Differential Equations on the Sierpinski Gasket

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Résumé

A recent discover allows to define differential operators on topological spaces that have a highly non classical structure. The aim is to introduce the Laplace operator on the Sierpinski Triangle and study some of its properties. The approach will basically be the same as in [1].

1. The graph of SG

We approximate the Sierpinski Gasket (SG) with its graph that we shall denote Γ. We let $\Gamma_m = (V_m, A_m)$ denote the graph of level m. (Informally, Γ can be viewed as a "hollow" version of SG).

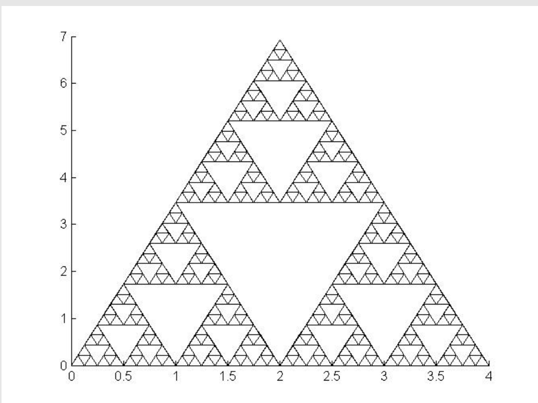

Figure 1: *Graph of SG at level 5.*

2. Measure and Energy on SG

In order to work towards the definition of a Laplacian we need to introduce a measure on SG. Any self-similar measure is suitable but to simplify the computations we mostly work with the uniform measure. This is the measure that assigns the weight of $(1/3)^m$ to each small triangle of SG at level m.

An important tool is the notion of graph energy.

Definition 1 *Suppose we have a function $f : V_m \to \mathbb{R}$, we define the graph energy of f on Γ_m as the quantity*

$$E_m(f) = \sum_{x \underset{m}{\sim} y} (f(x) - f(y))^2$$

where $x \underset{m}{\sim} y$ stands for "$(x, y) \in A_m$".

We use a more suitable energy that we define as

$$\mathcal{E}_m(f) = \left(\frac{5}{3}\right)^m E_m(f),$$

where $3/5$ is a renormalization factor. If f has values at any level m, then we set $\mathcal{E}(f) = \lim_{m\to\infty} \mathcal{E}_m(f)$ (when this limit exists).

A key element of our project is the notion of harmonic functions. We define the harmonic extension of f (as above) as the extension to level $m+1$ that minimizes the energy.

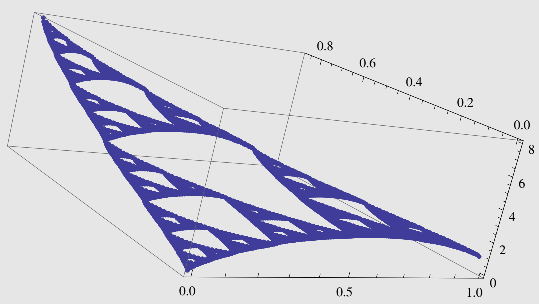

Figure 2: *Graph of the harmonic extension of level 7 for initial values 0, 1, 8.*

Definition 2 *A function $h : SG \to \mathbb{R}$ is said to be harmonic if*

$$\mathcal{E}(h) \leq \mathcal{E}(g),$$

for all g such that $g|_{V_0} = h|_{V_0}$.

We say that a function f has finite energy if $\mathcal{E}(f) < \infty$. We shall use the notation dom $\mathcal{E}$ for the set of functions $SG \to \mathbb{R}$ with finite energy.

3. Laplacian

By analogy with $I = [0, 1]$ we are able to state a weak formulation of the Laplacian on SG, that we will write Δ_μ to denote its dependence on the chosen measure.

Definition 3 *Let $u \in$ dom $\mathcal{E}$ and f be a continuous function. Then $u \in$ dom Δ_μ with $\Delta_\mu u = f$ if*

$$\mathcal{E}(u, v) = -\int_K f v d\mu$$

for all $v \in$ dom $\mathcal{E}$ that vanishes on V_0.

It is said to be the weak definition because of the boundary condition that v vanishes on V_0. In order to state a wider formulation we need to get rid of this condition. Thus we introduce the notion of normal derivatives. This yields the following theorem.

Theorem 1 *Let $u \in$ dom Δ_μ and $v \in$ dom $\mathcal{E}$. Then $\mathcal{E}(u, v)$ is equal to*

$$-\int_{SG} (\Delta_\mu u) v d\mu + \sum_{V_0} v(q) \partial_n u(q).$$

A pointwise formula of the Laplacian enables us to compute the Laplacian at a given vertex x in the graph (but not in V_0). We give this formula for the uniform measure :

$$\Delta_\mu u(x) = \frac{3}{2} \lim_{m\to\infty} 5^m \Delta_m u(x).$$

4. Examples of differential equations on SG

Let U be an open subset of $\mathbb{R}^n$ with some conditions on ∂U (smooth or piecewise smooth). The Dirichlet's problem is the following system:

$$\begin{cases} -\Delta u &= f \text{ on } U \\ u &\equiv 0 \text{ on } \partial U. \end{cases}$$

Here $f : U \to \mathbb{R}$ is a given function with some conditions. The theory we have covered enables us to solve this problem with the Green's function. More generally, it can be used to solve diffusion problems.

References

[1] Robert S. Strichartz. *Differential equations on fractals*. Princeton university press, 2006.