My Mathematical Life

A Tale of Cycles, Climbing, and Curves

David Ter-Borch Gram Lilienfeldt

Mathematical Institute

Leiden University

Universiteit Leiden

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A Christmas miracle

Born on the day after Christmas

30 years ago

the year was 1993

in the flattest country on Earth

with no mountains, but constant rain

and people on bicycles everywhere

the name of the country was of course...

Denmark

Next stop: France (2000-2011)




Mathematical awakening: Switzerland (2011-2016)

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

L-Series and Arithmetic

a thesis
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by
David Ter-Borch Gram Schjoldager Lilienfeldt
(david.lilienfeldt@epfl.ch)

under the supervision of:

Prof. Eva Bayer-Fluckiger
Swiss Federal Institute of Technology, Lausanne

Prof. Benedict Gross
Harvard University, Cambridge

EPFL, Lausanne
Fall Semester 2015-2016

Mathematical maturity: Canada (2016-2021)

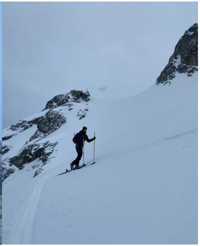
Postdoc'ing: Israel and the Netherlands (2021–???)



Mountaineering activities: Chamonix and beyond (1997–???)







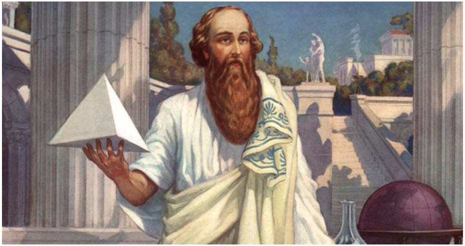
Check out my Youtube channel :)

A bit of mathematics

Number Theory: the study of properties of the integers and in particular of **prime numbers**.

Algebraic geometry: the study of the geometry of shapes cut out by systems of polynomial equations called **algebraic varieties**.

Arithmetic geometry: the study of rational solutions or integer solutions of systems of polynomial equations, i.e., the study of **rational points** on algebraic varieties.



Basic example (genus 0)

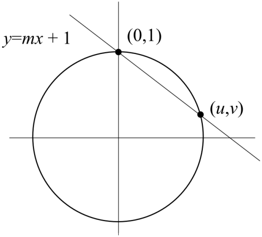
Find all integer solutions to the quadratic equation in 3 variables

$$X^2 + Y^2 = Z^2.$$

Equivalently, find all the rational solutions to the equation

$$x^2 + y^2 = 1 \qquad\qquad (x = X/Z, y = Y/Z).$$

This is the equation of the unit circle in the (x, y)-plane. Thus, the question is equivalent to asking for all the rational points on the unit circle.



Solutions are infinite and known as **Pythagorian triples**.

Elliptic curves (genus 1)

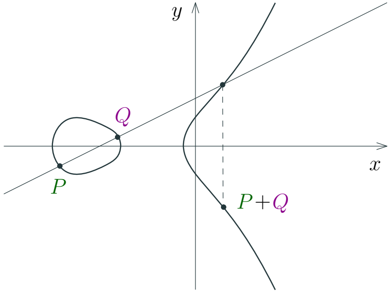
A deceptively simple looking cubic equation in 2 variables:

$$E\colon y^2 = x^3 + ax + b, \qquad a, b \in \mathbb{Q}$$

The set of solutions $E(\mathbb{Q})$ forms a **finitely generated** abelian group:

$$E(\mathbb{Q}) = E(\mathbb{Q})_{\mathrm{tors}} \times \mathbb{Z}^r$$

There are 15 possibilities for $E(\mathbb{Q})_{\mathrm{tors}}$ (Mazur 1978).



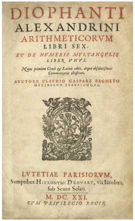
The rank r is predicted by the **Birch–Swinnerton-Dyer conjecture** to be equal to $\mathrm{ord}_{s=1} L(E/\mathbb{Q}, s)$.

Higher genus curves (genus ≥ 2)

Problem 17 Book VI of Diophantus' *Arithmetica*:

Find three squares which when added give a square, and such that the first one is the square-root of the second, and the second is the square-root of the third: $y^2 = x^8 + x^4 + x^2$

It is known that higher genus curves have **finitely** many rational points (Faltings 1983).



Wetherell (1997): the only positive rational solution is the one found by Diophantus himself: $(1/2, 9/16)$.

The proof uses a modern technique known as **Chabauty's method**.

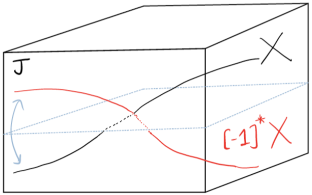
People are working hard to develop **Kim's non-abelian Chabauty program** in order to tackle more general higher genus curves.

Higher dimensions

So far we have concentrated on curves because it is simpler. But what about surfaces and higher dimensional algebraic varieties?

Algebraic cycles: formal linear combinations of algebraic subvarieties

Example: the Ceresa cycle obtained from a curve X embedded in its Jacobian $X \hookrightarrow J$ $$[X] - [X^{-}] = 1 - \mathit{cycle}$$



Modulo **rational equivalence**, algebraic cycles form an abelian group called the (null-homologous) **Chow group**.

It is conjectured to be **finitely generated** with rank equal to the order of vanishing of an ***L*-function** (Beilinson–Bloch 1984).

My contributions so far

- Triple product diagonal cycles on modular curves and applications to elliptic curves via Chow–Heegner points
- Generalized Heegner cycles and their Abel–Jacobi images (with Bertolini, Darmon, Prasanna) and their heights (with Shnidman)
- Torsion properties of Ceresa cycles of cyclic Fermat quotients (with Shnidman)
- The geometric quadratic Chabauty method for higher genus curves (with Coupek, Xiao, Yao)
- The polylogarithmic motivic Chabauty–Kim method for the thrice punctured projective line (with Jarossay, Saettone, Weiss, Zehavi)

Thank you for your attention!