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Differential equations on the Sierpinski Gasket

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Abstract

A recent discover allows to define differential operators on topological spaces that have a highly non classical structure. The aim of this project is to introduce (in an elementary way) the Laplace operator on the Sierpinski Triangle and study some of its properties. The approach will basically be the same as in [1] but the structure of our paper will slightly differ. We will also expose an algorithm that we produced to compute harmonic functions on the Gasket.

Contents

1 Preliminary

In this introductory section we shortly expose our approach to the problem, we introduce the Sierpinski Gasket by giving an easy construction algorithm and we introduce some notations and explain the concept of addresses.

1.1 Resume

The goal of this paper is to define a Laplacian on the Sierpinski Gasket. In order to do this we start by first studying some elementary properties of the Gasket in the first part. We introduce and explain concepts and notions such as self-similar measures and graph energy of functions. By the end of the first part we shall have defined harmonic functions on the Gasket which are key objects in this paper. This allows us to introduce the Laplacian on the Gasket in the second part. Here we develop some of the main properties of this differential operator. Finally, we open up to some applications of this theory.

1.2 Construction of the Sierpinski Gasket and notations

In order to begin our study of the Sierpinski Gasket and its properties we need to know what we are working with. We describe a small algorithm that generates the Gasket :
Consider an equilateral triangle in the plane with its interior. Take the midpoints of each segment and connect them. These lines define a smaller triangle which is "upside-down" compared to the original one. Remove this new triangle from the original one. We now have three smaller copies of the original triangle and a "hole" in the middle where the removed triangle used to be. Iterate this process on each of the smaller copies. Let T_m denote the figure at level m, then

$$SG = \bigcap_{m \in \mathbb{N}} T_m.$$

From now on we shall write SG for the Sierpinski Gasket to make the notations shorter.
SG is not necessarily constructed from an equilateral triangle. In fact, our algorithm works for any nondegenerate triangle in the plane, but we will only work on the symmetric version of SG.

We will sometimes work on the graph of SG. Let Γ_m denote the graph of level m with $m \in \mathbb{N}$. Let q_0, q_1 and q_2 denote respectively the lower left point, the lower right point and the upper point of SG. We define $V_0 = \{q_0,\ q_1,\ q_2\}$ and we call this set the boundary of SG. Two small neighboring triangles of T_m intersect at exactly one point. We put all such points and V_0 in a set that we call V_m. We let E_m denote the set of edges of the small triangles in T_m. We then write $\Gamma_m = (V_m,\ E_m)$. It is easy to see that $\#V_m = \frac{1}{2}(3^{m+1} + 3)$. If $\mathcal{T}_m$ denotes the set of small triangular cells contained in the graph of level m, then $\#\mathcal{T}_m = 3^m$.
Furthermore, the graph can be embedded in SG with V_* a dense subset. We write

$$\Gamma = \lim_{m \to \infty} \Gamma_m,$$

and let V_* be the set of vertices of Γ. We will work on SG by analogy with $I = [0, 1]$. Both I and SG are compact and from now on we will use K to denote either I or SG.

1.3 Addresses

To find our way around in SG we introduce the notion of address. To identify the cells of SG, we define three applications $F_i : SG \to SG$, for $i = 0,\ 1,\ 2$. If we consider the graph at level

1, then F_0SG, F_1SG and F_2SG correspond respectively to the lower left cell, the lower right cell and the upper cell. We generalize this notation to any cell of SG at any level m by using *words* w of length m. The notation then becomes $F_wSG = F_{w_1...w_m}SG = (F_{w_m} \circ ... \circ F_{w_1})SG$.

If x is a point in $V_m \setminus V_0$, we write $x = F_w q_i$, where $i = 0$, 1 or 2 and w is a word of length m. This notation is called the address of x. Note that the word w determines the cell of level m in which x lies (namely F_wSG) and q_i indicates whether x is the lower left vertex of, the lower right vertex or the upper vertex of this cell. But this address is not unique. In fact, we can also write $x = F_{w'}q_j$. If $w_m \neq i$, then $w_k = w'_k$ for all $k < m$, $w'_m = i$ and $w_m = j$. If $w_m = i$, then $x \in V_{m-1}$ and we reason by induction. We look at w_{m-1} and if $w_{m-1} = i$ we look at w_{m-2}. Since $x \in V_m \setminus V_0$, there exists $n < m$ an index of w such that $w_n \neq i$. Then, as seen before, $w'_k = w_k$ for all $k < n$, $w'_n = i$, $w_n = j$ and $w'_l = w_l = i$ for all $n < l \leq m$. The address of q_i ($i = 0$, 1, 2) is uniquely determined at each level m and is : $q_i = F_i^m q_i$. Notice that q_i has exactly 2 neighbors in V_m and that all x in $V_m \setminus V_0$ have exactly 4 neighbors in V_m. This is true for all $m \in \mathbb{N}$.

2 Measure and Energy on SG

2.1 Self-similar measures and integrals on SG

In this section we will define what self-similar measures are (particularly the uniform measure) and will state some of their interesting properties. The main purpose of defining measures is that to any measure is associated an integral, and that we will need one when we will define the Laplacian in section 3.

Definition 2.1. *We call a function μ : Cells(SG)[1] $\to \mathbb{R}$ a regular probability measure on SG if it satisfies the following conditions for any cell C :*

(i) $\mu(C) > 0$, *(positivity)*

(ii) If we have $C = \cup_{i=1}^N C_i$ with the C_i "almost" disjoints (i.e two cells can only intersect on a single point), then $\mu(C) = \sum_{i=1}^N \mu(C_i)$, *(additivity)*

(iii) $\mu(C) \to 0$ *when C gets smaller,* *(continuity)*

(iv) $\mu(SG) = 1$. *(probability)*

Notice that we can extend μ to the set of finite unions of almost disjoint cells, and keep the property that μ is a regular probability measure : if A is finite union of almost disjoint C_i, just define

$$\mu(A) \doteq \sum_i \mu(C_i)$$

and all four (extended) conditions are verified.

Moreover, instead of additivity, we can only require that

$$\mu(F_wSG) = \sum_i \mu(F_wF_iSG)$$

i.e. the additivity for the decomposition of a triangle into three smaller triangles of the next level.

[1]We use this notation for the set of cells of SG, i.e. for the set of subsets of SG which are similar to SG (actually it is equal to the set of $F_w(SG)$ where w is a word of $\{0, 1, 2\}$ of arbitrary length)

A particular case of measure in SG is what is called the *uniform measure* :

$$\mu(C) = \left(\frac{1}{3}\right)^m$$

for each cell C of level m. We can now make a more general definition :

Definition 2.2. *A self-similar measure on SG is a regular probability measure μ such that (if we use the notations $\mu(F_i SG) = \mu_i$ for $i = 0, 1, 2$) we have*

$$\mu(F_w SG) = \prod_{j=1}^{|w|} \mu_{w_j}$$

which implies the more visual statement that, when we apply F_i to a set $A \subset SG$, we multiply its measure by μ_i.

