TWISTED TRIPLE PRODUCT ROOT NUMBERS AND A CYCLE OF DARMON–ROTGER

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Abstract. We consider an algebraic cycle on the triple product of the prime level modular curve $X_0(p)$ with origins in work of Darmon and Rotger. It is defined over the quadratic extension of $\mathbb{Q}$ ramified only at p whose associated quadratic character χ is the Legendre symbol at p. We prove that it is null-homologous and describe actions of various groups on it. For any three normalised cuspidal eigenforms f_1, f_2, f_3 of weight 2 and level $\Gamma_0(p)$, we prove that the global root number of the twisted triple product L-function $L(f_1 \otimes f_2 \otimes f_3 \otimes \chi, s)$ is -1. Assuming conjectures of Beilinson and Bloch, and guided by the Gross–Zagier philosophy, this suggests that the Darmon–Rotger cycle could be non-torsion, although we do not currently have a proof of this.

1. Introduction

1.1. **The Darmon–Rotger cycle.** Let p be a prime number and fix a p-th root of unity ζ_p in $\bar{\mathbb{Q}}$. Let $K \subset \mathbb{Q}(\zeta_p)$ denote the unique quadratic extension of $\mathbb{Q}$ ramified only at p, and write $\mathrm{Gal}(K/\mathbb{Q}) = \{1, \tau\}$. Given an elliptic curve E and three pairwise distinct subgroups C_1, C_2, C_3 of order p with generators P_1, P_2, P_3, denote by e_p the Weil pairing on $E[p]$ and write

$$e_p(P_2, P_3) = \zeta_p^a, \qquad e_p(P_3, P_1) = \zeta_p^b, \qquad e_p(P_1, P_2) = \zeta_p^c.$$

Using this notation, define the invariant

$$o(E; C_1, C_2, C_3) := abc \in \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^{(2)},$$

where $(\mathbb{F}_p^\times)^{(2)}$ denotes the group of non-zero quadratic residues modulo p. This invariant does not depend on the choices of generators of the cyclic subgroups. Darmon–Rotger [4] defined two curves $\Delta_+, \Delta_- \subset X_0(p)^3_{\mathbb{Q}}$ respectively as the schematic closures of

$$\{(E, C_1), (E, C_2), (E, C_3) \colon o(E; C_1, C_2, C_3) = 1\}$$

and

$$\{(E, C_1), (E, C_2), (E, C_3) \colon o(E; C_1, C_2, C_3) \neq 1\}.$$

We refer to the difference of these two curves as the Darmon–Rotger cycle and prove the following:

Theorem 1.1. *The Darmon–Rotger cycle is null-homologous and its rational equivalence class gives rise to an element $\Xi := \Delta_+ - \Delta_-$ in the Chow group $\mathrm{CH}^2(X_0(p)^3_K)_0^{\tau=-1}$. The element Ξ is fixed by the action of the symmetric group S_3 if $p \equiv 1 \pmod 4$. If $p \equiv 3 \pmod 4$, then S_3 acts on Ξ by the sign character.*

In order to prove this, we give an alternative description of Δ_+ and Δ_- as images of certain maps $X(p) \longrightarrow X_0(p)^3$, where $X(p)$ denotes the modular curve of full level p structure over $\mathbb{Q}(\zeta_p)$.

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1.2. **Root numbers.** Consider $F := f_1 \otimes f_2 \otimes f_3 \in S_2(\Gamma_0(p))^{\otimes 3}$, the tensor product of three normalised cuspidal eigenforms with respect to the Hecke algebra $\mathbb{T} = \mathrm{End}_{\mathbb{Q}}(J_0(p)) \otimes \mathbb{Q}$. Let χ denote the Legendre symbol at p, which is the quadratic character associated to the extension $K/\mathbb{Q}$. We consider the triple product L-function $L(F, \chi, s)$ attached to the compatible family of 8-dimensional ℓ-adic $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-representations

$$V_\ell(F, \chi) := V_\ell(f_1) \otimes V_\ell(f_2) \otimes V_\ell(f_3) \otimes \chi,$$

where $V_\ell(f_i)$, for $i = 1, 2, 3$, is the usual 2-dimensional representation attached to a cuspform of weight 2. Define the completed L-function

$$\Lambda^*(F, \chi, s) := 2^4 p^{4s} (2\pi)^{3-4s} \Gamma(s-1)^3 \Gamma(s) L(F, \chi, s).$$

This function has analytic continuation to the entire s-plane and satisfies the functional equation

$$\Lambda^*(F, \chi, s) = W(F, \chi) \Lambda^*(F, \chi, 4-s),$$

where $W(F, \chi) \in \{\pm 1\}$ is the global root number [9, 12]. We prove the following:

Theorem 1.2. *The L-function $L(F, \chi, s)$ vanishes to odd order at its center $s = 2$.*

The proof consists in showing that $W(F, \chi) = -1$, which boils down to computing the epsilon factor of the 8-dimensional Weil–Deligne representation of $F \otimes \chi$ at p. The main difference with the untwisted case lies in the fact that the epsilon factor of this Weil–Deligne representation at p is equal to the epsilon factor of the Weil representation of $F \otimes \chi$ at p (see (17) and Remark 4.4).

Triple product L-functions for three newforms of weight 2 and same square-free level were studied in detail by Gross–Kudla [6]. In particular, they gave a formula for the global root number. In the case of prime level p, the global root number of $L(F, s)$ is $W(F) = a_p(f_1) a_p(f_2) a_p(f_3)$, where $a_p(f_i)$ denotes the p-th Fourier coefficient of f_i at the cusp ∞. The twisted case considered in this note corresponds to the triple product of three newforms of levels p, p, p^2 (see Remark 4.1), and thus falls outside the scope of [6].

In the case of a single normalised cuspidal eigenform $f \in S_2(\Gamma_0(p))$, the global root numbers are $W(f) = a_p(f)$ and $W(f, \chi) = -\chi(-1)$ [11, 13].

1.3. **The Beilinson–Bloch–Kato conjectures.** Let Y be a smooth and proper variety over a number field F. Taken together, the Beilinson–Bloch and Bloch–Kato conjectures [1, 2] predict that for each integer $i \geq 0$ and prime ℓ, the ℓ-adic étale Abel–Jacobi map

$$\mathrm{AJ}_\ell^i \colon \mathrm{CH}^i(Y)_0 \otimes \mathbb{Q}_\ell \longrightarrow H^1_f(F, H^{2i-1}_{\mathrm{et}}(Y_{\bar{F}}, \mathbb{Q}_\ell(i))) \tag{1}$$

from the null-homologous Chow group to the Bloch–Kato Selmer group is an isomorphism of $\mathbb{Q}_\ell$-vector spaces and moreover these spaces have dimension $\mathrm{ord}_{s=i} L(H^{2i-1}(Y), s)$, the L-function being the one attached to the compatible system of ℓ-adic $\mathrm{Gal}(\bar{F}/F)$-representations $H^{2i-1}_{\mathrm{et}}(Y_{\bar{F}}, \mathbb{Q}_\ell)$. The Beilinson–Bloch–Kato conjectures are compatible with algebraic correspondences.

In the setting of this note, we expect from the Beilinson–Bloch–Kato conjectures that

$$\mathrm{ord}_{s=2} L(F, \chi, s) = \dim_{K_F} t_F \, \mathrm{CH}^2(X_0(p)^3_K)_0^{\tau=-1} = \dim_{K_{F,\ell}} H^1_f(\mathbb{Q}, V_{F,\chi,\ell}(2)), \tag{2}$$

where $K_{F,\ell}$ is the completion of the Hecke field K_F of F at a fixed prime above ℓ, and the correspondence $t_F \in \mathrm{CH}^3(X_0(p)^6)_{K_F}$ is some choice of K_F-linear combination of tensor products of Hecke correspondences projecting to the 1-dimensional F-isotypic component of the $(\mathbb{T}^{\otimes 3} \otimes \mathbb{R})$-module $H^0(X^3, \Omega^3_{X^3}) \otimes \mathbb{R} = H^0(X, \Omega^1_X)^{\otimes 3} \otimes \mathbb{R}$. Theorem 1.2 implies that

$$\mathrm{ord}_{s=2} L(F, \chi, s) \geq 1. \tag{3}$$

In view of the Beilinson–Bloch–Kato conjectures, we thus expect to have the lower bound

$$\dim_{K_F} t_F \,\mathrm{CH}^2(X_0(p)^3_K)_0^{\tau=-1} \geq 1. \tag{4}$$

A natural question arises from (4) and Theorem 1.1: when are $t_F\Xi$ and $\mathrm{AJ}^2_\ell(t_F\Xi)$ non-trivial? It would be interesting to know the answer to this, especially given the canonical nature of the Darmon–Rotger cycle (it does not depend on a choice of base-point nor on a projector to make it null-homologous, as opposed to the modified diagonal cycle in Remark 1.3 below). At the moment, we have no answers to offer, only speculations.

