhanced elliptic curves for } \Gamma_0(N)\big\}/\!\sim,$$

and the quotient

$$Y_0(N) = \Gamma_0(N)\backslash\mathcal{H}.$$

**Theorem A.4** ([DS05, Thm 1.5.1 (a)])**.** *The moduli space for $\Gamma_0(N)$ is given by*

$$S_0(N) = \{[E_\tau, \langle 1/N + \Lambda_\tau\rangle] : \tau \in \mathcal{H}\}.$$

*Two classes $[E_\tau, \langle 1/N + \Lambda_\tau\rangle]$ and $[E_{\tau'}, \langle 1/N + \Lambda_{\tau'}\rangle]$ are equal if and only if $[\tau] = [\tau'] \in Y_0(N)$. In particular, there is a bijection*

$$\psi_0 : S_0(N) \to Y_0(N), \qquad [\mathbf{C}/\Lambda_\tau, \langle 1/N + \Lambda_\tau\rangle] \mapsto [\tau].$$

We see that $Y_0(N)$ naturally classifies equivalence classes of enhanced elliptic curves for $\Gamma_0(N)$. Instead of a pair $(E, C)$, we could also consider a cyclic $N$-isogeny, that is, a homomorphism of complex elliptic curves $\phi : E \to E'$ with a cyclic kernel of order $N$. Given a enhanced elliptic curve $(E, C)$, we can construct the cyclic $N$-isogeny $E \to E/C$ given by the quotient map. Conversely, given a cyclic $N$-isogeny $\phi : E \to E'$, the pair $(E, \ker(\phi))$ is an enhanced elliptic curve for $\Gamma_0(N)$. As such, we see that $Y_0(N)$ naturally classifies (equivalence classes of) cyclic $N$-isogenies as well.

One can compactify $Y_0(N)$ by adding finitely many cusps. In this way, we obtain a compact Riemann surface $X_0(N) = \Gamma_0(N)\backslash\mathcal{H}^*$, where $\mathcal{H}^* = \mathcal{H} \cup \mathbf{P}^1(\mathbf{Q})$. We call $X_0(N)$ a modular curve. It is possible to define $X_0(N)$ as a smooth projective curve over $\mathbf{Q}$ whose complex points are $\Gamma_0(N)\backslash\mathcal{H}^*$.

### Modular forms

Given a modular form $f \in S_2(\Gamma_0(N))$, one can define a holomorphic differential form on $X_0(N)(\mathbf{C})$ given by

$$\omega_f = 2\pi i f(\tau) d\tau \in \Omega_{X_0(N)}.$$

This is well-defined for two reasons. The modularity of $f$ guarantees that the form is invariant under the action of $\Gamma_0(N)$, and the fact that $f$ is a cusp form means that the differential form is holomorphic at the cusps of $X_0(N)$. Every holomorphic differential form on $X_0(N)(\mathbf{C})$ arises in this way, and so we may identify $S_2(\Gamma_0(N))$ with $\Omega_{X_0(N)}$. By the Riemann–Roch theorem, it follows that $\dim_{\mathbf{C}} S_2(\Gamma_0(N))$ is finite and given by the genus $g$ of $X_0(N)$.

Fix a cusp form $f \in S_2(\Gamma_0(N))$ with integer Fourier coefficients. If $f$ is a normalized eigenform (see [Dar04, Section 2.3]), it is possible to construct a special elliptic curve $E_f$ over $\mathbf{Q}$ using the Eichler–Shimura construction. The construction defines $E_f$ as a certain quotient of the Jacobian $J_0(N)$ of $X_0(N)$. This Jacobian can be defined as an abelian variety over $\mathbf{Q}$ of dimension $g$ and its points can be identified with the zero divisors on $X_0(N)$ quotiented by the principal divisors. The elliptic curve $E_f$ is special, as the $L$-series of $E_f$ is equal to the $L$-series $L(f, s)$ associated to $f$. For $\Re(s) > \frac{3}{2}$, this $L$-series is given by

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s},$$

where $a_f(n)$ is the $n^{\text{th}}$ Fourier coefficient of $f$.

**Remark.** Here we used a different normalization for the coefficients than in the rest of the thesis. This $L$-series is related to the $L$-series $L_f(s)$ given by (3.7) via $L(f, s) = L_f(s - \frac{1}{2})$. In particular, $L(f, s)$ has a functional equation with center $s = 1$ (see Proposition 3.5).