We also see that if μ is a self-similar measure, since A can be written as the almost disjoint union of $A \cap F_i SG$, we have

$$\mu(A) = \sum_i \mu(A \cap F_i SG).$$

This leads to

Theorem 2.1. *(Self-similar identity)*

$$\mu(A) = \sum_i \mu_i \mu(F_i^{-1} A).$$

Definition 2.3. *Let μ be a self-similar measure on SG and $f : SG \to \mathbb{R}$ be any continuous function. We then define the integral of f on SG as*

$$\int_{SG} f \, d\mu = \lim_{m \to \infty} \sum_{|w|=m} f(x_w) \mu(F_w SG)$$

where x_w is any point of $F_w SG$.

We must explain this definition a little. First, it is not hard to show that the limit always exist. Moreover, it does not matter which point of $F_w SG$ we pick because since f is continuous f tends to be constant on $F_w SG$ as m grows. We also notice that actually the term in the limit is a weighted average of f. We will often use the uniform measure because computations will be in general a lot easier with it than with the other self-similar measures. With it, in the integral, all values of $f(x_w)$ are given the same weight.
We now give a simple but useful result.

Theorem 2.2. *Let $f : SG \to \mathbb{R}$ be any continuous function, μ a self-similar measure on SG. Then*

$$\int_{SG} f \, d\mu = \sum_i \mu_i \int_{SG} f \circ F_i = \sum_i \int_{F_i SG} f \, d\mu$$

The proof uses the self-similar identity.

2.2 Graph energy and harmonic extension of a function

Definition 2.4. *Given a function $f : V_m \to \mathbb{R}$, we define the graph energy of f on Γ_m as the quantity*

$$E_m(f) = \sum_{x \underset{m}{\sim} y} (f(x) - f(y))^2$$

where $x \underset{m}{\sim} y$ stands for "x and y are neighbors in Γ_m".

We will say that $\tilde{f} : V_{m+1} \to \mathbb{R}$ extends harmonically f if $\tilde{f}$ if an extension of f (i.e. $\tilde{f}|_{V_m} = f$) and if $E_{m+1}(g) \geq E_{m+1}(\tilde{f})$ for any extension g of f on V_{m+1}.

2.2.1 Harmonic extension from V_0 to V_1

As an example, we give the harmonic extension of any function $f : V_0 \to \mathbb{R}$. Such a function consists of three real values , say a, b and c, which correspond to $f(q_0)$, $f(q_1)$ and $f(q_2)$. We want to find x, y and z three values for $\tilde{f}$ of the vertices in $V_1 \setminus V_0$ that minimize the graph energy of $\tilde{f}$ (see Figure 1 below).

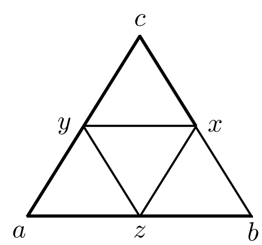

Figure 1: Values of f on V_0, and values of its extension on V_1.

After a computation we obtain a linear system where x, y, z are unknown (this linear system is detailed in "Properties of harmonic extension" below). We eventually find the $\frac{2}{5} - \frac{1}{5}$ rule : the value for $\tilde{f}$ at a certain vertex v of $V_1 \setminus V_0$ is given by the formula

$$\frac{2}{5}(n_1 + n_2) + \frac{1}{5}o$$

where n_1 and n_2 are the values of f on the V_0-neighbors[2] of v and o is the value of f on the remaining vertex of V_0 (the opposite one).

First, it is obvious that extending harmonically a function is a linear transformation, since it is just multiple applications of the $\frac{2}{5} - \frac{1}{5}$ rule. We can also state a result that we will use later :

[2]A vertex in $V_1 \setminus V_0$ has two neighbors in V_0 and two neighbors in $V_1 \setminus V_0$.
More generally, a vertex of V_{m+1} which is not in V_m has exactly two neighbors in $V_{m+1} \setminus V_m$ and two neighbors in V_m, whereas a vertex in the graph of level $m+1$ which is also a vertex in the graph of level m has all its four neighbors in $V_{m+1} \setminus V_m$. This statement is visually obvious.

Theorem 2.3. *(Mean value condition) If $f : V_m \to \mathbb{R}$ is a function and $\tilde{f} : V_{m+1} \to \mathbb{R}$ is its harmonic extension then for any point x in $V_{m+1} \setminus V_m$*

$$\sum_{y \underset{m+1}{\sim} x} (\tilde{f}(x) - \tilde{f}(y)) = 0 \tag{1}$$

or in other words $\tilde{f}(x)$ is the average of the values of f on the (four) neighbors of x in V_{m+1}.

Proof. This is actually part of the computation which leads to the $\frac{2}{5}-\frac{1}{5}$ rule. Let f be a function on V_m which takes real values (we will use the notations of figure (1), taking advantadge of SG's self-similarity by considering only a small triangle where the problem $m \rightsquigarrow m+1$ is reduced to a problem $0 \rightsquigarrow 1$).
We want to minimize the energy as a function of x, y and z. We obtain the system

$$\begin{cases} 4x &= b+c+y+z \\ 4y &= a+c+x+z \\ 4z &= a+b+x+y \end{cases}$$

and thus we found that x, y, z are the averages of the function values on the neighbors so our statement follows. $\square$

2.2.2 Generalisation

The method described above for extending harmonically a function f from $V_0 \to \mathbb{R}$ to $V_1 \to \mathbb{R}$ gives us a way for extending f from $V_m \to \mathbb{R}$ to $V_{m+1} \to \mathbb{R}$ (by self-similarity, and localness of energy[3] it is obvious that an algorithm $m \rightsquigarrow m+1$ can be just multiple applications of $0 \rightsquigarrow 1$ to all triangles that are similar to V_1).

We make another statement : for every $\varepsilon > 0$ we can find n an integer such that $h_n : V_n \to \mathbb{R}$(function computed by successive harmonic extensions of f) restricted to any "little triangle" T (i.e. a triangle similar to V_0) differs only of ε from a constant function. Thus the sequence h_n converges to a continuous function h.