In analogy with the situation for Heegner points [8], the Gross–Zagier philosophy might lead us to expect that

$$t_F\Xi \neq 0 \iff \mathrm{ord}_{s=2} L(F,\chi,s) = 1. \tag{5}$$

In particular, the existence of one triple product of normalised cuspidal eigenforms such that $L'(F,\chi,2) \neq 0$ would be enough to imply that Ξ has infinite order in $\mathrm{CH}^2(X_0(p)^3_K)_0$.

Remark 1.3. Over $\mathbb{Q}$, a naturally occurring cycle is the modified diagonal cycle

$$\Delta_{\mathrm{GKS}}(e) := P_{\mathrm{GKS}}(e)(\Delta) \in \mathrm{CH}^2(X_0(p)^3)_0$$

first considered in work of Gross–Kudla [6] and Gross–Schoen [7]. Here $e \in X_0(p)(\mathbb{Q})$ is a fixed base-point and $P_{\mathrm{GKS}}(e)$ is the Gross–Schoen projector (Definition 2.5) whose effect is to make the small diagonal $\Delta \subset X_0(p)^3$ null-homologous. When $\mathrm{ord}_{s=2} L(F,s)$ is even, $\mathrm{AJ}^2_\ell(t_F\Delta_{\mathrm{GKS}}(e))$ is trivial for all ℓ and base-point e [10] (hence $t_F\Delta_{\mathrm{GKS}}(e)$ is trivial assuming injectivity of (1)). When $\mathrm{ord}_{s=2} L(F,s) = 1$, $t_F\Delta_{\mathrm{GKS}}(\infty)$ is non-trivial by the height formula conjectured in [6] and announced in [15]. If in addition we have $\mathrm{ord}_{s=2} L(F,\chi,s) = 1$, then $\mathrm{ord}_{s=2} L(F/K,s) = 2$, and it is conceivable in view of (5) that

$$t_F \,\mathrm{CH}^2(X_0(p)^3_K)_0 = K_F \cdot t_F\Delta_{\mathrm{GKS}}(\infty) \oplus K_F \cdot t_F\Xi,$$

although this is pure speculation at the moment.

1.4. **Conventions.** By default, all Chow groups are with $\mathbb{Q}$-coefficients. All number fields are viewed as embedded in a fixed algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. Moreover, we fix a complex embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, as well as p-adic embeddings $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ for each rational prime p.

1.5. **Outline.** Section 2 describes the Darmon–Rotger cycle and contains the proof of Theorem 1.1. Section 3 provides the necessary background on Weil–Deligne representations and local factors needed for the proof of Theorem 1.2, which is given in Section 4.

2. Triple product cycles

We give a description of the Darmon–Rotger cycle using certain maps $X(p) \longrightarrow X_1(p)^3 \longrightarrow X_0(p)^3$, where $X_1(p)$ is the modular curve with $\Gamma_1(p)$-level structure. We use this description to prove Theorem 1.1.

2.1. **Cycles on $X_1(p)^3$.** Throughout this section we will assume that $X_0(p)$ has positive genus. Let $\bar{M}_p$ denote the fine moduli scheme representing isomorphisms classes of pairs (E,α_p) consisting of a generalised elliptic curve E together with a full level p structure $\alpha_p : E[p] \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})^2$. It is a smooth proper curve over $\mathbb{Q}$, whose base change to $\mathbb{Q}(\zeta_p)$ is the disjoint union of $p-1$ geometrically connected smooth proper curves $X^j(p)$ with $j \in \{1,\ldots,p-1\}$. The curve $X^j(p)$ classifies pairs $(E,(P,Q))$, where (P,Q) is a basis of $E[p]$ satisfying $e_p(P,Q) = \zeta_p^j$. We shall focus on $X(p) := X^1(p)$ (because considering the other components does not yield additional cycles).

Let $x_i = (a_i, b_i) \in \mathbb{F}_p^2 \setminus \{(0,0)\}$ with $i \in \{1,2,3\}$ and consider the map

$$\widetilde{\varphi}_{(x_1,x_2,x_3)} \colon \bar{M}_p \longrightarrow X_1(p)^3$$

defined over $\mathbb{Q}$ by

$$(E,(P,Q)) \mapsto ((E, a_1P + b_1Q), (E, a_2P + b_2Q), (E, a_3P + b_3Q)).$$

Denote by

$$\widetilde{\Delta}_{(x_1,x_2,x_3)} := \widetilde{\varphi}_{(x_1,x_2,x_3)}(X(p)) \in \mathrm{CH}^2(X_1(p)^3_{\mathbb{Q}(\zeta_p)})$$

the image of $X(p)$ under this map. We have a collection

$$\widetilde{\mathcal{C}} := \left\{\widetilde{\Delta}_{(x_1,x_2,x_3)} : (x_1,x_2,x_3) \in (\mathbb{F}_p^2 \setminus \{(0,0)\})^3\right\} \subset \mathrm{CH}^2(X_1(p)^3_{\mathbb{Q}(\zeta_p)})$$

that inherits from $X(p)$ and $X_1(p)^3$ various actions of groups, which we will now define and study.

2.1.1. *Action of the group* $\mathrm{SL}_2(\mathbb{F}_p)$. There is a natural left action of the group $\mathrm{SL}_2(\mathbb{F}_p)$ on $X(p)$, as can be seen, using the moduli interpretation, as follows: if $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{F}_p)$, then

$$\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \cdot (E,(P,Q)) := (E, (\alpha P + \beta Q, \gamma P + \delta Q)).$$

Because the determinant is one, the Weil pairing on the basis is preserved. The above action naturally induces a right action of $\mathrm{SL}_2(\mathbb{F}_p)$ on the set $\widetilde{\mathcal{C}}$ via

$$\widetilde{\Delta}_{x_1,x_2,x_3} \cdot \kappa := \widetilde{\varphi}_{(x_1,x_2,x_3)} \circ \kappa(X(p)),$$

but since $\mathrm{SL}_2(\mathbb{F}_p)$ acts by automorphisms this action is the trivial one. A quick calculation reveals that

$$\widetilde{\Delta}_{(x_1,x_2,x_3)} \cdot \kappa = \widetilde{\Delta}_{(x_1,x_2,x_3)\cdot\kappa}$$

where the right action of $\mathrm{SL}_2(\mathbb{F}_p)$ on the set $(\mathbb{F}_p^2 \setminus \{(0,0)\})^3$ is defined as follows. Let $\kappa = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{F}_p)$ and $(x_1,x_2,x_3) \in (\mathbb{F}_p^2 \setminus \{(0,0)\})^3$ with $x_i = (a_i, b_i)$, $i = 1,2,3$, then write the vector (x_1,x_2,x_3) as a 3×2 matrix and multiply on the right by κ:

$$\begin{aligned}(x_1,x_2,x_3)\cdot\kappa :&= \begin{pmatrix}a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3\end{pmatrix}\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \\ &= ((a_1\alpha + b_1\gamma, a_1\beta + b_1\delta), (a_2\alpha + b_2\gamma, a_2\beta + b_2\delta), (a_3\alpha + b_3\gamma, a_3\beta + b_3\delta)).\end{aligned}$$

It follows that the indexing set of the cycles can be taken to be

$$\widetilde{I} := (\mathbb{F}_p^2 \setminus \{(0,0)\})^3/\mathrm{SL}_2(\mathbb{F}_p).$$

We shall write $[x_1,x_2,x_3]$ for the image of (x_1,x_2,x_3) in $\widetilde{I}$. To understand the set $\widetilde{I}$ we introduce a determinant map

$$\mathrm{Det} : \widetilde{I} \longrightarrow (\mathbb{F}_p)^3$$

defined as follows. If (x_1,x_2,x_3) is a representative of a class in $\widetilde{I}$ with $x_i = (a_i,b_i)$ for $i \in \{1,2,3\}$, then

$$\mathrm{Det}([x_1,x_2,x_3]) := \left(\begin{vmatrix}a_2 & b_2 \\ a_3 & b_3\end{vmatrix}, \begin{vmatrix}a_3 & b_3 \\ a_1 & b_1\end{vmatrix}, \begin{vmatrix}a_1 & b_1 \\ a_2 & b_2\end{vmatrix}\right).$$