To summarize:

**Theorem A.5** ([Dar04, Thm. 2.10])**.** *Let $f \in S_2(\Gamma_0(N))$ be a normalized eigenform with integer Fourier coefficients. Then there exists an elliptic curve $E_f$ over $\mathbf{Q}$, given by the Eichler–Shimura construction, such that*

$$L(E_f, s) = L(f, s).$$

Due to the nature of the construction of the elliptic curve $E_f$, there is a *modular parametrization* map $\Phi_N : X_0(N) \to E_f$, sending a point $P$ to the equivalence class of $(P) - (\infty)$ viewed in $J_0(N)$ followed by the quotient to $E_f$. This is a non-constant morphism of curves defined over $\mathbf{Q}$.

## A.2 The Birch and Swinnerton-Dyer conjecture

As stated in the previous section, we would like to be able to analytically continue the $L$-series $L(E, s)$. For this, we use a converse to Theorem A.5, which is known as the modularity theorem. One way to phrase it, as in [DS05, Thm. 2.5.1], is that for any elliptic curve $E/\mathbf{Q}$, there exists some integer $N \geq 1$ and a surjective morphism of curves over $\mathbf{C}$,

$$\Phi_N : X_0(N)_{\mathbf{C}} \to E_{\mathbf{C}}.$$

An equivalent way to phrase it is the following.

**Theorem A.6** (Modularity Theorem, [Dar04, Thm. 2.12])**.** *Let $E/\mathbf{Q}$ be an elliptic curve of conductor $N$. Then there exists a newform $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$ such that*

$$L(E, s) = L(f, s).$$

*Furthermore, $E$ and $E_f$ are isogenous.*

From the theory of modular forms, it is known that $L(f, s)$ can be extended analytically to the complex plane. In fact, we prove a generalization of this in Proposition 3.5. By the modularity theorem, it now follows that the $L$-function $L(E, s)$ of any elliptic curve $E$ has an analytic continuation. There are currently no known methods to obtain this result without using the modularity theorem in some way. As $L(E, s)$ can be extended analytically, we can study its behavior around $s = 1$. We call the order of vanishing of $L(E, s)$ at $s = 1$ the *analytic rank* of the curve $E$. A central conjecture by Birch and Swinnerton-Dyer, and now a Millennium Prize Problem, hypothesizes that the analytic rank is equal to the algebraic rank.

**Conjecture A.7** (Birch and Swinnerton-Dyer)**.** *Let $E/\mathbf{Q}$ be an elliptic curve. Let $r$ denote the algebraic rank of $E(\mathbf{Q})$. Then $\mathrm{ord}_{s=1} L(E, s) = r$.*

The conjecture has been extended to also state a precise formula for the leading coefficient of the Taylor expansion of $L(E, s)$ at $s = 1$.

**Conjecture A.8** (Birch and Swinnerton-Dyer, [Sil09, Conjecture 16.5])**.** *Let $E/\mathbf{Q}$ be an elliptic curve with algebraic rank $r$. Then*

$$\lim_{s \to 1} \frac{L(E, s)}{(s-1)^r} = \frac{2^r \Omega \# Ш(E/\mathbf{Q}) R(E/\mathbf{Q}) \prod_p c_p}{(\# E(\mathbf{Q})_{\mathrm{tors}})^2}. \tag{A.2}$$

*Here $\Omega$ is the real period, $Ш(E/\mathbf{Q})$ is the Tate–Shafarevich group, $R(E/\mathbf{Q})$ is the regulator of $E$ and $c_p$ are Tamagawa numbers related to $E(\mathbf{Q}_p)$.*

One thing to note about this conjecture is that we do not know whether the Tate–Shafarevich group $Ш(E/\mathbf{Q})$ is finite. Of course, its finiteness would follow if the Birch and Swinnerton-Dyer conjecture were proven. At this point in time, the best result we have towards the conjecture is the following statement.

**Proposition A.9** (Gross, Zagier; Kolyvagin)**.** *Let $E$ be an elliptic curve with $\mathrm{ord}_{s=1} L(E, s) \leq 1$. Then the analytic rank and the algebraic rank of $E$ agree. Moreover, the Tate–Shafarevich group $Ш(E/\mathbf{Q})$ is finite.*

In the next section, we will discuss the idea behind the contribution of Gross and Zagier to Proposition A.9.