From these observations we see immediately that if we are given three values a, b, c for a function $f : V_0 \to \mathbb{R}$, we can obtain a function $h : V_* \to \mathbb{R}$ such that h extends f and $h|_{V_{m+1}}$ extends $h|_{V_m}$ harmonically for any m. Then because V_* is dense in SG (i.e. its closure is SG) we can extend h to SG, so we get $h : SG \to \mathbb{R}$ continuous[4] that minimizes the graph energy at any level m (where values of $h(q_0)$, $h(q_1)$ and $h(q_2)$ are given). We shall call such functions *harmonic functions.*

2.2.3 Harmonic extension algorithm

We have seen previously that given three real numbers a, b and c as values on V_0, a harmonic function $h : SG \to \mathbb{R}$ is uniquely determined. In particular, we can compute the values of h for all points in V_m for any m only from a, b and c, with a simple algorithm :

1. Use the $\frac{1}{5} - \frac{2}{5}$ rule to compute the values of x, y and z (see Figure 1). *($0 \rightsquigarrow 1$)*

[3] "Localness of energy" means here that for any n we have $E_n(f) = \sum_j E_{T_j}(f)$ where $E_{T_j}(f)$ means the energy of f on T_j, and where $\{T_j\}$ is a collection of triangles that covers Γ_m and such that two triangles intersect in at most one point.

[4] The h we obtain is continuous because as we said before on small triangles h tends to be constant.

2. Do the same on small triangles (q_0, F_1q_0, F_2q_0), (q_1, F_0q_1, F_2q_1) and (q_2, F_0q_2, F_1q_2). *(1 ⇝ 2)*

3. Repeat this until you reach level m

This allows us to compute, for instance, the graph of the harmonic extension of level 7 from $f(q_0) = 0$, $f(q_1) = 1$ and $f(q_2) = 8$ (represented in Figure 2 below).

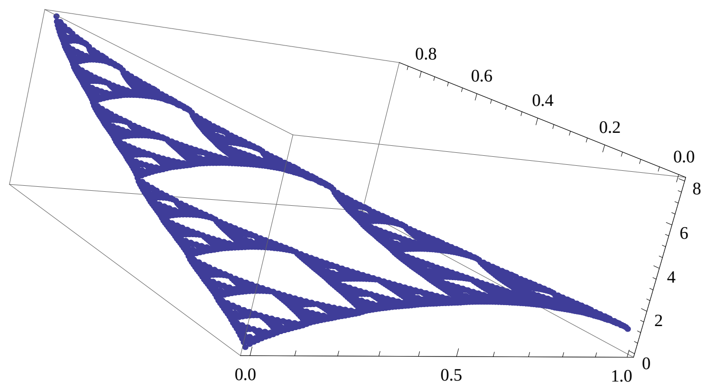

Figure 2: Graph of the harmonic extension of level 7 for initial values 0, 1, 8

2.3 Defining a new energy

We will see in this part that $E_m(f)$ does not behave well, and then we will define a new energy $\mathcal{E}_m(f)$ that is more convenient for us.

If h is harmonic, we obtain (with a bit of calculation) $E_{m+1}(h) = \frac{3}{5}E_m(h)$. In general, we have the inequality $E_{m+1}(f') \geq \frac{3}{5}E_m(f)$ for any f' such that $f'|_{V_m} = f$ (because a harmonic extension minimizes the energy among all possible extensions of a fonction defined at a given level). That is a big problem, because we cannot define a "nice" energy of a map $f : V_* \to \mathbb{R}$ as $\lim\limits_{m\to\infty} E_m(f)$, because this limit is always zero for any harmonic function f. This motivates us to define a "good" graph energy

$$\mathcal{E}_m(f) = \left(\frac{5}{3}\right)^m E_m(f).$$

We see that for any m, if f is harmonic then

$$\begin{aligned}\mathcal{E}_{m+1}(f) &= \left(\frac{5}{3}\right)^{m+1} E_{m+1}(f)\\ &= \left(\frac{5}{3}\right)^{m} E_{m}(f)\\ &= \mathcal{E}_m(f).\end{aligned}$$

So for a harmonic function the graph energy is the same at any level (and more generally $\mathcal{E}_{m+1}(f) \geq \mathcal{E}_m(f)$)
This leads us to

Definition 2.5. *For $f : SG \to \mathbb{R}$ we define the energy of f as*

$$\begin{aligned}\mathcal{E}(f) &= \lim_{m\to\infty} \mathcal{E}_m(f) \\ &= \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \sum_{x \underset{m}{\sim} y} (f(x) - f(y))^2\end{aligned}$$

(the energy is defined only if the limit above is finite).

We immediately see that if f is harmonic, $\mathcal{E}(f)$ is defined and is equal to $\mathcal{E}_0(f)$.

We also notice that if f is constant, then $(f(x) - f(y))^2 = 0$ $\forall x, y \in SG$, thus $\mathcal{E}_m(f) = 0$ for any f, and therefore $\mathcal{E}(f) = 0$.
We will now prove the converse ; suppose that $\mathcal{E}(f) = 0$. We know that

$$\mathcal{E}(f) = \lim_{m\to\infty} \mathcal{E}_m(f|_{V_m}) \geq 0$$

and that $\mathcal{E}_m(f)$ is increasing as m grows. So we must have $\mathcal{E}_m(f) = 0$ for any m. So if $a = f(p_0)$, $b = f(p_1)$ and $c = f(p_2)$ (where the p_i are the vertices of a triangle T of level m) then

$$2(a - b)^2 + 2(a - c)^2 + 2(b - c)^2 = 0$$

(we used localness of energy to "restrict" the energy to T) so $a = b = c$ and therefore f is constant on T.

2.4 Functions with finite energy

We know that in I, if $f : I \to \mathbb{R}$ is a function for which the Laplacian exists then

$$\int_0^1 f'(x)^2\, dx \ < \ \infty$$

or in other terms a function $f : I \to \mathbb{R}$ which has a Laplacian has finite energy. Since we proceed by analogy, this motivates us to move our interest towards functions with finite energy on SG.

Definition 2.6. *We will use the notation* $\operatorname{dom} \mathcal{E} = \{f : SG \to \mathbb{R} \mid \mathcal{E}(f) < \infty\}$

We have $\mathcal{E}(\alpha f) = \alpha^2 \mathcal{E}(f)$ (immediate from the definition). We then want to define the energy of two functions ; we proceed by analogy with the way of defining the inner product in $\mathbb{R}^n$ from the euclidean norm.

Definition 2.7. *The energy of a function at level m is a quadratic form. We define its associated bilinear form (which we shall call the energy of f and g at level m) by*

$$\mathcal{E}_m(f, g) = \frac{1}{4}(\mathcal{E}_m(f + g) - \mathcal{E}_m(f - g)).$$

It turns out that

$$\mathcal{E}_m(f, g) = \left(\frac{5}{3}\right)^m \sum_{x \underset{m}{\sim} y} (f(x) - f(y))(g(x) - g(y)).$$

We notice that $\mathcal{E}(f, g)$ is a bilinear product on $\operatorname{dom} \mathcal{E}$.