This map is well-defined as follows from the definition of the action of $\mathrm{SL}_2(\mathbb{F}_p)$. It is a bijection from the subset $\widetilde{I}^\times := \mathrm{Det}^{-1}((\mathbb{F}_p^\times)^3) \subset \widetilde{I}$ to $(\mathbb{F}_p^\times)^3$. If $[x_1,x_2,x_3] \in \widetilde{I}^\times$ with $\mathrm{Det}([x_1,x_2,x_3]) = (a,b,c)$, then we write $\widetilde{\Delta}_{a,b,c} := \widetilde{\Delta}_{(x_1,x_2,x_3)}$. We restrict our attention to the subcollection

$$\widetilde{\mathcal{C}}^\times := \left\{\widetilde{\Delta}_{a,b,c} : (a,b,c) \in (\mathbb{F}_p^\times)^3\right\} \subset \mathrm{CH}^2(X_1(p)^3_{\mathbb{Q}(\zeta_p)}).$$

2.1.2. *Action of the diamond operators.* The modular curve $X_1(p)$ carries a natural left action of the group $\mathbb{F}_p^\times$ via the so-called diamond operators. If $d \in \mathbb{F}_p^\times$, then in terms of the modular description we have $\langle d\rangle \cdot (E,P) = (E, dP)$. We get an induced action of $(\mathbb{F}_p^\times)^3$ on the triple product $X_1(p)^3$ described by

$$\langle d_1, d_2, d_3\rangle \cdot ((E_1,P_1),(E_2,P_2),(E_3,P_3)) = ((E_1,d_1P_1),(E_2,d_2P_2),(E_3,d_3P_3)).$$

This in turn induces a left action of $(\mathbb{F}_p^\times)^3$ on the collection of cycles $\widetilde{\mathcal{C}}$ via

$$\langle d_1,d_2,d_3\rangle \cdot \widetilde{\Delta}_{(x_1,x_2,x_3)} := \langle d_1,d_2,d_3\rangle \circ \widetilde{\varphi}_{(x_1,x_2,x_3)}(X(p)),$$

and this action preserves the subcollection $\widetilde{\mathcal{C}}^\times$ since

$$\langle d_1,d_2,d_3\rangle \cdot \widetilde{\Delta}_{a,b,c} = \widetilde{\Delta}_{d_2d_3a,d_1d_3b,d_1d_2c}.$$

The following lemma follows easily from this formula.

Lemma 2.1. *We have*

$$\mathrm{orb}_\diamond(\widetilde{\Delta}_{1,1,1}) = \left\{\widetilde{\Delta}_{a,b,c} \mid a,b,c \in \mathbb{F}_p^\times,\ abc \in (\mathbb{F}_p^\times)^{(2)}\right\}.$$

Here $(\mathbb{F}_p^\times)^{(2)}$ denotes the set of quadratic residues modulo p and thus the orbit of $\widetilde{\Delta}_{1,1,1}$ has size $\frac{(p-1)^3}{2}$. The stabiliser of $\widetilde{\Delta}_{1,1,1}$ for this action is given by $\{\langle 1,1,1\rangle, \langle -1,-1,-1\rangle\}$. As a consequence, there are 2 orbits for the action of the diamond operators on $\widetilde{\mathcal{C}}^\times$:

$$\widetilde{\mathcal{C}}^\times = \mathrm{orb}_\diamond(\widetilde{\Delta}_{1,1,1}) \sqcup \mathrm{orb}_\diamond(\widetilde{\Delta}_{1,1,a}),$$

where $a \in \mathbb{F}_p^\times$ is a choice of a non-quadratic residue modulo p.

2.1.3. *Action of the Galois group* $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. We identify $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ with $\mathbb{F}_p^\times$ so that the element of the Galois group σ_i indexed by $i \in \mathbb{F}_p^\times$ raises ζ_p to the i-th power. Recall that the curve $\bar{M}_p$ is defined over $\mathbb{Q}$. When base-changed to $\mathbb{Q}(\zeta_p)$, the Galois group of $\mathbb{Q}(\zeta_p)$ permutes the $p-1$ connected components $X^j(p)$ of this curve transitively. Using this, $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ acts on $\widetilde{\mathcal{C}}$ via

$$\widetilde{\Delta}^{\sigma_i}_{(x_1,x_2,x_3)} := \widetilde{\varphi}_{(x_1,x_2,x_3)}(\sigma_i(X(p))) = \widetilde{\varphi}_{(x_1,x_2,x_3)}(X^i(p)).$$

It is then not difficult to prove the following:

Lemma 2.2. *For all $i \in \mathbb{F}_p^\times$ and $(a,b,c) \in (\mathbb{F}_p^\times)^3$, we have $\widetilde{\Delta}^{\sigma_i}_{a,b,c} = \widetilde{\Delta}_{ia,ib,ic}$.*

2.2. **Cycles on $X_0(p)^3$.** There is a natural degree $(p-1)/2$ covering of curves $\pi : X_1(p) \longrightarrow X_0(p)$ defined over $\mathbb{Q}$ given by mapping (E,P) to $(E,\langle P\rangle)$. It gives rise to a map on triple products $\pi^3 : X_1(p)^3 \longrightarrow X_0(p)^3$. Define, for $(x_1,x_2,x_3) \in ((\mathbb{F}_p \times \mathbb{F}_p) \setminus \{(0,0)\})^3$, the map

$$\varphi_{(x_1,x_2,x_3)} := \pi^3 \circ \widetilde{\varphi}_{(x_1,x_2,x_3)} : \bar{M}_p \longrightarrow X_0(p)^3,$$

as well as the cycle

$$\Delta_{(x_1,x_2,x_3)} := \varphi_{(x_1,x_2,x_3)}(X(p)) \in \mathrm{CH}^2(X_0(p)^3_{\mathbb{Q}(\zeta_p)}).$$

The cycles $\Delta_{(x_1,x_2,x_3)}$ are invariant under the action of the diamond operators on the triples (x_1,x_2,x_3). By taking images under π^3 of the cycles in $\widetilde{\mathcal{C}}^\times$ indexed by the set $\widetilde{I}^\times$, we obtain a collection of cycles $\mathcal{C}^\times$ in $\mathrm{CH}^2(X_0(p)^3_{\mathbb{Q}(\zeta_p)})$ indexed by the double coset space $(\mathbb{F}_p^\times)^3 \backslash \tilde{I}^\times$, which has cardinality 2 by Lemma 2.1. The 2 cycles are the schematic closures of:

- $\Delta_+ := \pi^3(\widetilde{\Delta}_{1,1,1}) = \{((E,\langle P\rangle),(E,\langle Q\rangle),(E,\langle P+Q\rangle))\}$
- $\Delta_- := \pi^3(\widetilde{\Delta}_{1,1,a}) = \{((E,\langle P\rangle),(E,\langle Q\rangle),(E,\langle aP+Q\rangle))\}$ (a is a non-quadratic residue).

These cycles first appeared in work of Darmon–Rotger [4, p. 30]. It is clear that Δ_+ and Δ_- match the descriptions given in the introduction.

2.2.1. *Field of definition.*

Lemma 2.3. *The cycles Δ_+ and Δ_- are defined over the quadratic field*

$$K := \mathbb{Q}\left(\sqrt{\chi(-1)p}\right) \subset \mathbb{Q}(\zeta_p),$$

where χ denotes the Legendre symbol modulo p. The non-trivial element τ of $\mathrm{Gal}(K/\mathbb{Q})$ interchanges Δ_+ and Δ_-.

Proof. Let $G(\chi)$ denote the Gauss sum associated to χ given by the expression

$$G(\chi) := \sum_{n=0}^{p-1} \zeta_p^{n^2}.$$

The equality $G(\chi)^2 = \chi(-1)p$ goes back to Gauss and implies that K is the quadratic subfield of the cyclotomic field $\mathbb{Q}(\zeta_p)$. We have

$$\sigma_i(G(\chi)) = \sum_{n=0}^{p-1} \zeta_p^{in^2} = G(\chi) \iff i \in (\mathbb{F}_p^\times)^{(2)},$$

and as a consequence $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/K) \simeq (\mathbb{F}_p^\times)^{(2)}$ and $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^{(2)}$. Thus τ acts as σ_a where $a \in \mathbb{F}_p^\times$ is not a square. It follows from Lemmas 2.2 and 2.1 that both cycles in $\mathcal{C}^\times$ are fixed by $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/K)$ and moreover that

$$\Delta_+^\tau = \pi^3(\widetilde{\Delta}_{1,1,1}^\tau) = \pi^3(\widetilde{\Delta}_{a,a,a}) = \pi^3(\widetilde{\Delta}_{1,1,a}) = \Delta_-.$$

□

2.2.2. *Action of the symmetric group S_3.* The symmetric group S_3 acts on $X_0(p)^3$ and $X_1(p)^3$ by permuting the coordinates. This induces a left action of S_3 on the various cycles given by

$$\sigma \cdot \Delta_{(x_1,x_2,x_3)} := \sigma \circ \varphi_{x_1,x_2,x_3}(X(p)) = \Delta_{(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})}.$$

(And similarly for cycles in $\widetilde{\mathcal{C}}$.) Let $[x_1, x_2, x_3] \in \widetilde{I}^\times$ with determinant $(a, b, c) \in (\mathbb{F}_p^\times)^3$. For all $\sigma \in S_3$, observe that

$$\prod_{i=1}^{3} \mathrm{Det}([x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}])_i = \mathrm{sgn}(\sigma)abc,$$

where $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ. The following result now follows from Lemma 2.1.