## A.3 Heegner points

In this section we will discuss Heegner points. Using the modularity of elliptic curves, they allow for the construction of algebraic points on an elliptic curve over $\mathbf{Q}$. Under the right conditions, it turns out that these points are also $\mathbf{Q}$-rational. In this way, we learn something about the Mordell–Weil group $E(\mathbf{Q})$.

Fix some elliptic curve $E$ over $\mathbf{Q}$ of conductor $N$. Let $K$ be an imaginary quadratic field of discriminant $D$ coprime to $N$. We are interested in points $x \in X_0(N)(\mathbf{C})$ that correspond to an $N$-isogeny $E_1 \to E_2$, where both $E_1$ and $E_2$ have complex multiplication by $\mathcal{O}_K$. This means that $\mathrm{End}(E_1)$ and $\mathrm{End}(E_2)$ are both isomorphic to $\mathcal{O}_K$. In order to guarantee the existence of these points, we will need the necessary and sufficient *Heegner hypothesis*: if $p$ is a prime number dividing $N$, then $p$ is split in $K$. Given such a point $x$, we can then use the modular parametrization map $\Phi_N : X_0(N) \to E$ to obtain a point on $E(\mathbf{C})$. In fact, something stronger holds by complex multiplication theory: if $H$ is the Hilbert class field of $K$, then $x \in X_0(N)(H)$ and so $\Phi_N(x)$ lies in $E(H)$. The Hilbert class field of $K$ is the largest unramified abelian extension of $K$ and satisfies $\mathrm{Gal}(H/K) \cong \mathrm{Pic}(\mathcal{O}_K) = \mathrm{Cl}_K$. We call a point on $E(H)$ of the form $\Phi_N(x)$ a Heegner point (of conductor 1). See [Dar04, Chapter 3] for the construction of Heegner points of larger conductor.

Let $P = \Phi_N(x) \in E(H)$ be a Heegner point. We now define

$$P_K = \mathrm{Tr}_{H/K}(P) = \sum_{\sigma \in \mathrm{Gal}(H/K)} P^\sigma \in E(K).$$

We want to know wether $P_K$ is torsion or not. Under the right assumptions, this is decided by the work of Gross and Zagier.

**Theorem A.10** (Gross, Zagier, [GZ86, Thm. V.2.1])**.** *Let $E$ be an elliptic curve over $\mathbf{Q}$ of and let $K$ be an imaginary quadratic field satisfying the Heegner hypothesis relative to $E$. Let $P \in E(H)$ be a Heegner point. Then for some non-zero $\alpha \in \mathbf{R}^\times$,*

$$L'(E/K, 1) = \alpha \cdot \hat{h}(P_K),$$

*where $\hat{h}(P_K)$ is the Néron–Tate height of $P_K$.*

They prove this in two parts, one algebraic and one analytic. The first, algebraic part of their paper is devoted to computing height pairings on $J_0(N)(H) \otimes \mathbf{C}$ of the form $\langle c, T_m c^\sigma \rangle$. Here $c$ is the divisor class of $(P) - (\infty)$, $T_m$ is a Hecke operator and $\sigma$ is an element of $\mathrm{Gal}(H/K)$. Via the isomorphism $\mathrm{Gal}(H/K) \cong \mathrm{Cl}_K$, this element $\sigma$ corresponds to an ideal class $\mathcal{A}$. In the second, analytic part, they compute explicit Fourier coefficients $a_{m,\mathcal{A}}$ of a cusp form $\Phi_\mathcal{A} \in S_2(\Gamma_0(N))$ that satisfies

$$L'_\mathcal{A}(f, 1) = \frac{8\pi^2}{\sqrt{|D|}} \langle f, \Phi_\mathcal{A} \rangle, \tag{A.3}$$

for any newform $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$. Here $L_\mathcal{A}(f, s)$ is a Rankin–Selberg $L$-function (see Section 5.1).

From the formulas they obtain, it follows that $a_{m,\mathcal{A}} = u^2 \langle c, T_m c^\sigma \rangle$, where $2u$ is the number of units in $\mathcal{O}_K$. Given a class group character $\chi : \mathrm{Cl}_K \to \mathbf{C}^\times$, one can sum over all the classes and obtain

$$L'(f, \chi, 1) := \sum_{\mathcal{A} \in \mathrm{Cl}_K} \chi(\mathcal{A}) L'_\mathcal{A}(f, 1) = \frac{8\pi^2 \langle f, f \rangle}{hu^2 \sqrt{|D|}} \hat{h}(c_{\chi,f}),$$