We now come to a result which will prove to be useful later :

Lemma 2.4. *Let f, g be two functions $V_m \to \mathbb{R}$, $\tilde{f}$ the harmonic extension of f, g' any extension of g on V_{m+1} (possibly not harmonic). Then we have*

$$\mathcal{E}_{m+1}(\tilde{f}, g') = \mathcal{E}_m(f, g).$$

Proof. We notice first that

$$\mathcal{E}_{m+1}(\tilde{f}, \tilde{g}) = \mathcal{E}_m(f, g)$$

where $\tilde{g}$ is the harmonic extension of g.
Indeed, by definition (and because harmonic extension is a linear transformation)

$$\begin{aligned}\mathcal{E}_{m+1}(\tilde{f}, \tilde{g}) &= \frac{1}{4}(\mathcal{E}_{m+1}(\widetilde{f+g}) - \mathcal{E}_{m+1}(\widetilde{f-g})) \\ &= \frac{1}{4}(\mathcal{E}_m(f+g) - \mathcal{E}_m(f-g)) \\ &= \mathcal{E}_m(f, g).\end{aligned}$$

Thus to finish the proof we only need to show that $\mathcal{E}_{m+1}(\tilde{f}, g') = \mathcal{E}_{m+1}(\tilde{f}, \tilde{g})$ i.e. that $\mathcal{E}_{m+1}(\tilde{f}, g^*) = 0$ where $g^* = g' - \tilde{g}$. Note that $g^*|_{V_m} = 0$ (we will say that g^* vanishes on V_m).
We have

$$E_{m+1}(\tilde{f}, g^*) = \sum_{x \underset{m+1}{\sim} y} (\tilde{f}(x) - \tilde{f}(y))(g^*(x) - g^*(y)).$$

We now fix x. We consider the terms of the sum of type $g^*(x)(\tilde{f}(x) - \tilde{f}(y))$. Either x is in V_m (and then $g^*(x) = 0$ because g^* vanishes on V_m so the entire term is zero) or x is in $V_{m+1} \setminus V_m$ and then $\tilde{f}(y) = f(y)$ for all neighbors of x (because they are in V_m and $\tilde{f}$ is an extension of f). So the term is

$$\sum_{y \underset{m+1}{\sim} x} (\tilde{f}(x) - f(y))$$

which is equal to zero by (1). Hence $E_{m+1}(\tilde{f}, g^*) = 0$ and therefore $\mathcal{E}_m(\tilde{f}, g^*) = 0$. $\square$

3 Laplacian

3.1 Weak formulation

Recall. *For all functions u defined on V_*, we say $u \in \operatorname{dom} \mathcal{E}$ if $\mathcal{E}(u) < \infty$. Furthermore, we say $u \in \operatorname{dom}_0 \mathcal{E}$ if $u \in \operatorname{dom} \mathcal{E}$ and u vanishes on the boundary.*

We are now ready to define a Laplacian Δ on both our self-similar spaces, I and SG. In fact, all we need is : the bilinear energy $\mathcal{E}(u, v)$ and a regular probability measure μ. Most of the time we will use the standard self-similar measure μ for simplicity, but it is important to note that we are free to use any (self-similar) measure. We will write the Laplacian Δ_μ to denote its dependence on the chosen measure or simply Δ when using the standard self-similar measure. The Laplacian Δ is then called the *standard* Laplacian.

The idea behind the definition is the integration-by-parts formula. Suppose u and v are two C^2 functions on I and v vanishes at the endpoints (i.e. $v(0) = v(1) = 0$). Then the integration-by-parts formula gives us

$$\int_0^1 u''(x)v(x)dx = -\int_0^1 u'(x)v'(x)dx.$$

The converse also holds. Let u be a C^1 function on I and suppose there exists a continuous function f such that

$$\int_0^1 f(x)v(x)dx = -\int_0^1 u'(x)v'(x)dx \tag{2}$$

for all such v, then $u \in C^2$ and $u'' = f$.
Note that on I, $\mathcal{E}(u,v) = \displaystyle\int_0^1 u'(x)v'(x)dx$, for all u, v in C^1.
We now rewrite (2), with the energy, as

$$\mathcal{E}(u,v) = -\int_0^1 f(x)v(x)dx \tag{3}$$

for all $v \in \mathrm{dom}_0\ \mathcal{E}$. This tells us that $u \in C^2$ and $u'' = f$ if and only if $u \in \mathrm{dom}\ \mathcal{E}$ and (3) holds.
The following definition is valid for $K = I$ as well as SG.

Definition 3.1. *Let $u \in \mathrm{dom}\ \mathcal{E}$ and f be a continuous function. Then $u \in \mathrm{dom}\ \Delta_\mu$ with $\Delta_\mu u = f$ if*

$$\mathcal{E}(u,v) = -\int_K fv d\mu \tag{4}$$

for all $v \in \mathrm{dom}_0\ \mathcal{E}$.

This definition is the weak formulation. In section 3.3 we will deal with the boundary terms and we will then give a more general definition.

In order for our definition to be interesting, it needs to hold for a certain number of cases. At first it is not clear that dom Δ_μ contains non-trivial functions. Of course, $0 \in \mathrm{dom}\ \Delta_\mu$ and $\Delta_\mu 0 = 0$. Also, all constant functions are contained in dom Δ_μ.
This theorem shows that all harmonic functions h are contained in dom Δ_μ and that $\Delta_\mu h = 0$. Note that this is true for all μ.

