Algebraic cycles and Diophantine geometry

Generalised Heegner cycles, quadratic Chabauty & diagonal cycles

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Diophantine Geometry

Let C denote a smooth projective curve over a number field K.

Diophantine geometry: the study of the set $C(K)$ using geometry.

Question: How big is $C(K)$ and can it be determined explicitly?

The size of $C(K)$ is dictated by the genus $g := \dim H^0(C, \Omega^1_C)$.

g	#C(K)	Type	Proof
0	infinite	projective line	Hilbert-Hurwitz (1890)
1	finite or infinite	elliptic curve	Mordell-Weil (1929)
			Faltings (1983)
≥ 2	finite	higher genus	Vojta (1991)
			Lawrence-Venkatesh (2020)

Questions in higher genus

Question: How to explicitly determine the finite set $C(K)$?

Problem: The proofs of Mordell's conjecture are not effective.

Let J be the Jacobian of C, $r = \mathrm{rank}_{\mathbb{Z}} J(K)$, $\rho =$ Picard number of J.

$\mathbb{Q}$	**Method**	**Rank condition**	**Authors**
	CC	$r < g$	Chabauty, Coleman
	QC	$r < g + \rho - 1$	Kim, BD, BDMTV
	GQC	$r < g + \rho - 1$	Edixhoven-Lido

Let $\delta = \mathrm{rank}_{\mathbb{Z}} \mathcal{O}_K^\times$ and $d = [K : \mathbb{Q}]$.

K	**Method**	**Rank condition**	**Authors**
	RoSC	$r \leq (g-1)d$	Siksek
	RoSQC	$r + \delta(\rho - 1) \leq (g + \rho - 2)d$	Dogra, BBBM

Theorem (Čoupek–L.–Xiao–Yao (2020))

Generalisation of GQC to K under $r + \delta(\rho - 1) \leq (g + \rho - 2)d$.

Questions in genus 1

Let E be an elliptic curve over a number field K.

Algebra

$$E(K) = E(K)_{\mathrm{tors}} \oplus \mathbb{Z}^{r_{\mathrm{alg}}(E/K)}$$

Analysis

$$\prod_{N_{K/\mathbb{Q}}(\mathfrak{q}) \leq X} \frac{\#E(\mathbb{F}_{\mathfrak{q}})}{N_{K/\mathbb{Q}}(\mathfrak{q})} \overset{?}{\sim} \log(X)^{r_{\mathrm{alg}}(E/K)}$$

Birch and Swinnerton-Dyer Conjecture (1960's)

$$r_{\mathrm{alg}}(E/K) = \mathrm{ord}_{s=1} L(E/K, s) =: r_{\mathrm{an}}(E/K).$$

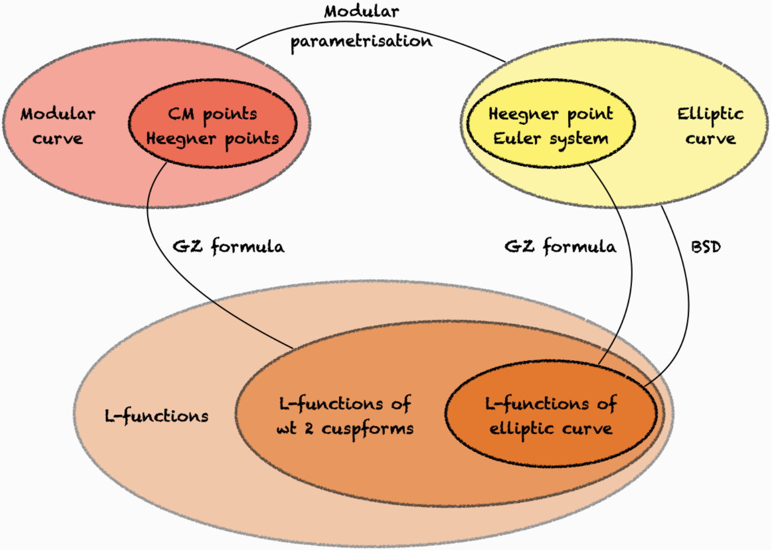
Theorem (Gross-Zagier-Kolyvagin (1980's))

$$r_{\mathrm{an}}(E/\mathbb{Q}) \in \{0, 1\} \implies r_{\mathrm{an}}(E/\mathbb{Q}) = r_{\mathrm{alg}}(E/\mathbb{Q}).$$