where $c_{\chi,f}$ is the projection of

$$c_\chi = \sum_{\sigma \in \mathrm{Gal}(H/K)} \chi^{-1}(\sigma) c^\sigma \in J_0(N)(H) \otimes \mathbf{C},$$

onto the $f$-isotypical component of $J_0(N)(H) \otimes \mathbf{C}$ for the action of the Hecke algebra $\mathbf{T}$ (see [GZ86, p. 230]). Here $\chi$ is interpreted as a character on $\mathrm{Gal}(H/K)$ via the Artin map $\mathrm{Cl}_K \cong \mathrm{Gal}(H/K)$. In the case that $\chi$ is trivial and $f$ corresponds to $E$ via modularity, the $L$-series $L(f, \chi, s)$ coincides with $L(E/K, s)$. The fact that $\hat{h}(P_K) = \hat{h}(c_{1,f}) \cdot \deg(\Phi_N)$ now leads to Theorem A.10.

Given an elliptic curve $E/\mathbf{Q}$ of conductor $N$ and analytic rank one, we can always find a number field $K$ satisfying the Heegner hypothesis such that $\mathrm{ord}_{s=1} L(E/K, s) = 1$ [Dar04, p. 41]. In that case, $E(K)$ must have at least algebraic rank one by Theorem A.10. Using the known behavior of $P_K$ under the action of $\mathrm{Gal}(K/\mathbf{Q})$ (see [Dar04, Prop. 3.11]), one can deduce that $P_K$ lies in $E(\mathbf{Q})$ up to torsion if and only if $L(E, s)$ has odd order of vanishing at $s = 1$. Under our assumptions, this order of vanishing is odd and so we deduce that the algebraic rank of $E(\mathbf{Q})$ is at least one.

In order to conclude that $E(\mathbf{Q})$ has an algebraic rank of exactly one and to deduce the analytic rank zero part of Proposition A.9, a result of Kolyvagin is needed [Kol88]. This result requires the full power of the Euler system of Heegner points.

In Chapter 5, we compute the values and derivatives of the Rankin–Selberg $L$-function $L_{\mathcal{A}}(f, s)$, thereby deducing parts of the analytic side of [GZ86]. Instead of Rankin's method, which is used by Gross and Zagier, we use the Goldfeld–Zhang method [GZ99]. As our approach is limited to modular forms $f \in S_k(\Gamma_0(N))$ with weight $k > 2$, we are unable to deduce (A.3).

# B Changes to the Goldfeld–Zhang method

In this appendix, we give a list of adjustments and corrections we had to make while deriving a formula for the holomorphic kernel, when compared to [GZ99]. This includes both differences in definitions and corrections to formulas.

(1) [GZ99, Section 3]: When defining the holomorphic kernel, the inner product $\langle \Phi_{s,g}, f\rangle$ should be $\langle f, \Phi_{\bar{s},g}\rangle$ instead, as $f \mapsto \langle \Phi_{s,g}, f\rangle$ is not a linear map. As a result, all formulas for the coefficients of the holomorphic kernel should be conjugated.

(2) [GZ99, Proposition 3.6]: The formula for $\gamma(m, n; s)$ when $n > m$ should contain $\frac{m}{m-n}$ instead of $\frac{n}{m-n}$. Note that we do not derive this in the thesis.

(3) [GZ99, Proposition 4.2]: $L_g(s, \frac{a}{c})$ does not have a pole at $s = \frac{1-\ell}{2}$ with residue $-b(0)$. Instead, $L_g^*(s, \frac{a}{c})$ satisfies that description. Moreover, the functional equation should contain the factor $i^\ell$ instead of $i^{-\ell}$. See Proposition 3.5.

(4) [GZ99, Definition 5.2]: We use $\tau(\chi_\delta)$ instead of $G(\delta)$. This is because we define $S^\delta(s, B)$ differently, see point (6) below. Note that $G(\delta) = \tau(\overline{\chi_\delta})$.

(5) [GZ99, Lemma 5.3]: The statement of the lemma is incorrect and should use $\epsilon_{\delta'}^{-1}(r)$ instead of $\epsilon_{\delta'}(r)$ in the generalized Ramanujan sum. From their proof, it is clear that this is a typing error.

(6) [GZ99, Equation (6.4)]: We define $S^\delta(s, B)$ with $\epsilon_{\delta'}(r)$ instead of $\epsilon_{\delta'}^{-1}(r)$. See Definition 4.2. This difference stems from a mistake in the proof of [GZ99, Thoerem 6.5], where they use the functional equation of $L_g(s, \frac{\bar{r}}{c})$, but write $\bar{a}$ where $r$ should be used instead.