Theorem 3.1. *If h is harmonic then $h \in \mathrm{dom}\ \Delta_\mu$ and $\Delta_\mu h = 0$. If $u \in \mathrm{dom}\ \Delta_\mu$ and $\Delta_\mu u = 0$ then u is harmonic.*

Proof. Suppose h is harmonic. Then, by lemma 2.4, $\forall v \in \mathrm{dom}_0\ \mathcal{E}$, $\mathcal{E}(h,v) = \mathcal{E}_0(h,v)$. But $\mathcal{E}_0(h,v) = 0$ because v vanishes on the boundary. This gives us

$$0 = \mathcal{E}(h,v) = -\int_K \Delta_\mu hv d\mu$$

$\Rightarrow \Delta_\mu h = 0$ and $h \in \mathrm{dom}\ \Delta_\mu$.
To show the converse, we consider a function u that satisfies $u \in \mathrm{dom}\ \Delta_\mu$ and $\Delta_\mu u = 0$. We make a special choice for v : for some point $x \in V_m \setminus V_0$, we let $\Psi_x^{(m)}$ denote the piecewise harmonic spline that satisfies $\Psi_x^{(m)}(y) = \delta_{xy}$, for all $y \in V_m$. Note that $\Psi_x^{(m)} \in \mathrm{dom}_0\ \mathcal{E}$ since $x \notin V_0$. We have $\mathcal{E}(u, \Psi_x^{(m)}) = 0$ because $\Delta_\mu u = 0$. By lemma 2.4 (here $\Psi_x^{(m)}$ is the one that is harmonic), $\mathcal{E}(u, \Psi_x^{(m)}) = \mathcal{E}_m(u, \Psi_x^{(m)})$. But the equation $\mathcal{E}_m(u, \Psi_x^{(m)}) = 0$ implies that

$$\sum_{y \underset{m}{\sim} x} (u(x) - u(y)) = 0$$

which is exactly the condition that u is harmonic. $\square$

Now that we know dom Δ_μ contains harmonic functions we might try to expand this property to a wider class based on harmonic functions. To do this we may try powers or polynomials of harmonic function. But this does not work for $K = SG$. In fact, we will see that $u \in$ dom $\Delta_\mu \Rightarrow u^2 \notin$ dom Δ_μ for the standard measure on SG.

3.2 Normal derivatives

To formulate our weak definition of the Laplacian we used the integration-by-parts formula assuming v vanished at the endpoints of I. We would like to move towards a wider definition of the Laplacian and therefore our next goal is to remove the condition that v vanishes at the endpoints. Without this assumption, the integration-by-parts formula yields

$$\int_0^1 u''(x)v(x)dx = -\int_0^1 u'(x)v'(x)dx + (u'(1)v(1) - u'(0)v(0)). \tag{5}$$

To restore the symmetry between the boundary points we define the normal derivatives such that $\partial_n u(1) = u'(1)$ and $\partial_n u(0) = -u'(0)$ to measure the rate of change in a direction moving outside of I. With this definition, (5) becomes

$$\mathcal{E}(u, v) = -\int_I (\Delta u) v d\mu + \sum_{\partial I} v \partial_n u \tag{6}$$

where μ is the standard measure. This is actually a special case of the Gauss-Green formula that we shall state in section 3.3.

For $K = I$ the standard Laplacian is a second derivative and the normal derivatives are first derivatives. It is important to observe that at any rate the existence of normal derivatives is weaker than the existence of a Laplacian.

What we are looking for is a formula for $\mathcal{E}(u, v)$ on SG that is similar to (6). We now give a definition of normal derivatives which also covers the case $K = SG$. Here we let r denote the renormalization factor. On SG, $r = \frac{3}{5}$ and on I, $r = \frac{1}{2}$.

Definition 3.2. *Let $x \in V_0$ and u be a continuous function of K. We define*

$$\partial_n u(x) = \lim_{m\to\infty} r^{-m} \sum_{y \underset{m}{\sim} x} (u(x) - u(y)) \tag{7}$$

to be the normal derivative of u at x and we say that $\partial_n u(x)$ exists if the limit exists.

We make the following statement without giving a proof since this one is trivial.

Proposition 3.2. *The normal derivative is a linear operator.*

In other words, if u and v are two functions that both have existing normal derivatives, then $\partial_n(u + v) = \partial_n u + \partial_n v$.

For I the normal derivative is clearly the ordinary first derivative. On SG this is not the case, but we can explicit (7) which then becomes

$$\partial_n u(q_i) = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m (2u(q_i) - u(F_i^m q_{i+1}) - u(F_i^m q_{i-1})).$$

The factor 2 comes form the fact that on the boundary x only has 2 neighbors.

We want to compute the normal derivatives of harmonic functions on SG. Therefore we introduce the notion of eigenvalue-functions. It is convenient to represent the harmonic extension algorithm by the three matrices A_0, A_1 and A_2. Using the "$\frac{2}{5}-\frac{1}{5}$" rule it is easy to see that

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}, \ A_1 = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}, \ A_2 = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of these matrices are respectively 1, $\frac{3}{5}$ and $\frac{1}{5}$. The eigenvectors associated to these values are respectively (for A_2) :

$$h_c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ h_s = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ h_a = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

It is appropriate to speak about eigenfunctions rather than eigenvectors since the vectors h_s, h_a and h_c uniquely determine functions by harmonic extension.
h_s is the symmetric eigenfunction, h_a the anti-symmetric, or skew-symmetric eigenfunction and h_c the constant eigenfunction. For simplicity we will compute the normal derivatives of these three functions at q_2.
For the constant function, $h_c(q_2) = 1$ and $h_c(F_2^m q_0) = h_c(F_2^m q_1) = 1$ for all m, which implies that $2h_c(q_2) - h_c(F_2^m q_0) - h_c(F_2^m q_0) = 0$ for all m. It follows that $\partial h_c(q_2) = 0$. For the skew-symmetric function, $h_a(q_2) = 0$, $h_a(F_2^m q_0) = (\frac{1}{5})^m$ and $h_a(F_2^m q_1) = -(\frac{1}{5})^m$ where we used the "$\frac{2}{5}-\frac{1}{5}$" rule. Using the same argument as for h_c, it follows that $\partial_n h_a(q_2) = 0$. For the symmetric function, $h_s(q_2) = 0$, $h_s(F_2^m q_0) = h_s(F_2^m q_1) = (\frac{3}{5})^m$ and from this it follows that $(\frac{5}{3})^m(2h_s(q_2) - h_s(F_2^m q_0) - h_s(F_2^m q_0)) = -2$ which brings us to $\partial_n h_s(q_2) = -2$.
Notice that we did not need to take the limit to compute these normal derivatives because the sequence is constant. This is true for any harmonic function h by linearity. Indeed,

Theorem 3.3. *If h is harmonic on SG, then*

$$\partial_n h(q_2) = 2h(q_2) - h(q_0) - h(q_1).$$

Proof. The three eigenfunctions h_s, h_a and h_c form a basis of the space of harmonic functions on SG. This implies that

$$\exists \alpha, \beta, \gamma \in \mathbb{R} \text{ such that } h = \alpha h_s + \beta h_a + \gamma h_c.$$

Using the linearity of the normal derivative, we get

$$\partial_n h(q_2) = \alpha \partial_n h_s(q_2) + \beta \partial_n h_a(q_2) + \gamma \partial_n h_c(q_2) = -2\alpha.$$

On the other hand we have

$$2h(q_2) - h(q_1) - h(q_0) = 2\gamma - (\alpha - \beta + \gamma) - (\alpha + \beta + \gamma) = -2\alpha.$$

Thus

$$\partial_n h(q_2) = 2h(q_2) - h(q_0) - h(q_1).$$

□

Note that the definition of the normal derivative can be localized. Let $F_w K$ be any cell of K and let $x = F_w q_i$ be a boundary point. Then $\partial_n u(F_w q_i) = r^{-|w|} \partial_n (u \circ F_w)(q_i)$.