Proposition 2.4. *If $p \equiv 1 \pmod 4$, then the action of S_3 fixes Δ_+ and Δ_-. If $p \equiv 3 \pmod 4$, then any transposition in S_3 permutes Δ_+ and Δ_-.*

2.3. **Homological triviality.** We give two proofs, the first of which uses the Gross–Schoen projector [7].

Definition 2.5. Let C be a smooth projective geometrically connected curve over a number field k and let e be a k-rational point of Y. For any non-empty subset T of $\{1, 2, 3\}$, let T' denote the complementary set. Write $p_T : C^3 \longrightarrow C^{|T|}$ for the natural projection map and let $q_T(e) : C^{|T|} \longrightarrow C^3$ denote the inclusion obtained by filling in the missing coordinates using the point e. Let $P_T(e)$

denote the graph of the morphism $q_T(e) \circ p_T : C^3 \longrightarrow C^3$ viewed as a codimension 3 cycle on the product $C^3 \times C^3$. Define the Gross–Schoen projector by

$$P_{\mathrm{GKS}}(e) := \sum_T (-1)^{|T'|} P_T(e) \in \mathrm{CH}^3(C^3 \times C^3),$$

where the sum is taken over all subsets of $\{1,2,3\}$. This is an idempotent in the ring of correspondences of C^3 by [7, Proposition 2.3] with the property that it annihilates the cohomology groups $H^i(C^3(\mathbb{C}), \mathbb{Z})$ for $i \in \{4,5,6\}$ and maps $H^3(C^3(\mathbb{C}), \mathbb{Z})$ onto the Künneth summand $H^1(C(\mathbb{C}), \mathbb{Z})^{\otimes 3}$ by [7, Corollary 2.6].

Consider the Gross–Schoen projector on $X_0(p)^3$, with base-point some rational point $e \in X_0(p)(\mathbb{Q})$. This idempotent correspondence acts on cohomology and annihilates $H^4(X_0(p)^3(\mathbb{C}), \mathbb{Z})$, the target of the Betti cycle class map $\mathrm{cl}^2_B \colon \mathrm{CH}^2(X_0(p)^3_{\mathbb{Q}}) \longrightarrow H^4(X_0(p)^3(\mathbb{C}), \mathbb{Z})$. Hence, for any cycle $Z \in \mathrm{CH}^2(X_0(p)^3_{\mathbb{Q}})$, the cycle $P_{\mathrm{GKS}}(e)_*(Z)$ is null-homologous and belongs to $\mathrm{CH}^2(X_0(p)^3_{\mathbb{Q}})_0$.

Theorem 2.6. *The Darmon–Rotger cycle $\Xi = \Delta_+ - \Delta_-$ satisfies the equality $\Xi = P_{\mathrm{GKS}}(e)_*(\Xi)$ for any base-point $e \in X_0(\mathbb{Q})$. In particular, it is null-homologous.*

We will use the following lemma.

Lemma 2.7. *Let $i < j \in \{1,2,3\}$ and denote by $\mathrm{pr}_{ij} : X_0(p)^3 \longrightarrow X_0(p)^2$ the natural projection to the product of the i-th and j-th components. There exist elements $[x_1,x_2,x_3]$ and $[y_1,y_2,y_3]$ of $\widetilde{I}^\times$ satisfying*

$$\prod_{k=1}^{3} \mathrm{Det}([x_1,x_2,x_3])_k \in (\mathbb{F}_p^\times)^{(2)} \qquad \textit{and} \qquad \prod_{k=1}^{3} \mathrm{Det}([y_1,y_2,y_3])_k \notin (\mathbb{F}_p^\times)^{(2)},$$

and such that there is an equality $\mathrm{pr}_{ij} \circ \varphi_{(x_1,x_2,x_3)} = \mathrm{pr}_{ij} \circ \varphi_{(y_1,y_2,y_3)}$ of maps $X(p) \longrightarrow X_0(p)^2$.

Proof. Fix some $a \notin (\mathbb{F}_p^\times)^{(2)}$.
If $i = 1$ and $j = 2$, then we may take

$$(x_1,x_2,x_3) = ((1,0),(0,1),(-1,-1)) \qquad \text{and} \qquad (y_1,y_2,y_3) = ((1,0),(0,1),(-a,-1)).$$

If $i = 1$ and $j = 3$, then we may take

$$(x_1,x_2,x_3) = ((-1,0),(1,-1),(0,1)) \qquad \text{and} \qquad (y_1,y_2,y_3) = ((-1,0),(a,-1),(0,1)).$$

If $i = 2$ and $j = 3$, then we may take

$$(x_1,x_2,x_3) = ((-1,-1),(1,0),(0,1)) \qquad \text{and} \qquad (y_1,y_2,y_3) = ((-1,-a),(1,0),(0,1)).$$

□

Remark 2.8. The maps $\varphi_{(x_1,x_2,x_3)}$ and $\varphi_{(y_1,y_2,y_3)}$ associated with the specific choices made in the above proof will be denoted $\varphi_+(ij)$ and $\varphi_-(ij) = \varphi_-(ij;a)$ respectively.

Proof of Theorem 2.6. Observe that

$$P_{\mathrm{GKS}}(e)_*(\Xi) = \Xi - P_{12}(e)_*(\Xi) - P_{13}(e)_*(\Xi) - P_{23}(e)_*(\Xi) + P_1(e)_*(\Xi) + P_2(e)_*(\Xi) + P_3(e)_*(\Xi).$$

Let $i < j \in \{1,2,3\}$ and consider $P_{ij}(e)_*(\Xi)$. Let $k \in \{1,2,3\}$ be the remaining element distinct from i and j. The correspondence $P_{ij}(e)$ is the graph of the function

$$q_{ij}(e) \circ \mathrm{pr}_{ij} : X_0(p)^3 \longrightarrow X_0(p)^3,$$

which replaces the k-th coordinate by the point e, and $P_{ij}(e)_*(\Xi)$ is the image of Ξ under $q_{ij}(e)\circ\mathrm{pr}_{ij}$. Choose $[x_1, x_2, x_3]$ and $[y_1, y_2, y_3]$ of $\widetilde{I}^\times$ satisfying the properties of Lemma 2.7 for the fixed i and j. The first condition ensures that

$$\varphi_{(x_1,x_2,x_3)}(X(p)) = \Delta_+ \quad \text{and} \quad \varphi_{(y_1,y_2,y_3)}(X(p)) = \Delta_-,$$

while the second condition implies that

$$P_{ij}(e)_*(\Delta_+) = q_{ij}(e)\circ \mathrm{pr}_{ij}\circ\varphi_{(x_1,x_2,x_3)}(X(p)) = q_{ij}(e)\circ \mathrm{pr}_{ij}\circ\varphi_{(y_1,y_2,y_3)}(X(p)) = P_{ij}(e)_*(\Delta_-).$$

As a consequence, we have $P_{ij}(e)_*(\Xi) = 0$.

Let $i \in \{1,2,3\}$ and consider $P_i(e)_*(\Xi)$. Let $j,k \in \{1,2,3\}$ such that $\{i,j,k\} = \{1,2,3\}$. The correspondence $P_i(e)$ is the graph of the map $q_i(e)\circ\mathrm{pr}_i : X_0(p)^3 \longrightarrow X_0(p)^3$, which replaces the j-th and k-th coordinates by the point e, and $P_i(e)_*(\Xi)$ is the image of Ξ under $q_i(e)\circ\mathrm{pr}_i$. This map can be written as the composition

$$q_i(e)\circ\mathrm{pr}_i = (q_{ik}(e)\circ\mathrm{pr}_{ik})\circ(q_{ij}(e)\circ\mathrm{pr}_{ij}),$$

hence in terms of correspondences we have $P_i(e) = P_{ik}(e)\circ P_{ij}(e)$. It follows from the previous paragraph that $P_i(e)_*(\Xi) = 0$.