(7) [GZ99, Theorem 6.5]: In part (a), the coefficients should be conjugated. In part (b), the formula should contain $i^\ell$ instead of $i^{-\ell}$ and no factor $\epsilon_{D/\delta'}^{-2}(\delta)$. Moreover, the formula holds only for $s$ in a certain vertical strip. See Theorem 4.4.

(8) [GZ99, Proposition 7.1]: Due to the difference in the definition of $S^\delta(s, B)$, occurrences of $\epsilon_{\delta'}$ should be replaced by $\epsilon_{\delta'}^{-1}$. In particular, $\epsilon^\delta$ becomes $\epsilon^{-1}$ and $L_{N_2}^\delta(2s)$ becomes $L_{N_2}(2s)$. See Proposition 4.6. We use the notation $N^{[\delta']}$ instead of $N_2$.

(9) [GZ99, Proposition 8.3]: In the case that $x > 1$, the hypergeometric function should have $\ell$ as the third argument instead of $k$.

(10) [GZ99, Proposition 9.1]: The formula for $\tilde{\phi}_{s,g}$ should be conjugated, not contain $\mu\left(\frac{N_2}{e}\right)$, and contain $i^\ell$ instead of $i^{-\ell}$. The formula for $T_{m,N_2}$ should use $S_{N_2}^\delta$ instead of $S_e^\delta$. Also, $\langle \tilde{\Phi}_{s,g}, f\rangle$ should be $\langle f, \tilde{\Phi}_{\bar{s},g}\rangle$. The assumption that $\chi$ must be a real character can be dropped. Lastly, the formula holds only for $s$ in a certain vertical strip. See Proposition 4.11.

(11) [GZ99, Proposition 10.1]: See Theorem C.1 for the corrected formula for the coefficients.

(12) [GZ99, Theorem 11.5]: See Theorem 4.13 for the corrected formula for the coefficients.

# C Modified holomorphic kernel for D = 1

We give an adjusted version of [GZ99, Thm. 10.1]. In particular, it contains an additional term that we believe to have been overlooked.

**Theorem C.1.** *Let $k$ and $\ell$ be integers with $k \equiv \ell \equiv 0 \pmod 2$. Let $N \geq 1$ be some level and $g \in M_\ell(\mathrm{SL}_2(\mathbf{Z}))$. Let $\gamma > 0$ be a bound on the coefficients of $g$ and its Atkin–Lehner translates as in (4.3) and assume that $k > 2\gamma + 3$. Let $\tilde{\Phi}_{\bar{s},g}$ be as in Proposition 4.11 with $D = 1$. Then $\tilde{\Phi}_{\bar{s},g}$ has Fourier coefficients whose complex conjugates for $\frac{3-k}{2} + \gamma < \Re(s) < \frac{k-1}{2} - \gamma$ are given by*

$$\begin{aligned}\overline{\tilde{\phi}_{\bar{s},g}(m)} &= i^{k+\ell}\frac{N^{2s-1}}{(2\pi)^{2s}}\frac{b(m)}{m^s}\frac{\Gamma(1-s)\Gamma(s)}{\Gamma(\frac{\ell-k}{2}+1-s)\Gamma(\frac{\ell+k}{2}-s)}\zeta(2s)\\
&+ i^{k+\ell}(2\pi)^{2s-2}\frac{b(m)}{m^{1-s}}\frac{\Gamma(1-s)\Gamma(s)}{\Gamma(\frac{\ell-k}{2}+s)\Gamma(\frac{\ell+k}{2}-1+s)}\zeta(2-2s)\\
&+ i^{k+\ell}N^{s-1}\sum_{\substack{n\geq 1, n\neq m\\ n\equiv m \pmod N}} b(n) \sum_{d_1\cdot d_2 = \left|\frac{m-n}{N}\right|}\frac{1}{d_1^s d_2^{1-s}}\left|\frac{m}{n}-1\right|^{1-s}\tilde{I}_s\left(\frac{m}{n}\right)\\
&+ i^{k+\ell}\frac{b(0)}{\Gamma(\ell)}m^{s-\frac{\ell+1}{2}}\sum_{d\mid\frac{m}{N}} d^{1-2s}.\end{aligned}$$