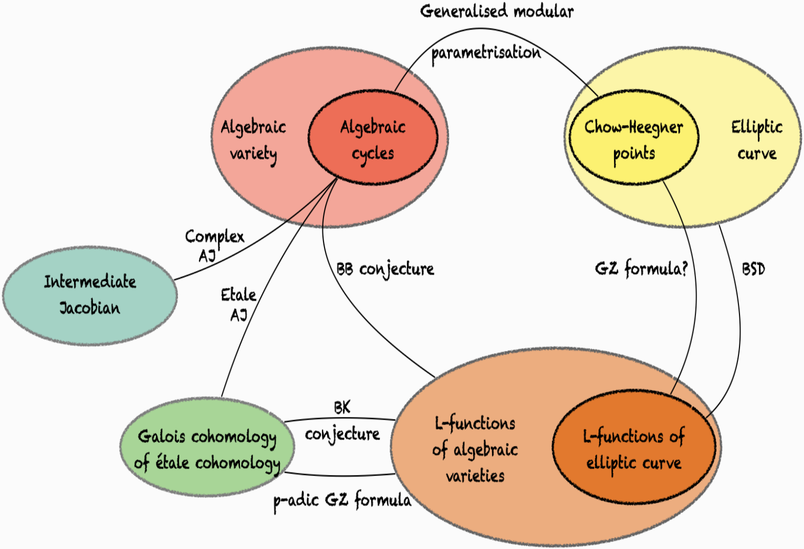
S. Zhang (2001): Generalisation to K totally real.

Status: Open for general number fields and higher rank.

The Gross-Zagier-Kolyvagin strategy over $\mathbb{Q}$



Algebraic cycles and Chow-Heegner points



Generalised Heegner cycles

Bertolini-Darmon-Prasanna (2013) introduced a collection of generalised Heegner cycles on varieties $X_r := W_r \times A^r$:

$$\{\Delta_\varphi \in \mathrm{CH}^{r+1}(X_r)_0(\bar{\mathbb{Q}}), \quad \varphi : A \longrightarrow A'\}.$$

Functoriality of complex Abel-Jacobi maps:

$$\begin{array}{ccc} \mathrm{CH}^{r+1}(X_r)_0(\mathbb{C}) & \xrightarrow{\mathrm{AJ}_{X_r}} & J^{r+1}(X_r) \\ \Pi_*^? \big\downarrow & & \big\downarrow (\Phi_{\mathrm{dR}}^*)^\vee \\ A(\mathbb{C}) & \xrightarrow{\mathrm{AJ}_A} & \mathrm{Jac}(A)(\mathbb{C}) \end{array} \qquad \Phi_{\mathrm{dR}}^*(\omega_A) = c_r \cdot \omega_{\theta_A} \wedge \eta_A^r.$$

Upshot: We can obtain information about Chow-Heegner points by computing $\mathrm{AJ}_{X_r}(\Delta_\varphi)$.

Theorem (Bertolini-Darmon-L.-Prasanna (2019))

Let $\varphi : A \longrightarrow \mathbb{C}/\langle 1, \tau\rangle$ be an isogeny of degree $d_\varphi = \deg(\varphi)$.
For all $f \in S_{r+2}(\Gamma_1(N))$ and $0 \leq j \leq r$,

$$\mathrm{AJ}_{X_r}(\Delta_\varphi)(\omega_f \wedge \omega_A^j \eta_A^{r-j}) = \frac{(-d_\varphi)^j (2\pi i)^{j+1}}{(\tau - \bar{\tau})^{r-j}} \int_{i\infty}^{\tau} (z-\tau)^j (z-\bar{\tau})^{r-j} f(z) dz.$$

Application: Used by Bertolini-Darmon-Prasanna to provide computational evidence for the rationality of Chow-Heegner points.

Theorem (Bertolini-Darmon-L.-Prasanna (2019))

The collection of generalised Heegner cycles generates a subgroup of infinite rank in the group of codimension $r+1$ null-homologous cycles in X_r over K^{ab} modulo both rational and algebraic ($r \geq 2$) equivalence. In particular,

$$\dim_{\mathbb{Q}} \mathrm{CH}^{r+1}(X_r)_0(K^{ab}) \otimes \mathbb{Q} = \infty.$$

Diagonal cycles over $\mathbb{Q}$

Consider the modular curve $X_0(p)$. Let $F = f_1 \otimes f_2 \otimes f_3 \in S_2(\Gamma_0(p))^{\otimes 3}$.

Beilinson-Bloch Conjecture (1980's)

$$\mathrm{ord}_{s=2} L(F/K, s) = \dim_{K_F} (t_F)_*(\mathrm{CH}^2(X_0(p)^3)_0(K) \otimes K_F).$$

When $K = \mathbb{Q}$ and $W(F) = -1$:

Gross-Kudla Conjecture (1992), Yuan-Zhang-Zhang (20??)

$$L'(F/\mathbb{Q}, 2) = \Omega_F \cdot \langle (t_F)_*(\Delta_{\mathrm{GKS}}), (t_F)_*(\Delta_{\mathrm{GKS}}) \rangle^{BB}.$$

When $K = \mathbb{Q}$ and $W(F) = +1$:

Theorem (L. (2021))

The Abel-Jacobi image $\mathrm{AJ}_{X_0(p)^3}((t_F)_*(\Delta_{\mathrm{GKS}}(e)))$ *is torsion in* $J^2(X_0(p)^3/\mathbb{C})$ *for any base point* $e \in X_0(p)(\mathbb{Q})$.

