L-Series and Arithmetic

by David Ter-Borch Gram Schjoldager Lilienfeldt
under the supervision of
Prof. Eva Bayer-Fluckiger, EPFL
Prof. Benedict H. Gross, Harvard University

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Objectives

▶ Acquire the necessary background to understand the statement of Stark's conjecture as well as the motivation behind it.

▶ Study special cases of Stark's conjecture and present the refined conjecture of Stark in the abelian rank one case.

▶ Acquire the necessary background to present the conjecture of Birch and Swinnerton-Dyer.

Historical Overview

▶ 19th century: Proof of the analytic class number formula by Pierre Gustave Lejeune Dirichlet and Richard Dedekind.

▶ 1923 – 1931: Definition of Artin L-functions by Emil Artin and proof of main properties in the attempt to produce a non-abelian class field theory.

▶ 1965: Formulation by Peter Swinnerton-Dyer and Bryan Birch of the Birch and Swinnerton-Dyer Conjecture, a generalization of the class number formula in the context of elliptic curves.

▶ 1970's: Formulation by Harold Stark of Stark's conjecture, a generalization of the class number formula to Artin L-functions. In the rank one abelian case he proposed a refined conjecture with a connection to Hilbert's twelfth problem.

▶ 1986: Formulation by Benedict Gross of a refinement of the Birch and Swinnerton-Dyer Conjecture using ideas of Stark. He proved together with Don Zagier the Gross-Zagier Formula giving important insight in the rank one case.

L-Series and L-Functions

An **L-series** is an infinite series $\sum \frac{a_n}{n^s}$ in the complex variable s, usually convergent in some right half-plane of $\mathbf{C}$. If this function can be extended to all of $\mathbf{C}$ via analytic continuation, then the extended function is called an **L-function**. There are two classes of L-functions in number theory:

▶ **Hecke type L-functions** which enjoy desirable analytic properties: they are defined as an L-series, have an Euler product, are meromorphic on $\mathbf{C}$ and satisfy a functional equation;
· L-functions attached to abelian characters (Dirichlet, Weber and Hecke L-functions),
· L-functions of Hecke forms.

▶ **Artin type L-functions** which encode arithmetic data: they are defined as an Euler product in some right half-plane;
· Dedekind zeta-functions attached to a number field.
· Artin L-functions attached to a finite Galois extension of number fields which encode arithmetic data concerning the splitting of primes.
· Hasse-Weil L-functions attached to an elliptic curve defined over a number field which encode arithmetic data concerning reductions and rational points of the curve.

▷ The interplay between the complex analytic and arithmetic worlds happens when one connects the two above types of L-functions. This is the importance of **reciprocity laws**:

· **Quadratic Reciprocity** connects Dedekind zeta-functions to Dirichlet L-functions.
· **Artin Reciprocity** connects Artin L-functions of 1-dimensional characters to Weber L-functions.
· **The Modularity Theorem** connects Hasse-Weil L-functions of elliptic curves defined over $\mathbf{Q}$ to L-functions of Hecke forms of weight 2.

The Class Number Formula

▶ Denote by ζ_k the Dedekind zeta-function of the number field k.

▶ It satisfies a functional equation centered at $s = 1/2$ and extends meromorphically to $\mathbf{C}$ with a simple pole at $s = 1$.

Theorem (Class Number Formula)**.** *As s tends to 0 we have*

$$\zeta_k(s) \sim -\frac{h_k R_k}{\omega_k} s^{r_1+r_2-1}$$

where:
· r_1 *is the number of embeddings* $k \hookrightarrow \mathbf{R}$;
· r_2 *is the number of pairs of embeddings* $k \hookrightarrow \mathbf{C}$;
· h_k *is the ideal class number of* k;
· ω_k *is the number of roots of unity in* k;
· R_k *is the regulator of* k.

The BSD Conjecture

▶ Let E/k be an elliptic curve defined over a number field k and denote by $L(E/k, s)$ the Hasse-Weil L-function of E over k.

▶ It is conjectured to satisfy a functional equation centered at $s = 1$ and to extend meromorphically to $\mathbf{C}$.

Conjecture (BSD)**.** *$L(E/k, s)$ extends meromorphically to a neighborhood of the point $s = 1$ and as s tends to 1 we have*

$$L(E/k, s) \sim P(E/k)R(E/k)|Ш(k, E)|(s-1)^n$$

where
· *n is the rank of the Mordell-Weil group $E(k)$;*
· *$P(E/k)$ is the global period of E;*
· *$R(E/k)$ is the regulator of E;*
· *$Ш(k, E)$ is the Tate-Shafarevitch group of E.*

Stark's Conjectures

▶ Let K/k be a finite Galois extension of number fields with Galois group G.

▶ Let S be a finite set of places of k containing the archimedean ones and let S_K denote the finite set of places of K above the ones in S.

▶ Let U_{K,S_K} denote the S_K-unit group of K and let X_{K,S_K} denote the degree zero elements of the free abelian group on S_K.

▶ Let $f : \mathbf{C}X_{K,S_K} \longrightarrow \mathbf{C}U_{K,S_K}$ be a left $\mathbf{C}[G]$-module isomorphism defined over $\mathbf{Q}$.

▶ Let V be a finite-dimensional complex linear representation of G with character χ and denote by $L_S(s, \chi)$ the S-modified Artin L-function of χ.

▶ $L_S(s, \chi)$ extends meromorphically to $\mathbf{C}$ and has a functional equation centered at $s = 1/2$.

▶ We write its Taylor expansion about $s = 0$ as

$$L(s, \chi) = c_S(\chi)s^{r_S(\chi)} + O(s^{r_S(\chi)+1}).$$

▶ One computes that

$$r_S(\chi) = \dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{C}[G]}(V^\vee, \mathbf{C}X_{K,S_K}).$$

▶ Denote by $R_S(\chi, f)$ the Stark regulator associated to χ and f.

▶ Define

$$A_S(\chi, f) = \frac{R_S(\chi, f)}{c_S(\chi)} \in \mathbf{C}.$$

Conjecture (Stark)**.** *With the above notations, for all $\alpha \in \mathrm{Aut}(\mathbf{C})$ we have*

$$A_S(\chi, f)^\alpha = A_S(\chi^\alpha, f).$$

✓ Independence of the choice of f
✓ Independence of the choice of S
✓ Trivial character
✓ The case $r_S(\chi) = 0$

The Rank One Abelian Stark Conjecture

▶ Suppose that K/k is abelian.

▶ Suppose that S satisfies the following:

- S contains all archimedean places of k as well as all finite places that ramify in K
- S contains at least one place that splits completely in K
- $|S| \geq 2$.

Conjecture ($\mathrm{St}(K/k, S)$)**.** *There exists an S_K-unit ϵ called a Stark Unit such that $K(\epsilon^{1/\omega_K})/k$ is abelian and for all irreducible characters χ of G we have*

$$L'_S(0, \chi) = -\frac{1}{\omega_K} \sum_{\sigma \in G} \chi(\sigma) \log |\epsilon^\sigma|_w$$

where w is any place above v.

✓ Independence of the choice of v
✓ S contains two places that split
✓ $\mathrm{St}(k/k, S)$
✓ S contains two complex archimedean places
✓ $\mathrm{St}(K/k, S)$ for $k = \mathbf{Q}$ or k an imaginary quadratic field.