*Furthermore, $\tilde{\Phi}_{s,g}$ satisfies the functional equation*

$$\tilde{\Phi}_{1-s,g} = N^{1-2s}\tilde{\Phi}_{s,g}.$$

*Proof.* We remark that $g$ has the trivial nebentypus $\chi_0$ modulo 1. As $N^{[D]} = N$, we have by Proposition 4.11, for $1 < \Re(s) < \frac{k-1}{2} - \gamma$, that

$$\overline{\tilde{\phi}_{\bar{s},g}(m)} = \frac{b(m)}{m^s}\sum_{e\mid N}\frac{e}{N}\frac{\Gamma(\frac{k-\ell}{2}+s)L^{(e)}(\chi_0,2s)}{\Gamma(\frac{k+\ell}{2}-s)(2\pi)^{2s}e^{1-2s}} + 2\pi i^k T_{m,N}(s).$$

To obtain the first term, we first calculate

$$\begin{aligned}\sum_{e\mid N}\frac{e}{N}\frac{L^{(e)}(\chi_0,2s)}{e^{1-2s}} &\overset{(3.3)}{=} \sum_{e\mid N}\frac{e^{2s}}{N}\zeta(2s)\prod_{p\mid e}\left(1-\frac{1}{p^{2s}}\right) = \frac{\zeta(2s)}{N}\sum_{e\mid N}e^{2s}\sum_{d\mid e}\mu(d)d^{-2s}\\
&= \frac{\zeta(2s)}{N}\sum_{d\mid N}\mu(d)\sum_{e\mid\frac{N}{d}}e^{2s} = \frac{\zeta(2s)}{N}\sum_{e\mid N}e^{2s}\sum_{d\mid\frac{N}{e}}\mu(d) = \zeta(2s)N^{2s-1}.\end{aligned}$$

This yields the first term, as

$$\begin{aligned}\frac{b(m)}{m^s}\sum_{e\mid N}\frac{e}{N}\frac{\Gamma(\frac{k-\ell}{2}+s)L^{(e)}(\chi_0,2s)}{\Gamma(\frac{k+\ell}{2}-s)(2\pi)^{2s}e^{1-2s}} &= \frac{b(m)}{m^s}\frac{\Gamma(\frac{k-\ell}{2}+s)}{\Gamma(\frac{k+\ell}{2}-s)(2\pi)^{2s}}\zeta(2s)N^{2s-1}\\
&= i^{k+\ell}\frac{N^{2s-1}}{(2\pi)^{2s}}\frac{b(m)}{m^s}\frac{\Gamma(1-s)\Gamma(s)}{\Gamma(\frac{\ell-k}{2}+1-s)\Gamma(\frac{k+\ell}{2}-s)}\zeta(2s).\end{aligned}$$

Here we used that

$$\Gamma\left(\frac{k-\ell}{2}+s\right)\Gamma\left(\frac{\ell-k}{2}+1-s\right)=i^{k+\ell}\Gamma(1-s)\Gamma(s).$$

Next, recall that

$$2\pi i^k T_{m,N}(s)=i^{k+\ell}\sum_{n=0}^{\infty}b(n)S_N^1(s,m-n)\tilde{V}_s(n,m). \tag{C.1}$$

By Proposition 4.6, we have

$$S_N^1(s,m-n)=\begin{cases}\sum_{d|\frac{m-n}{N}}d^{1-2s} & \text{if } m\neq n, N|(m-n),\\ \zeta(2s-1) & \text{if } m=n,\\ 0 & \text{else.}\end{cases}$$

Taking the term in (C.1) for $n=m$ and using Proposition 4.8, yields the second term:

$$\begin{aligned}i^{k+\ell}\frac{b(m)}{m^{1-s}}\zeta(2s-1)\tilde{I}_s(1)&=i^{k+\ell}\frac{b(m)}{m^{1-s}}\zeta(2s-1)\frac{\Gamma(2s-1)}{\Gamma(\frac{\ell-k}{2}+s)\Gamma(\frac{\ell+k}{2}+s-1)}\\&=i^{k+\ell}(2\pi)^{2s-2}\frac{b(m)}{m^{1-s}}\frac{\Gamma(1-s)\Gamma(s)}{\Gamma(\frac{\ell-k}{2}+s)\Gamma(\frac{\ell+k}{2}+s-1)}\zeta(2-2s).\end{aligned}$$