Diagonal cycles over $\mathbb{Q}(\sqrt{\chi(-1)p})$

Let $\chi =$ Legendre symbol at p, $K = \mathbb{Q}(\sqrt{\chi(-1)p})$, $\mathrm{Gal}(K/\mathbb{Q}) = \{1, \tau\}$.

Construction: There exist maps $\varphi_\pm : X(p) \longrightarrow X_0(p)^3$ such that

$$\Xi := \varphi_+(X(p)) - \varphi_-(X(p)) \in \mathrm{CH}^2(X_0(p)^3)_0(K)^{\tau=-1}.$$

Theorem (L. (2021))

The global root number of $L(F \otimes \chi, s)$ is -1.

BB Conjecture $\implies \dim_{K_F}(t_F)_*(\mathrm{CH}^2(X_0(p)^3)_0(K)^{\tau=-1} \otimes K_F) \geq 1.$

Conjecture (L. (2021))

$$(t_F)_*(\Xi) \neq 0 \quad \iff \quad \mathrm{ord}_{s=2} L(F \otimes \chi, s) = 1.$$

Diagonal Chow-Heegner points over $\mathbb{Q}$

When $f = f_3$ has rational coefficients, and $g = f_2 = f_3$, we can define a modular parametrisation

$$\Pi_{g,f,*} : \mathrm{CH}^2(X_0(p)^3)_0(K) \longrightarrow E_f(K).$$

Theorem (Darmon-Rotger-Sols (2012))

If $W(f) = -1$, $W(\mathrm{Sym}^2(g) \otimes f) = +1$, *then* $\Pi_{g,f,*}(\Delta_{\mathrm{GKS}}) \in E_f(\mathbb{Q})$ *has infinite order iff* $L'(f,1) \neq 0$, $L(\mathrm{Sym}^2(g^\sigma) \otimes f, 2) \neq 0$ *for all* $\sigma : K_g \hookrightarrow \mathbb{C}$.

Theorem (L. (2021))

If $W(f) = +1$, *then* $\Pi_{g,f,*}(\Delta_{\mathrm{GKS}}(e)) \in E_f(\mathbb{Q})$ *is torsion for all* $e \in X_0(p)(\mathbb{Q})$.

Diagonal Chow-Heegner points over $\mathbb{Q}(\sqrt{\chi(-1)p})$

Recall the cycle $\Xi = \varphi_+(X(p)) - \varphi_-(X(p)) \in \mathrm{CH}^2(X_0(p)^3)(K)^{\tau=-1}$.

When $p \equiv 3 \pmod 4$, $K = \mathbb{Q}(\sqrt{-p})$ and $W(E_f^\chi) = +1$.

Theorem (L. (2021))

If $p \equiv 3 \pmod 4$, then $\Pi_{g,f,*}(\Xi) \in E_f(\mathbb{Q}(\sqrt{-p}))^{\tau=-1}$ is torsion.

When $p \equiv 1 \pmod 4$, $K = \mathbb{Q}(\sqrt{p})$ and $W(E_f^\chi) = -1$.

Conjecture (L. (2021))

When $p \equiv 1 \pmod 4$, $\Pi_{g,f,*}(\Xi) \in E_f(\mathbb{Q}(\sqrt{p}))^{\tau=-1}$ has infinite order iff $L'(f \otimes \chi, 1) \neq 0$, $L(\mathrm{Sym}^2(g^\sigma) \otimes f \otimes \chi, 2) \neq 0$ for all $\sigma : K_g \hookrightarrow \mathbb{C}$.

Future work: Address these conjectures using Abel-Jacobi maps.

Thank you for your attention!

Appendix

Theorem (Čoupek–L.–Xiao–Yao (2020))

Let C be a smooth, proper, geometrically connected curve of genus $g \geq 2$ defined over K, and satisfying the condition

$$r + \delta(\rho - 1) \leq (g + \rho - 2)d.$$

Let $R := \mathbb{Z}_p\langle z_1, \ldots, z_{r+\delta(\rho-1)}\rangle$ be the p-adically completed polynomial algebra over $\mathbb{Z}_p$. There exists an ideal I of R, which is explicitly computable modulo p, such that if $\overline{A} := (R/I) \otimes \mathbb{F}_p$ is a finite dimensional $\mathbb{F}_p$-vector space, then the set of rational points $C(K)$ is finite and

$$|C(K)| \leq \dim_{\mathbb{F}_p} \overline{A}.$$