We conclude that $\Xi = P_{\mathrm{GKS}}(e)_*(\Xi)$. □

Remark 2.9. A perhaps more direct way to see that the cycle Ξ is null-homologous is to consider its image under the de Rham cycle class map, namely

$$\mathrm{cl}^2_{\mathrm{dR}}(\Xi) = \mathrm{cl}^2_{\mathrm{dR}}(\Delta_+) - \mathrm{cl}^2_{\mathrm{dR}}(\Delta_-) \in H^4_{\mathrm{dR}}(X_0(p)^3/\mathbb{C}),$$

where we recall that

$$\int_{X_0(p)(\mathbb{C})^3} \mathrm{cl}^2_{\mathrm{dR}}(\Delta_\pm)\wedge\alpha = \int_{\Delta_\pm}\alpha, \qquad \text{for all } \alpha\in H^2_{\mathrm{dR}}(X_0(p)^3/\mathbb{C}).$$

By the Künneth decomposition for $H^2_{\mathrm{dR}}(X_0(p)^3/\mathbb{C})$, any component of α can at most involve de Rham classes coming from two of the three components of $X_0(p)^3$; indeed, the components are either of the form $\mathrm{pr}_i^*(\beta)$ for some $\beta\in H^2_{\mathrm{dR}}(X_0(p)/\mathbb{C})$ and $i\in\{1,2,3\}$, or of the form $\mathrm{pr}_j^*(\gamma)\wedge\mathrm{pr}_k^*(\delta)$ for some $\gamma,\delta\in H^1_{\mathrm{dR}}(X_0(p)/\mathbb{C})$ and $j<k\in\{1,2,3\}$. Using the notations of Remark 2.8, observe that

$$\int_{\Delta_\pm}\mathrm{pr}_i^*(\beta) = \int_{X(p)}(\mathrm{pr}_i\circ\varphi_\pm(ij))^*(\beta)$$
$$\int_{\Delta_\pm}\mathrm{pr}_j^*(\gamma)\wedge\mathrm{pr}_k^*(\delta) = \int_{X(p)}(\mathrm{pr}_{jk}\circ\varphi_\pm(jk))^*(\gamma\wedge\delta).$$

Since

$$\mathrm{pr}_i\circ\varphi_+(ij) = \mathrm{pr}_i\circ\varphi_-(ij) : X(p)\longrightarrow X_0(p)$$
$$\mathrm{pr}_{jk}\circ\varphi_+(jk) = \mathrm{pr}_{jk}\circ\varphi_-(jk) : X(p)\longrightarrow X_0(p)^2,$$

this implies that $\mathrm{cl}^2_{\mathrm{dR}}(\Delta_+) = \mathrm{cl}^2_{\mathrm{dR}}(\Delta_-)$ in $H^4_{\mathrm{dR}}(X_0(p)^3/\mathbb{C})$.

3. Weil–Deligne representations and local factors

This section provides background material on Weil–Deligne representations and epsilon factors following [5, 13].

3.1. **Weil–Deligne representations.** Let q denote a prime number. The embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_q$ fixed in Section 1.4 realises $\mathrm{Gal}(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)$ as the decomposition subgroup at q of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. It sits in the short exact sequence

$$1 \longrightarrow I_q \longrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}_q/\mathbb{Q}_q) \xrightarrow{r} \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 1$$

where I_q denotes the inertia subgroup at q and r denotes the natural reduction map. The group $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is topologically generated by the Frobenius automorphism $\phi : x \mapsto x^q$ and is isomorphic to the profinite completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$. We denote by φ the inverse of the Frobenius automorphism ϕ. The Weil group at q, denoted $W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)$, is defined as the pre-image under r of the infinite cyclic subgroup of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ generated by ϕ. We endow it with the coarsest topology for which $r : W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q) \longrightarrow \langle\phi\rangle$ and $I_q \hookrightarrow W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)$ are both continuous and for which $W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)$ is a topological group. A representation of the Weil group is a continuous homomorphism of groups

$$\sigma_q : W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q) \longrightarrow \mathrm{GL}(V)$$

where V is a finite-dimensional complex vector space.

Example 3.1. Examples of Weil representations include all finite-dimensional complex representations of Galois groups of finite extensions of $\mathbb{Q}$. Also, we identify all characters of $\mathbb{Q}_q^\times$ with characters of $W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)$ via the Artin isomorphism

$$\mathbb{Q}_q^\times \simeq W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)^{\mathrm{ab}} \tag{6}$$

normalised so that it maps q to the image in $W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)^{\mathrm{ab}}$ of an inverse Frobenius element of $W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)$. Another example of a Weil representation is given by the character

$$\omega_q : W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q) \longrightarrow \mathbb{C}^\times \tag{7}$$

defined by $\omega_q(I_q) = 1$ (i.e., it is unramified) and $\omega_q(\Phi) = q^{-1}$ where Φ is an inverse Frobenius element of $\mathrm{Gal}(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)$ (i.e., an element satisfying $r(\Phi) = \varphi$). Under the isomorphism (6) the character ω_q corresponds to the q-adic norm character $\|\cdot\|_q : \mathbb{Q}_q^\times \longrightarrow \mathbb{C}^\times$ normalised such that $\|q\|_q = q^{-1}$.

Definition 3.2. A Weil–Deligne representation is a pair $\sigma'_q = (\sigma_q, N_q)$ where σ_q is a Weil representation on a finite-dimensional complex vector space V and N_q is a nilpotent endomorphism of V satisfying

$$\sigma_q(g) \circ N_q \circ \sigma_q(g)^{-1} = \omega_q(g) N_q, \qquad \text{for all } g \in W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q). \tag{8}$$

There is also a theory of Weil–Deligne representations at archimedean places, but we will not need it in our calculations.

Example 3.3. Fix an embedding $\iota : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$. Consider the ℓ-adic cyclotomic character

$$\omega_{\mathrm{cyc},\ell} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^\infty})/\mathbb{Q}) \longrightarrow \mathbb{Z}_\ell^\times$$

where ζ_{ℓ^∞} denotes a compatible system $(\zeta_{\ell^n})_n$ of primitive ℓ^n-th roots of unity. If σ is an element in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, then $\sigma(\zeta_{\ell^n}) = \zeta_{\ell^n}^{m_n}$ for some compatible $m_n \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times$ and $\omega_{\mathrm{cyc},\ell}(\sigma) = (m_n)_n \in \mathbb{Z}_\ell^\times$. This character is unramified at q since the extension $\mathbb{Q}(\zeta_{\ell^\infty})$ of $\mathbb{Q}$ is ramified only at ℓ. The Weil–Deligne representation at q of $\omega_{\mathrm{cyc},\ell}$ is then the Weil representation $\iota \circ \omega_{\mathrm{cyc},\ell}|_{W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)}$. If Φ is a geometric Frobenius element of $W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)$, then $\omega_{\mathrm{cyc},\ell}(\Phi) = q^{-1} \in \mathbb{Z}_\ell^\times$ and thus

$$\iota \circ \omega_{\mathrm{cyc},\ell}|_{W(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)} = \omega_q,$$

where ω_q is defined by (7). In particular, the Weil–Deligne representation of $\omega_{\mathrm{cyc},\ell}$ at q is independent of ι and ℓ.

Example 3.4. Let (e_0, e_1) denote the standard basis of $\mathbb{C}^2$. The special representation of the Weil–Deligne group at q of dimension 2, denoted $\mathrm{sp}(2)$, is the representation (σ_q, N) defined by the matrices

$$\sigma_q := \begin{pmatrix} 1 & 0 \\ 0 & \omega_q \end{pmatrix} \qquad \text{and} \qquad N := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is an admissible, indecomposable, reducible 2-dimensional Weil–Deligne representation.