Here we used the functional equation of $\zeta(s)$ (given by (3.6)) together with the reflection property of $\Gamma(s)$. For the third term, we consider the terms for $n\geq 1$ with $n\neq m$ in (C.1). This yields

$$\begin{aligned}i^{k+\ell}\frac{b(n)}{n^{1-s}}\sum_{d|\frac{m-n}{N}}d^{1-2s}\tilde{I}_s\left(\frac{m}{n}\right)&=i^{k+\ell}\frac{b(n)}{n^{1-s}}\sum_{d_1d_2=\left|\frac{m-n}{N}\right|}d_1^{-s}\cdot\left(\left|\frac{m-n}{N}\right|\cdot d_2^{-1}\right)^{1-s}\tilde{I}_s\left(\frac{m}{n}\right)\\&=i^{k+\ell}N^{s-1}\cdot b(n)\sum_{d_1d_2=\left|\frac{m-n}{N}\right|}\frac{1}{d_1^sd_2^{1-s}}\left|\frac{m}{n}-1\right|^{1-s}\tilde{I}_s\left(\frac{m}{n}\right).\end{aligned}$$

The fourth term is simply given by the term corresponding to $n=0$ in (C.1):

$$i^{k+\ell}\frac{b(0)}{\Gamma(\ell)}m^{s-\frac{\ell+1}{2}}\sum_{d|\frac{m}{N}}d^{1-2s}.$$

Using the fact that $k>2\gamma+3$, one can now show that the series in the third term converges locally absolutely and uniformly for $\frac{3-k}{2}+\gamma<\Re(s)<\frac{k-1}{2}-\gamma$. See also the proof of Theorem 4.13. This proves the first part of the theorem.

For the functional equation, we note that the first two terms of $\tilde{\phi}_{1-s,g}(m)$ are swapped up to multiplication by $N^{1-2s}$ under the transformation $s\mapsto 1-s$. Each term in the sum of the third term is kept invariant under the transformation $s\mapsto 1-s$ due to the functional equation of $\tilde{I}_s$, see Proposition 4.7. Lastly, for the fourth term, a simple calculation reveals that

$$i^{k+\ell}\frac{b(0)}{\Gamma(\ell)}m^{\frac{1-\ell}{2}-s}\sum_{d|\frac{m}{N}}d^{2s-1}=i^{k+\ell}\frac{b(0)}{\Gamma(\ell)}m^{\frac{1-\ell}{2}-s}\sum_{d|\frac{m}{N}}\left(\frac{m}{dN}\right)^{2s-1}=i^{k+\ell}N^{1-2s}\frac{b(0)}{\Gamma(\ell)}m^{s-\frac{\ell+1}{2}}\sum_{d|\frac{m}{N}}d^{1-2s}.$$

□

## References


[BS66] A. I. Borevich and I. R. Shafarevich. *Number theory*. Vol. 20. Pure and Applied Mathematics. Translated from Russian by Newcomb Greenleaf. Academic Press, New York-London, 1966, pp. x+435.

[Bre+01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. "On the modularity of elliptic curves over $\mathbf{Q}$: wild 3-adic exercises". In: *J. Amer. Math. Soc.* 14.4 (2001), pp. 843–939. ISSN: 0894-0347,1088-6834. DOI: 10.1090/S0894-0347-01-00370-8.

[Dar04] Henri Darmon. *Rational points on modular elliptic curves*. Vol. 101. CBMS Regional Conference Series in Mathematics. Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2004, pp. xii+129. ISBN: 0-8218-2868-1.

[Del74] Pierre Deligne. "La conjecture de Weil. I". In: *Inst. Hautes Études Sci. Publ. Math.* 43 (1974), pp. 273–307. ISSN: 0073-8301,1618-1913. URL: http://www.numdam.org/item?id=PMIHES_1974__43__273_0.

[DS05] Fred Diamond and Jerry Shurman. *A first course in modular forms*. Vol. 228. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xvi+436. ISBN: 0-387-23229-X.

[Erd+81a] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi. *Higher transcendental functions. Vol. I.* Based on notes left by Harry Bateman, With a preface by Mina Rees, With a foreword by E. C. Watson, Reprint of the 1953 original. Robert E. Krieger Publishing Co., Inc., Melbourne, FL, 1981, pp. xiii+302. ISBN: 0-89874-069-X.

[Erd+81b] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi. *Higher transcendental functions. Vol. II.* Based on notes left by Harry Bateman, Reprint of the 1953 original. Robert E. Krieger Publishing Co., Inc., Melbourne, FL, 1981, pp. xviii+396. ISBN: 0-89874-069-X.