3.3 The Gauss-Green formula

In section 3.2 we saw a particular form of this formula. We now give the full formula.

Theorem 3.4. *Let Ω be a nice (in some sense) open set of $\mathbb{R}^n$ and u, v be two C^2 functions. Then the Gauss-Green formula is*

$$\int_\Omega \nabla u \cdot \nabla v dx = -\int_\Omega (\Delta u) v dx + \int_{\partial\Omega} v(\partial_n u) d\sigma \tag{8}$$

where $d\sigma$ is the surface measure on $\partial\Omega$.

By analogy, we get

Theorem 3.5. *Let $u \in \mathrm{dom}\ \Delta_\mu$ and $v \in \mathrm{dom}\ \mathcal{E}$. Then*

$$\mathcal{E}(u,v) = -\int_{SG} (\Delta_\mu u) v d\mu + \sum_{V_0} v(q)\partial_n u(q).$$

We have now defined the Laplacian on SG without the condition on the boundary.

Another result is easily obtained from the Gauss-Green formula. We expose it in the following theorem.

Theorem 3.6. *Let u, v be defined as in theorem* (3.4). *Then*

$$\int_{SG} (u\Delta_\mu v - v\Delta_\mu u) d\mu = \sum_{V_0} (u\partial_n v - v\partial_n u).$$

The proof is a computation.

3.4 Pointwise formula

Now that we have a clear definition of the Laplacian, we would like to be able to compute the Laplacian of a function at a given point. For this we need the pointwise formula. In this paragraph we will proceed to finding this formula on SG.

Let u be a function in $\mathrm{dom}\ \Delta_\mu$ and v be a function defined on SG that happens to vanish at the boundary. The weak formulation then yields

$$\mathcal{E}(u,v) = -\int_{SG} (\Delta_\mu u) v d\mu. \tag{9}$$

We want a formula for $\Delta_\mu u(x)$ with $x \in V_*$. We make a special choice for the function v by replacing it with the piecewise harmonic spline $\Psi_x^{(m)}$ where x is a point in $V_m \setminus V_0$. As seen in the proof of theorem 3.1, $\Psi_x^{(m)}(y) = \delta_{xy}\ \forall y \in V_m \setminus x$ and $\Psi_x^{(m)} \in \mathrm{dom}_0\ \mathcal{E}$.

By lemma 2.4,

$$\begin{aligned}\mathcal{E}(u, \Psi_x^{(m)}) &= \mathcal{E}_m(u, \Psi_x^{(m)}) \\ &= \left(\frac{5}{3}\right)^m \sum_{y \underset{m}{\sim} y'} (u(y) - u(y'))(\Psi_x^{(m)}(y) - \Psi_x^{(m)}(y')) \\ &= \left(\frac{5}{3}\right)^m \sum_{y \underset{m}{\sim} x} (u(x) - u(y)) \\ &= \left(\frac{5}{3}\right)^m (4u(x) - \sum_{y \underset{m}{\sim} x} u(y)) \\ &= -\left(\frac{5}{3}\right)^m \Delta_m u(x).\end{aligned}$$

Let f be a continuous function such that $f := \Delta_\mu u$. We rewrite (9), where $v = \Psi_x^{(m)}$, as

$$\left(\frac{5}{3}\right)^m \Delta_m u(x) = \int_{SG} f \Psi_x^{(m)} d\mu.$$

We now divide both sides of this equation by $\displaystyle\int_{SG} \Psi_x^{(m)} d\mu$. This is a non-zero constant so it is legal. This operation yields the equation

$$\frac{(\frac{5}{3})^m \Delta_m u(x)}{\int_{SG} \Psi_x^{(m)} d\mu} = \frac{\int_{SG} f \Psi_x^{(m)} d\mu}{\int_{SG} \Psi_x^{(m)} d\mu}. \tag{10}$$

Let σ_m be the function defined as $\sigma_m := \dfrac{\Psi_x^{(m)}}{\int_{SG} \Psi_x^{(m)} d\mu}$. Clearly, $\displaystyle\int_{SG} \sigma_m d\mu = 1$.
In what follows in the construction of the pointwise formula we shall give the main idea without entering in details.
As $m \to \infty$, supp σ_m becomes very "small", f becomes "constant" and we can therefore take it out of the integral. This translates as

$$\lim_{m\to\infty} \int_{SG} f(x)\sigma_m d\mu = \lim_{m\to\infty} f(x) \underbrace{\int_{SG} \sigma_m d\mu}_{=1} = f(x).$$

With this information we rewrite (10) as

$$\Delta_\mu u(x) = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \left(\int_{SG} \Psi_x^{(m)} d\mu\right)^{-1} \Delta_m u(x).$$

These computations also work for the case $I = K$, hence the theorem we now give.

Theorem 3.7. *Let $K = I$ or $K = SG$, x be a point in $V_* \setminus V_0$ and u be a function of* dom Δ_μ. *The pointwise formula is*

$$\Delta_\mu u(x) = \lim_{m\to\infty} r^{-m} \left(\int_{SG} \Psi_x^{(m)} d\mu\right)^{-1} \Delta_m u(x).$$

If we consider the standard measure on K it is easy to compute the factor $(\int_{SG} \Psi_x^{(m)} d\mu)^{-1}$ exactly.
If $K = I$, $\Psi_x^{(m)}$ is simply a tent function. So $\displaystyle\int_{SG} \Psi_x^{(m)} d\mu$ is simply the area of a triangle of heigth 1 and basis $2.\frac{1}{2^m}$. It follows immediately that $\left(\displaystyle\int_{SG} \Psi_x^{(m)} d\mu\right)^{-1} = 2^m$. Furthermore, $r^{-m} = 2^m$. So the pointwise formula turns out to be

$$\Delta_\mu u(x) = \lim_{m\to\infty} \frac{u(x+\frac{1}{2^m}) - 2u(x) + u(x-\frac{1}{2^m})}{(\frac{1}{2^m})^2} = u''(x).$$

If $K = SG$, $\Psi_x^{(m)}$ is supported on to cells of level m. If $F_w K$ is one of these cells with vertices x, y and z then $\Psi_x^{(m)} + \Psi_y^{(m)} + \Psi_z^{(m)}$ restricted to $F_w K$ is identically 1 (it is harmonic and takes on the value 1 at all three vertices). Thus

$$\int_{F_w K} (\Psi_x^{(m)} + \Psi_y^{(m)} + \Psi_z^{(m)}) d\mu = \mu(F_w K) = \frac{1}{3^m}.$$

By symmetry, all three summands have equal integrals, so $\displaystyle\int_{F_w K} \Psi_x^{(m)} = \frac{1}{3^{m+1}}$. Together with the contribution of the other m-cell, we get

$$\int_{SG} \Psi_x^{(m)} = \frac{2}{3^{m+1}}$$

and the pointwise formula becomes

$$\Delta_\mu u(x) = \frac{3}{2} \lim_{m\to\infty} 5^m \Delta_m u(x).$$

A pointwise formula can also be computed with a non-standard self-similar measure, but it then depends on x.