Example 3.5. Let $f = \sum_{n\geq 1} a_n(f)q^n \in S_2(\Gamma_0(p))$ be a normalised cuspidal eigenform. Let $\mathfrak{l}$ denote the prime of K_f above ℓ determined by the field embeddings fixed in Section 1.4. Attached to this data is a 2-dimensional $\mathfrak{l}$-adic Galois representation

$$V_\ell(f) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(K_{f,\mathfrak{l}}). \tag{9}$$

See [3, Theorem 3.1] for the precise definition and properties. Note that we suppressed the dependencies of the various embeddings from the notation, as these have been fixed from the beginning. Let q be a prime different from ℓ and fix an embedding $\iota_\ell : K_{f,\mathfrak{l}} \hookrightarrow \mathbb{C}$. Following [13, §4], one can associate to $V_\ell(f)$ a 2-dimensional Weil–Deligne representation $\sigma'_{f,\ell,\iota_\ell,q} = (\sigma_{f,\ell,\iota_\ell,q}, N_{f,\ell,\iota_\ell,q})$. It turns out that the isomorphism class of the representation $\sigma'_{f,\ell,\iota_\ell,q}$ is independent of ℓ and ι_ℓ and we shall simply write $\sigma'_{f,q} = (\sigma_{f,q}, N_{f,q})$. This is the Weil–Deligne representation of f at q.

Proposition 3.6. *The Weil–Deligne representations of f satisfy the following:*

- *If $q \neq p$, then $N_{f,q} = 0$ and $\sigma_{f,q} \simeq \xi_q \oplus \xi_q^{-1}\omega_q^{-1}$ for some unramified character ξ_q. Here ω_q is the Weil–Deligne representation of the ℓ-adic cyclotomic character defined by (7) and Example 3.3.*
- *Let λ be the unramified quadratic character of $W(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ defined by $\lambda(\Phi) = a_p(f)$, where Φ denotes an inverse Frobenius element. Then $\sigma'_{f,p} \simeq \lambda\omega_q^{-1} \otimes \mathrm{sp}(2)$, so that, in particular, $N_{f,q} \neq 0$ and $\sigma'_{f,q}$ is ramified. Here $\mathrm{sp}(2)$ is the special representation in Example 3.4.*

Proof. Using [3, Theorem 3.1], the proofs in [13, §14, §15] adapt to this setting and give the above descriptions of the Weil–Deligne representations of f. In particular, these are independent of the choices of a prime ℓ and an embedding $\iota_\ell : K_{f,\mathfrak{l}} \hookrightarrow \mathbb{C}$. □

3.2. **Local factors.** Epsilon factors were first introduced by Deligne and their properties are summarised in [5, §5]. We will follow the exposition in [13] to collect the essential properties needed for the purposes of this paper.

If q is a finite place, let $\sigma'_q = (\sigma_q, N_q)$ be a Weil–Deligne representation with associated finite-dimensional complex vector space V. Let $\psi_q : \mathbb{Q}_q \longrightarrow \mathbb{C}^\times$ denote an additive character and let $\mathrm{d}x_q$ denote the choice of a Haar measure on $\mathbb{Q}_p$. The epsilon factor associated to σ'_q depends on ψ_q and $\mathrm{d}x_q$ and is given by

$$\epsilon'(\sigma'_q, \psi_q, \mathrm{d}x_q) := \epsilon(\sigma_q, \psi_q, \mathrm{d}x_q)\delta(\sigma'_q) \in \mathbb{C}^\times, \tag{10}$$

where

$$\delta(\sigma'_q) := \det(-\Phi \mid V^{I_q}/(V^{I_q} \cap \ker N_q)), \tag{11}$$

and $\epsilon(\sigma_q, \psi_q, \mathrm{d}x_q)$ is the epsilon factor of the Weil representation σ_q, which we will now describe.

Explicit formulas for the epsilon factor of a character at a finite place are given as follows. Let μ be a character of $\mathbb{Q}_q^\times$ identified with a 1-dimensional representation of the Weil group via (6). Let $n(\psi_q)$ denote the largest integer n such that ψ_q is trivial on $q^{-n}\mathbb{Z}_q$. Let $a(\mu)$ denote the conductor of μ, i.e., $a(\mu)=0$ if μ is unramified and otherwise $a(\mu)$ is the smallest positive integer m such that μ is trivial on $1+q^m\mathbb{Z}_q$. Then

$$\epsilon(\mu,\psi_q,\mathrm{d}x_q)=\begin{cases}\int_{q^{-(n(\psi_q)+a(\mu))}\mathbb{Z}_q^\times}\mu^{-1}(x)\psi_q(x)\mathrm{d}x_q & \text{if } \mu \text{ is ramified}\\ \mu\omega_q^{-1}(q^{n(\psi_q)})\int_{\mathbb{Z}_q}\mathrm{d}x_q & \text{if } \mu \text{ is unramified.}\end{cases} \tag{12}$$

The epsilon factor of a Weil representation is completely determined by the following result.

Theorem 3.7. *Let k be either $\mathbb{R},\mathbb{C}$ or $\mathbb{Q}_q$ for some finite place q. There is a unique function ϵ, which to any Weil representation σ, any non-trivial additive character $\psi: k\longrightarrow\mathbb{C}^\times$ and any choice of a Haar measure $\mathrm{d}x$ on k, associates a complex number $\epsilon(\sigma,\psi,\mathrm{d}x)\in\mathbb{C}^\times$ satisfying:*

i) $\epsilon(*,\psi,\mathrm{d}x)$ is multiplicative in short exact sequences.

ii) If L/k is any finite extension of k in $\bar{k}$ and σ_L is a Weil representation of L, then for any choice of Haar measure $\mathrm{d}x_L$ on L, we have

$$\epsilon\left(\mathrm{ind}_{W(\bar{k}/L)}^{W(\bar{k}/k)}\sigma_L,\psi,\mathrm{d}x\right)=\epsilon\left(\sigma_L,\psi\circ\mathrm{tr}_{L/k},\mathrm{d}x_L\right)\left(\frac{\epsilon(\mathrm{ind}_{W(\bar{k}/L)}^{W(\bar{k}/k)}1_L,\psi,\mathrm{d}x)}{\epsilon(1_L,\psi\circ\mathrm{tr}_{L/k},\mathrm{d}x_L)}\right)^{\dim\sigma_L}.$$

iii) If $\dim\sigma=1$, then $\epsilon(\sigma,\psi,\mathrm{d}x)$ is given by explicit formulas (formula (12) in the case $k=\mathbb{Q}_q$, and see [5] for the formulas in the archimedean case, which we will not need).

Proof. This is [5, Theorem 4.1]. □

Definition 3.8. Let k be either $\mathbb{R},\mathbb{C}$ or $\mathbb{Q}_q$ for some finite place q. Given a Weil–Deligne representation $\sigma'=(\sigma,N)$ of k, the choice of an additive character $\psi: k\longrightarrow\mathbb{C}^\times$ and a Haar measure $\mathrm{d}x$ on k, we define the root number

$$W(\sigma',\psi):=\frac{\epsilon'(\sigma',\psi,\mathrm{d}x)}{|\epsilon'(\sigma',\psi,\mathrm{d}x)|}.$$

As the notation suggests, the root number is independent of the choice of a Haar measure $\mathrm{d}x$, as can be seen from [13, §11 Proposition (ii)]. Moreover, if the Weil–Deligne representation σ'_q at a finite prime q is essentially symplectic, then the local root number at q is independent of the additive character ψ and belongs to $\{\pm1\}$ by [13, §12]. We shall simply write $W(\sigma'_q)$ in this case.

We end this subsection by collecting a few results concerning epsilon factors of Weil representations at finite places.

Proposition 3.9. *Let σ_q be a Weil representation at a prime q. If μ is an unramified character of $\mathbb{Q}_q^\times$, $\psi:\mathbb{Q}_q\longrightarrow\mathbb{C}^\times$ is a non-trivial additive character, and $\mathrm{d}x$ is Haar measure on $\mathbb{Q}_q$, then*

$$\epsilon(\sigma_q\otimes\mu,\psi,\mathrm{d}x)=\mu(q^{n(\psi)\dim(\sigma_q)+a(\sigma_q)})\epsilon(\sigma_q,\psi,\mathrm{d}x).$$

Here $a(\sigma_q)$ is the conductor of σ_q defined in [13, §10].

Proof. This is [13, §11 Proposition (iii)]. □

The following proposition gives an explicit formula for the epsilon factor of a ramified character of conductor 1. Note that if $\psi:\mathbb{Q}_q\longrightarrow\mathbb{C}^\times$ is an additive character with $n(\psi)=0$, then $\psi|_{\mathbb{Z}_q^\times}=1$ but $\psi|_{q^{-1}\mathbb{Z}_q^\times}\neq1$. Thus there exists $c\in\mathbb{F}_q^\times$ such that $\psi(1/q)=\exp((2\pi ic)/q)$. In this case, we write ψ_c for ψ.