[GZ99] Dorian Goldfeld and Shouwu Zhang. "The holomorphic kernel of the Rankin-Selberg convolution". In: *Asian J. Math.* 3.4 (1999), pp. 729–747. ISSN: 1093-6106,1945-0036. DOI: 10.4310/AJM.1999.v3.n4.a1.

[GZ86] Benedict H. Gross and Don B. Zagier. "Heegner points and derivatives of $L$-series". In: *Invent. Math.* 84.2 (1986), pp. 225–320. ISSN: 0020-9910,1432-1297. DOI: 10.1007/BF01388809.

[IK04] Henryk Iwaniec and Emmanuel Kowalski. *Analytic number theory*. Vol. 53. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004, pp. xii+615. ISBN: 0-8218-3633-1. DOI: 10.1090/coll/053.

[Kol88] V. A. Kolyvagin. "Finiteness of $E(\mathbf{Q})$ and $Ш(E, \mathbf{Q})$ for a subclass of Weil curves". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 52.3 (1988), pp. 522–540, 670–671. ISSN: 0373-2436. DOI: 10.1070/IM1989v032n03ABEH000779.

[Li79] Wen Ch'ing Winnie Li. "$L$-series of Rankin type and their functional equations". In: *Math. Ann.* 244.2 (1979), pp. 135–166. ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01420487.

[LS24] David T.-B. G. Lilienfeldt and Ari Shnidman. *Derivatives of Rankin-Selberg L-functions and heights of generalized Heegner cycles*. 2024. arXiv: 2408.04375 [math.NT]. URL: https://arxiv.org/abs/2408.04375.

[Maz78] B. Mazur. "Rational isogenies of prime degree (with an appendix by D. Goldfeld)". In: *Invent. Math.* 44.2 (1978), pp. 129–162. ISSN: 0020-9910,1432-1297. DOI: 10.1007/BF01390348.

[Mer96] Loïc Merel. "Bornes pour la torsion des courbes elliptiques sur les corps de nombres". In: *Invent. Math.* 124.1-3 (1996), pp. 437–449. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s002220050059.

[Miy06] Toshitsune Miyake. *Modular forms*. English. Springer Monographs in Mathematics. Translated from the 1976 Japanese original by Yoshitaka Maeda. Springer-Verlag, Berlin, 2006, pp. x+335. ISBN: 978-3-540-29592-1.

[Nel13] Paul D. Nelson. "Stable averages of central values of Rankin-Selberg $L$-functions: some new variants". In: *J. Number Theory* 133.8 (2013), pp. 2588–2615. ISSN: 0022-314X,1096-1658. DOI: 10.1016/j.jnt.2013.01.001.

[Neu99] Jürgen Neukirch. *Algebraic number theory*. Vol. 322. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder. Springer-Verlag, Berlin, 1999, pp. xviii+571. ISBN: 3-540-65399-6. DOI: 10.1007/978-3-662-03983-0.

[Ran39] R. A. Rankin. "Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions. II. The order of the Fourier coefficients of integral modular forms". In: *Proc. Cambridge Philos. Soc.* 35 (1939), pp. 351–372. ISSN: 0008-1981.

[Sel40] Atle Selberg. "Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist". In: *Arch. Math. Naturvid.* 43 (1940), pp. 47–50. ISSN: 0365-4524.

[Sil09] Joseph H. Silverman. *The arithmetic of elliptic curves*. Second. Vol. 106. Graduate Texts in Mathematics. Springer, Dordrecht, 2009, pp. xx+513. ISBN: 978-0-387-09493-9. DOI: 10.1007/978-0-387-09494-6.

[SW71] Elias M. Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Vol. 32. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1971, pp. x+297.

[Sze75] Gábor Szegő. *Orthogonal polynomials*. Fourth. Vol. XXIII. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 1975, pp. xiii+432.

[Tit39] E. C. Titchmarsh. *The theory of functions*. Second. Oxford University Press, Oxford, 1939, pp. x+454.

[Wei48] André Weil. "On some exponential sums". In: *Proc. Nat. Acad. Sci. U.S.A.* 34 (1948), pp. 204–207. ISSN: 0027-8424. DOI: 10.1073/pnas.34.5.204.

[Wil95] Andrew Wiles. "Modular elliptic curves and Fermat's last theorem". In: *Ann. of Math. (2)* 141.3 (1995), pp. 443–551. ISSN: 0003-486X,1939-8980. DOI: 10.2307/2118559.