3.5 Local behavior of functions

In this part we will only work on SG. We will study the local behavior of harmonic functions.

Definition 3.3. *Let $f : SG \to \mathbb{R}$ be a function defined on SG. The oscillation of f on a cell $K_m = F_w SG$ (w is a word of length m) is*

$$Osc_{K_m}(f) = \max_{x\in K_m} f(x) - \min_{x\in K_m} f(x).$$

This next lemma will soon be useful.

Lemma 3.8. *Let f, g be two functions defined on SG. Then*

(i) $\max\ (f) + \min\ (g) \le \max\ (f+g) \le \max\ (f) + \max\ (g)$,

(ii) $\min\ (f) + \min\ (g) \le \min\ (f+g) \le \min\ (f) + \max\ (g)$.

Proof. We will only prove (i) since the proof of (ii) is similar.

$$\begin{aligned}\max\,(f+g) &= \max\,\{f(x)+g(x) : x \in SG\}\\ &\geq \max\,\{f(x)+\min\,(g) : x \in SG\}\\ &= \max\,(f)+\min\,(g).\end{aligned}$$

The second inequality is proved using the same method. $\square$

We now give a result concerning the local behavior of harmonic functions that will prove to be important for the next steps of our study.

Lemma 3.9. *Let h be a harmonic function and consider the m-level cell $F_2^m SG$.*

(i) If $\partial_n h(q_2) \neq 0$, then $Osc_{F_2^m SG}(h)$ is a $\Theta\left(\left(\frac{3}{5}\right)^m\right)$.

(ii) If $\partial_n h(q_2) = 0$, then $Osc_{F_2^m SG}(h)$ is a $O\left(\left(\frac{1}{5}\right)^m\right)$.

Proof. Let h be a harmonic function and consider the eigenfunctions h_s, h_a and h_c.
As seen before, we have

$$\exists \alpha, \beta, \gamma \in \mathbb{R} \text{ such that } h = \alpha h_s + \beta h_a + \gamma h_c.$$

Let us have a closer look at the eigenfunctions, and more precisely at the oscillation of these functions on the given cell $F_2^m SG$.

- The symmetric eigenfunction :
  Notice that $\max\limits_{F_2^m SG} h_s = \left(\frac{3}{5}\right)^m$ and $\min\limits_{F_2^m SG} h_s = 0$. Hence, $\mathrm{Osc}_{F_2^m SG}(h_s) = \left(\frac{3}{5}\right)^m$.

- The skew-symmetric eigenfunction :
  Notice that $\max\limits_{F_2^m SG} h_a = \left(\frac{1}{5}\right)^m$ and $\min\limits_{F_2^m SG} h_a = -\left(\frac{1}{5}\right)^m$. Hence, $\mathrm{Osc}_{F_2^m SG}(h_a) = 2\left(\frac{1}{5}\right)^m$.

- The constant eigenfunction :
  Notice that $\max\limits_{F_2^m SG} h_c = 1$ and $\min\limits_{F_2^m SG} h_c = 1$. Hence, $\mathrm{Osc}_{F_2^m SG}(h_c) = 0$.

By using lemma (3.8) we know that

$$\begin{aligned}\mathrm{Osc}_{F_2^m SG}(h) \leq |\alpha|\,\mathrm{Osc}_{F_2^m SG}(h_s) + |\beta|\,\mathrm{Osc}_{F_2^m SG}(h_a) + |\gamma|\,\mathrm{Osc}_{F_2^m SG}(h_c)\\ \Rightarrow \mathrm{Osc}_{F_2^m SG}(h) \leq |\alpha|\left(\frac{3}{5}\right)^m + 2\,|\beta|\left(\frac{1}{5}\right)^m.\end{aligned}$$

By using lemma (3.8) again, we also get

$$\mathrm{Osc}_{F_2^m SG}(h) \geq |\alpha|\left(\frac{3}{5}\right)^m - 2\,|\beta|\left(\frac{1}{5}\right)^m.$$

This tells us that :
– if $\alpha \neq 0$, then $\mathrm{Osc}_{F_2^m SG}(h)$ is a $\Theta\left(\left(\frac{3}{5}\right)^m\right)$,
– if $\alpha = 0$, then $\mathrm{Osc}_{F_2^m SG}(h)$ is a $O\left(\left(\frac{1}{5}\right)^m\right)$.
Yet, we have shown previously that $\partial_n h(q_2) = -2\alpha$ and thus the lemma has been proved. $\square$

We will now broaden lemma (3.9) to the class of functions in dom Δ, but the second result will be slightly weaker than previously. We give this lemma without proof.

Lemma 3.10. *Let $u \in$ dom Δ.*

(i) If $\partial_n u(q_2) \neq 0$, then $Osc_{F_2^m SG}(u)$ is a $\Theta\left(\left(\frac{3}{5}\right)^m\right)$,

(ii) If $\partial_n u(q_2) = 0$, then $Osc_{F_2^m SG}(u)$ is a $O\left(m\left(\frac{1}{5}\right)^m\right)$.

With these tools we are now able to show the following result.

Theorem 3.11. *If u, $v \in$ dom Δ and u, v are nonconstant, then $uv \notin$ dom Δ.*

Proof. Since u, v are nonconstant functions in dom Δ, there exists a point $x_0 \in V_*$ such that $\partial_n(uv)(x_0) \neq 0$. By using lemma (2) we treat different cases:

$\partial_n u$	$\partial_n v$	Oscillation	decay rate compared to decay rate of $\Theta\left(\left(\frac{3}{5}\right)^m\right)$
$\neq 0$	$\neq 0$	$\Theta\left(\left(\frac{3}{5}\right)^{2m}\right)$	too fast
$= 0$	$\neq 0$	$O\left(m\left(\frac{3}{25}\right)^m\right)$	too fast
$= 0$	$= 0$	$O\left(m^2\left(\frac{1}{25}\right)^m\right)$	way too fast

This proves that Osc(uv) is not a $\Theta\left(\left(\frac{3}{5}\right)^m\right)$. By lemma (2), this shows that $uv \notin$ dom Δ. $\square$

The next result is a direct consequence of this theorem.