Proposition 3.10. *Let μ be a ramified character of $\mathbb{Q}_q^\times$ identified with a 1-dimensional representation of the Weil group via (6). Let $\psi : \mathbb{Q}_q \longrightarrow \mathbb{C}^\times$ denote an unramified additive character, i.e., $n(\psi) = 0$, and $\mathrm{d}x$ denote the Haar measure on $\mathbb{Q}_p$ such that $\int_{\mathbb{Z}_p} \mathrm{d}x = 1$. Suppose that $a(\mu) = 1$. Let $c \in \mathbb{F}_q^\times$ such that $\psi = \psi_c$. Then the following formula holds:*

$$\epsilon(\mu, \psi, \mathrm{d}x) = \mu(c)\mu(q)G(\mu^{-1})$$

where $G(\mu^{-1}) = \sum_{b \in \mathbb{F}_q^\times} \mu^{-1}(b) e^{\frac{2\pi i b}{q}}$ is the Gauss sum of the character μ^{-1}.

Proof. This can be extracted from the proof of [11, Theorem 3.2 (2)]. □

Corollary 3.11. *With the same notations and assumptions as in Proposition 3.10, we have the formula*

$$\epsilon(\mu, \psi, \mathrm{d}x)\epsilon(\mu^{-1}, \psi, \mathrm{d}x) = q\mu(-1).$$

Proof. Applying the result of Proposition 3.10 to μ and μ^{-1} leads to

$$\epsilon(\mu, \psi, \mathrm{d}x)\epsilon(\mu^{-1}, \psi, \mathrm{d}x) = G(\mu^{-1})G(\mu).$$

By standard properties of Gauss sums, we have $G(\mu^{-1}) = \mu(-1)\overline{G(\mu)}$. Using the fact that $|G(\mu)|^2 = q$, we obtain the desired result. □

4. Triple product root numbers

We work with the modular curve $X_0(p)$ of prime level and assume that the genus of $X_0(p)$ is positive (i.e., $p = 11$ or $p > 13$). Let

$$f_1 = \sum_{n \geq 1} a_n(f_1)q^n, \qquad f_2 = \sum_{n \geq 1} a_n(f_2)q^n, \qquad f_3 = \sum_{n \geq 1} a_n(f_3)q^n$$

be three normalised cuspidal eigenforms of level $\Gamma_0(p)$, and let $F := f_1 \otimes f_2 \otimes f_3$. Recall that χ denotes the Legendre symbol at p. Let $L(F, \chi, s)$ be the L-function associated to the compatible family of 8-dimensional ℓ-adic Galois representations

$$\{V_\ell(F, \chi) = V_\ell(f_1) \otimes V_\ell(f_2) \otimes V_\ell(f_3) \otimes \chi\}_\ell,$$

where the representations $V_\ell(f_i)$ for $i \in \{1, 2, 3\}$ are the ones in (9).

Following the recipe in [14], one may lift χ to a unitary Hecke character $\chi_{\mathbb{A}} : \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times \longrightarrow \mathbb{C}^\times$ by setting $\chi_{\mathbb{A}}(g) = \prod_v \chi_v(g_v)$ where v runs over all places of $\mathbb{Q}$ and

$$\chi_\infty(g_\infty) = \begin{cases} 1 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1, g_\infty > 0 \\ -1 & \text{if } \chi(-1) = -1, g_\infty < 0 \end{cases} \qquad \chi_\ell(g_\ell) = \begin{cases} \chi(\ell)^{\mathrm{ord}_\ell(g_\ell)} & \text{if } \ell \neq p \\ \chi(j)^{-1} & \text{if } g_p \in p^k(j + p\mathbb{Z}_p). \end{cases}$$

The collection of ℓ-adic characters $\{\chi_\ell : \mathbb{Q}_\ell^\times \longrightarrow \mathbb{C}^\times\}_\ell$ is characterised by the following:

- For $\ell \neq p$, χ_ℓ is unramified with $\chi_\ell(\ell) = \left(\frac{\ell}{p}\right)$;
- χ_p is tamely ramified, $\chi_p(p) = 1$, and $\chi_p|_{\mathbb{Z}_p^\times} = \left(\frac{\cdot}{p}\right)$.

The Weil–Deligne representation of $F \otimes \chi$ at a prime q is the 8-dimensional representation

$$\sigma'_{F,\chi,q} = \sigma'_{f_1,q} \otimes \sigma'_{f_2,q} \otimes \sigma'_{f_3,q} \otimes \chi_q,$$

where $\sigma'_{f_i,q}$ is described in Example 3.5. Concretely, we have

$$\sigma'_{F,\chi,q} = (\sigma_{F,\chi,q}, N_{F,\chi,q}) = (\sigma_{f_1,q} \otimes \sigma_{f_2,q} \otimes \sigma_{f_3,q} \otimes \chi_q, N_{f_1,q} \otimes 1 \otimes 1 + 1 \otimes N_{f_2,q} \otimes 1 + 1 \otimes 1 \otimes N_{f_3,q}).$$

Following the recipe in [5], we attach to $F \otimes \chi$ a completed L-function

$$\Lambda(F,\chi,s) := 2^4(2\pi)^{3-4s}\Gamma(s-1)^3\Gamma(s)L(F,\chi,s).$$

(This is based on the fact that the Hodge numbers of $F \otimes \chi$ are given by $h^{3,0} = 1$ and $h^{2,1} = 3$.)

Following general recipes for motives [5], we define the conductor, the global epsilon factor, and the global root number of $F \otimes \chi$. The conductor of $F \otimes \chi$ is defined to be

$$\operatorname{cond}(F,\chi) := \prod_q q^{a(\sigma'_{F,\chi,q})} \in \mathbb{N}, \tag{13}$$

where the product is over all finite places q. Consider $\psi = \prod_v \psi_v : \mathbb{A}_\mathbb{Q}/\mathbb{Q} \longrightarrow \mathbb{C}$ an additive character of the adèles and let $\mathrm{d}x$ denote the normalised Haar measure on the adèles such that $\int_{\mathbb{A}_\mathbb{Q}/\mathbb{Q}} \mathrm{d}x = 1$. It decomposes as a product of local Haar measures $\mathrm{d}x_v$ which satisfy $\int_{\mathbb{Z}_v} \mathrm{d}x_v = 1$ for almost all finite places v. The global epsilon factor of $F \otimes \chi$ is then defined to be

$$\epsilon(F,\chi) := \prod_v \epsilon'(\sigma'_{F,\chi,v}, \psi_v, \mathrm{d}x_v),$$

which is independent of the choice of ψ and $\mathrm{d}x$. Moreover, $\epsilon'(\sigma'_{F,\chi,v}, \psi_v, \mathrm{d}x_v) = 1$ for almost all v (in fact, for all $v \neq \infty, p$ as we will see in the proof of Theorem 4.2 below). The global root number is defined to be

$$W(F,\chi) = \prod_v W(\sigma'_{F,\chi,v}, \psi_v, \mathrm{d}x_v). \tag{14}$$

The completed L-function

$$\Lambda^*(F,\chi,s) := \operatorname{cond}(F,\chi)^{\frac{s}{2}}\Lambda(F,\chi,s) \tag{15}$$

admits analytic continuation to the entire complex plane and satisfies the functional equation

$$\Lambda^*(F,\chi,s) = W(F,\chi)\Lambda^*(F,\chi,4-s). \tag{16}$$

Remark 4.1. Notice that $F \otimes \chi$ is equal to the tensor product of the three newforms f_1, f_2 and $f_3^{(p)}$, where $f_3^{(p)} = f_3 \otimes \chi$. The L-function $\Lambda^*(F,\chi,s)$ is the triple product L-function associated to the triple $(f_1, f_2, f_3^{(p)})$. The first two forms have level $\Gamma_0(p)$ while the form $f_3^{(p)}$ has level $\Gamma_0(p^2)$. Hence, the analytic properties and functional equation of $\Lambda^*(F,\chi,s)$ fall outside the scope of [6] where the case of three newforms of the same square-free level is treated. However, as explained in [9], the analytic properties and functional equation in this case follow from [12].

Theorem 4.2. $W(F,\chi) = -1$.

Proof. The local root number at infinity of $F \otimes \chi$ is the same as the one of F since twisting by finite order characters does not affect Hodge structures. In particular, it is equal to -1 by [6]. We thus focus on the local root numbers at the finite places. For any prime ℓ, we choose an additive character ψ_ℓ with $n(\psi_\ell) = 0$, as well as the Haar measure $\mathrm{d}x_\ell$ normalised such that $\int_{\mathbb{Z}_\ell} \mathrm{d}x_\ell = 1$.

Let q be a prime distinct from p. By Proposition 3.6 we have, for $i \in \{1,2,3\}$,

$$\sigma'_{f_i,q} = \sigma_{f_i,q} = \xi_{i,q} \oplus \xi_{i,q}^{-1}\omega_q^{-1}$$

for some unramified characters $\xi_{i,q}$. The character χ_q is also unramified. We therefore obtain

$$\begin{aligned}\sigma'_{F,\chi,q} = \sigma_{F,\chi,q} = {} & \xi_{1,q}\xi_{2,q}\xi_{3,q}\chi_q \oplus \xi_{1,q}\xi_{2,q}^{-1}\xi_{3,q}\omega_q^{-1}\chi_q \oplus \xi_{1,q}^{-1}\xi_{2,q}\xi_{3,q}\omega_q^{-1}\chi_q \oplus \xi_{1,q}^{-1}\xi_{2,q}^{-1}\xi_{3,q}\omega_q^{-2}\chi_q \\ & \oplus \xi_{1,q}\xi_{2,q}\xi_{3,q}^{-1}\omega_q^{-1}\chi_q \oplus \xi_{1,q}\xi_{2,q}^{-1}\xi_{3,q}^{-1}\omega_q^{-2}\chi_q \oplus \xi_{1,q}^{-1}\xi_{2,q}\xi_{3,q}^{-1}\omega_q^{-2}\chi_q \oplus \xi_{1,q}^{-1}\xi_{2,q}^{-1}\xi_{3,q}^{-1}\omega_q^{-3}\chi_q.\end{aligned}$$

Since all characters involved are unramified, Theorem 3.7 i) and (12) imply, given the choice of ψ_q and $\mathrm{d}x_q$, that $\epsilon'(\sigma'_{F,\chi,q}, \psi_q, \mathrm{d}x_q) = 1$, and in particular $W(\sigma'_{F,\chi,q}) = 1$.

For each $i \in \{1,2,3\}$, let λ_i be the unramified quadratic character of $W(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ defined by $\lambda_i(\Phi) = a_p(f_i)$, where Φ denotes an inverse Frobenius element. We will sometimes view it as a character of $\mathbb{Q}_p^\times$ via the identification (6). Let $\lambda = \lambda_1\lambda_2\lambda_3$ denote the product of these characters. By Proposition 3.6, the Weil–Deligne representation of $F \otimes \chi$ at p is given by

$$\sigma'_{F,\chi,p} = \chi_p \lambda \omega_p^{-3} \otimes \mathrm{sp}(2)^{\otimes 3}.$$

Let V denote the complex vector space associated to it. The character χ_p is tamely ramified, i.e., $a(\chi_p) = 1$. Suppose, by contradiction, that $V^{I_p} \neq 0$. Then there is a non-zero vector $v \in V$ which is fixed by the action of the inertia I_p. But $\sigma_{F,\chi,p}(g)(v) = \chi_p(g)v$ for all $g \in I_p$ since $\sigma_{F,\chi,p} = \sigma_{f_1,p} \otimes \sigma_{f_2,p} \otimes \sigma_{f_3,p} \otimes \chi_p$ and $\sigma_{f_1,p} \otimes \sigma_{f_2,p} \otimes \sigma_{f_3,p}$ is unramified. As $v \in V^{I_p}$, we must have $\chi_p(g)v = v$ which implies that $\chi_p(g) = 1$ since $v \neq 0$. Since this holds for all $g \in I_p$, it contradicts the fact that χ_p is ramified. Hence $V^{I_p} = 0$ and as a consequence $\delta(\sigma'_{F,\chi,p}) = 1$, which implies that

$$\epsilon'(\sigma'_{F\chi,p}, \psi_p, \mathrm{d}x_p) = \epsilon(\sigma_{F,\chi,p}, \psi_p, \mathrm{d}x_p). \tag{17}$$

If (e_0, e_1) denotes the standard basis of $\mathbb{C}^2$, then $\mathrm{sp}(2)$ is the representation (σ_p, N) defined in Example 3.4 by the matrices

$$\sigma_p := \begin{pmatrix} 1 & 0 \\ 0 & \omega_p \end{pmatrix} \qquad \text{and} \qquad N := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let us denote by $V_i = \mathbb{C}^2$ the complex vector space associated to $\sigma'_{f_i,p}$ and by $(e_0^{(i)}, e_1^{(i)})$ its standard basis for each $i \in \{1,2,3\}$. Then $V = V_1 \otimes_{\mathbb{C}} V_2 \otimes_{\mathbb{C}} V_3 = \mathbb{C}^8$ is the space of $\sigma_{M,p}$ and an ordered basis for it is given by

$$\begin{aligned} B := (&e_0^{(1)} \otimes e_0^{(2)} \otimes e_0^{(3)}, e_0^{(1)} \otimes e_0^{(2)} \otimes e_1^{(3)}, e_0^{(1)} \otimes e_1^{(2)} \otimes e_0^{(3)}, e_0^{(1)} \otimes e_1^{(2)} \otimes e_1^{(3)}, \\ &e_1^{(1)} \otimes e_0^{(2)} \otimes e_0^{(3)}, e_1^{(1)} \otimes e_0^{(2)} \otimes e_1^{(3)}, e_1^{(1)} \otimes e_1^{(2)} \otimes e_0^{(3)}, e_1^{(1)} \otimes e_1^{(2)} \otimes e_1^{(3)}). \end{aligned} \tag{18}$$

With respect to the basis B, the representation

$$\mathrm{sp}(2)^{\otimes 3} = (\sigma_p^{\otimes 3}, N^{\otimes 3} := N \otimes 1 \otimes 1 + 1 \otimes N \otimes 1 + 1 \otimes 1 \otimes N)$$

is given by the matrices

$$\sigma_p^{\otimes 3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_p^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_p^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_p^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_p^3 \end{pmatrix} \qquad \text{and} \qquad N^{\otimes 3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

We conclude that

$$\sigma_{F,\chi,p} \simeq \chi_p\lambda\omega_p^{-3} \oplus \chi_p\lambda\omega_p^{-2} \oplus \chi_p\lambda\omega_p^{-2} \oplus \chi_p\lambda\omega_p^{-1} \oplus \chi_p\lambda\omega_p^{-2} \oplus \chi_p\lambda\omega_p^{-1} \oplus \chi_p\lambda\omega_p^{-1} \oplus \chi_p\lambda. \tag{19}$$

By Theorem 3.7 i) and Proposition 3.9, we obtain

$$\epsilon(\sigma_{F,\chi,p}, \psi_p, \mathrm{d}x_p) = \lambda^8 \omega_p^{-12} (p^{(n(\psi)\dim(\chi_p) + a(\chi_p))}) \epsilon(\chi_p, \psi, \mathrm{d}x)^8 = p^{12} \epsilon(\chi_p, \psi_p, \mathrm{d}x_p)^8,$$

since $a(\chi_p) = 1$ and λ is a quadratic character. By Corollary 3.11, we see that

$$\epsilon(\sigma_{F,\chi,p}, \psi_p, \mathrm{d}x_p) = p^{12}(p\chi_p(-1))^4 = p^{16}.$$

In conclusion, we deduce that $W(\sigma'_{F,\chi,p}) = 1$, which completes the proof. □

Remark 4.3. We proceed to extract the conductor $\mathrm{cond}(F,\chi)$ from the proof. When q is distinct from p, we saw that $\sigma'_{F,\chi,q}$ is unramified, hence $a(\sigma'_{F,\chi,q}) = 0$. At the prime p, we have

$$a(\sigma'_{F,\chi,p}) = a(\sigma_{F,\chi,p}) + \dim V^{I_p}/(V^{I_p} \cap \ker N_{F,\chi,p}) = a(\sigma_{F,\chi,p})$$
$$= a(\chi_p\lambda\omega_p^{-3}) + 3a(\chi_p\lambda\omega_p^{-2}) + 3a(\chi_p\lambda\omega_p^{-1}) + a(\chi_p\lambda) = 8a(\chi_p) = 8.$$

We conclude that

$$\mathrm{cond}(F,\chi) = \prod_{\ell} \ell^{a(\sigma'_{F,\chi,p})} = p^8.$$

Remark 4.4. By contrast, the same techniques applied to the untwisted case lead to the equalities

$$\delta(\sigma'_{F,p}) = -p^{10}\lambda^5(\Phi) = -p^{10}a_p(f_1)a_p(f_2)a_p(f_3)$$

and $\epsilon(\sigma_{F,p},\psi_p,\mathrm{d}x_p) = 1$. This in turn implies that $W(F) = a_p(f_1)a_p(f_2)a_p(f_3)$, in agreement with [6]. Moreover, the conductor in this case is $\mathrm{cond}(F) = p^5$.

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