Corollary 3.12. *If $u \in$ dom Δ and u is nonconstant, then $u^2 \notin$ dom Δ.*

Proof. Write $v \doteq u$ and apply theorem (3.11). $\square$

Furthermore, we have

Theorem 3.13. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonlinear differentiable function and u be a nonconstant function in* dom Δ. *Then $f(u)$ is not in* dom Δ.

Proof. Without loss of generality we let $f(0) = 0$ (proving the statement for $f(x)$ is the same as proving the statement for $g(x) = f(x) - f(0)$). To prove the theorem we use the Taylor expansion of f in a close neighborhood of 0 up to the second order : there exists constants α, β and a function g such that

$$f(t) = \alpha t + \beta t^2 + g(t).$$

Pick a point x_0 in K_m such that $u(x_0) = 0$ and where K_m is an m-level cell of SG. The goal is to see the local behavior of Osc$(f \circ u)$ on this cell. Using the Taylor expansion above, we write

$$f(u(x)) = \alpha u(x) + \beta u^2(x) + g(u(x)).$$

Notice that $\alpha u(x)$ is in dom Δ and that $(f \circ u)(x_0) = 0$.
Let y_1 and y_2 denote the two neighbors of x_0 in K_m. We now compute the normal derivative

of $f \circ u$ at x_0.

$$\begin{aligned}
\partial_n(f\circ u)(x_0) &= \lim_{m\to\infty}\left(\frac{5}{3}\right)^m (2\overbrace{(f\circ u)(x_0)}^{=0} - (f\circ u)(y_1) - (f\circ u)(y_2)) \\
&= \lim_{m\to\infty}\left(\frac{5}{3}\right)^m \left(\alpha(-u(y_1)-u(y_2)) + \underbrace{\beta(-u^2(y_1)-u^2(y_2)) - g(u(y_1)) - g(u(y_2))}_{\to\ 0 \text{ faster than the first term}}\right) \\
&= \alpha\partial_n u(x_0).
\end{aligned}$$

The last equality comes from the fact that $u(x_0) = 0$.
From this computation we can conclude that

$$\partial_n u(x_0) \neq 0 \Longleftrightarrow \partial_n(f\circ u)(x_0) \neq 0.$$

Recall that u is nonconstant, so $\partial_n u(x_0) \neq 0$. Since $u \in \text{dom}\,\Delta$, by lemma (3.10), $\text{Osc}_{K_m}(u)$ is a $\Theta\left(\left(\frac{3}{5}\right)^m\right)$. Hence, $\text{Osc}_{K_m}(f\circ u)$ is also a $\Theta\left(\left(\frac{3}{5}\right)^m\right)$. But,

$$\begin{aligned}
\text{Osc}_{K_m}(f\circ u) &\leq |(f\circ u)(x) - (f\circ u)(x_0)| \\
&\leq \left|\beta u^2(x) + g(u(x))\right| \\
&\leq \beta C_1\left(\frac{3}{5}\right)^{2m} + C_2\left(\frac{3}{5}\right)^{3m} \\
&\Rightarrow \text{Osc}_{K_m}(f\circ u) \text{ is a } O\left(\left(\frac{3}{5}\right)^{2m}\right)
\end{aligned}$$

This cannot be possible unless $f(u) \notin \text{dom}\,\Delta$. $\square$

Actually this theorem has quite negative consequences. If $\{\rho_i\}$ is a partition of unity and f is a functon we want to study and ρ_i and f both have Laplacians, then $f\circ\rho_i$ does not have a Laplacian. Therefore, we cannot use partitions of unity for partial differential equations on SG as we would usually do on manifolds.

3.6 Gluing functions on SG

Suppose we have three continuous functions $f_0 : F_0SG \to \mathbb{R}$, $f_1 : F_1SG \to \mathbb{R}$ and $f_2 : F_2SG \to \mathbb{R}$. If we know that (we will use the notations in 1) $f_1(x) = f_2(x)$, $f_0(y) = f_2(y)$ and $f_0(z) = f_1(z)$ (i.e. matching of the f_i on the three intersection points), we can "glue" them to form another function $f : SG \to \mathbb{R}$. We then know that f is continuous (by a basic result in topology called the gluing lemma). We notice that if f_i are of finite energy, then f also has finite energy. We then state without proof

Theorem 3.14. *Under the hypothesis from above, and if we suppose that the f_i have Laplacian with the extra condition that $\partial_n f_i(p) + \partial_n f_j(p) = 0$ for any intersection point p (i.e the normal derivatives sum to 0 on intersection points), then the function g obtained by gluing the Δf_i is equal to Δf.*

4 Examples of differential equations on SG

In this section we will introduce the Green's function and expose the Dirichlet's problem on SG.

Let U be an open subset of $\mathbb{R}^n$ with some conditions on ∂U (smooth or piecewise smooth). The Dirichlet's problem is the following system:

$$\begin{cases} -\Delta u & = & f \text{ on } U \\ u & \equiv & 0 \text{ on } \partial U. \end{cases}$$

Here $f : U \to \mathbb{R}$ is a given function with some conditions.
We do not prove the following result.

Theorem 4.1. *The homogeneous Dirichlet's problem admits a unique solution.*

This unique solution is given by

$$u(x) = \int_U G(x, y) f(y) dy,$$

where G denotes the Green's function. This is also true for the problem on SG.

Furthermore, the inhomogeneous Dirichlet's problem $-\Delta_\mu u = f,\ u(q_i) = a_i,\ i = 0,\ 1,\ 2,$ has a unique solution

$$u(x) = \int_U G(x, y) f(y) d\mu(y) + h(x),$$

where h is the harmonic function satisfying $h(q_i) = a_i$.

Theorem 4.2. *If $u \in \mathrm{dom}_0\ \mathcal{E}$ and u minimizes the energy*

$$\frac{1}{2}\mathcal{E}(u, u) + \int_{SG} f u d\mu,$$

then u solves the Dirichlet's problem.

Conclusion :

We have successfully defined a Laplacian on SG and studied its main properties. Nevertheless, it is important to note that many other differential operators exist and that these can be compared with our Laplacian. What we have developed in this paper can be used to solve differential equations on SG (including the heat equation and other diffusion problems). But the tools that we usually use on manifolds are not all available in our case. For example, we cannot work with partitions of unity, as we have seen.

References

[1] Robert S. Strichartz. *Differential equations on fractals.* Princeton university press, 2006.