Hodge groups of $K3$ surfaces

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Dedicated to I. R. Shafarevich on the occasion of his 60. birthday

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The purpose of this paper is to study the rational Hodge structure attached to the second rational cohomology group $H^2(Y, \mathbb{Q})$ of a smooth irreducible projective surface Y over the field $\mathbb{C}$ of complex numbers. We shall prove that the Hodge group of Y acts irreducibly on the $\mathbb{Q}$-lattice $V(Y)$ of transcendental cycles if the geometric genus P_g is one (§ 1). As a corollary, we get a geometric analogue of Tate's conjecture (algebraicity of invariant cycles) for families of surfaces with $P_g = 1$, whose global monodromy group is infinite (§ 3) and for non-isotrivial families of $K3$ surfaces (§ 4). We shall also prove that the action of the automorphism group of a surface on the lattice of transcendental cycles $T(Y)$ factors through a finite quotient group (§ 1). Besides, we shall study the centralizer E_Y of the Hodge group in $\operatorname{End} V(Y)$ (§§ 1, 2). For $K3$ surfaces the algebra E_Y plays the same role as the endomorphism algebra for Abelian varieties. We explicitly compute the Hodge group of a $K3$ surface Y in terms of E_Y (§ 2).

0. Notations, conventions, preliminaries

0. 0. Let V be a finite-dimensional vector space over $\mathbb{Q}$. We write $V_{\mathbb{R}}$ for the $\mathbb{R}$-space $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$ and $V_{\mathbb{C}}$ for the $\mathbb{C}$-space $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.
We have natural inclusions

$$V \subset V_{\mathbb{R}} \subset V_{\mathbb{C}}, \quad \operatorname{Aut}(V) \subset \operatorname{Aut}(V_{\mathbb{R}}) \subset \operatorname{Aut}(V_{\mathbb{C}}),$$
$$\operatorname{End}(V) \subset \operatorname{End}(V_{\mathbb{R}}) = \operatorname{End} V \otimes_{\mathbb{Q}} \mathbb{R} \subset \operatorname{End}(V_{\mathbb{C}}) = \operatorname{End} V \otimes_{\mathbb{Q}} \mathbb{C} = \operatorname{End}(V_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}.$$

The group $\operatorname{Aut} \mathbb{C}$ of all automorphisms of the field $\mathbb{C}$ acts compatibly on $V_{\mathbb{C}}$ and $\operatorname{End}(V_{\mathbb{C}})$. In particular,

$$\sigma(uv) = u(\sigma v), \quad \sigma[u, v] = [u, \sigma v], \quad \sigma(zv) = (\sigma z)(\sigma v)$$

for $u \in \operatorname{End} V$, $v \in \operatorname{End}(V_{\mathbb{C}})$, $z \in \mathbb{C}$, $\sigma \in \operatorname{Aut} \mathbb{C}$.

We write $x \longrightarrow \bar{x}$ for the usual complex conjugation map of V_C. We have

$$V_R = \{x \in V_C \mid x = \bar{x}\}.$$

If $W \subset V_C$ is a $\mathbb{C}$-subspace such that $W = \bar{W}$ then $W = (W_0)_C = W_0 \otimes_R \mathbb{C}$ for the $\mathbb{R}$-subspace $W_0 = W \cap V_R$.

0. 1. Let A be a $\mathbb{C}$-subspace of $\operatorname{End}(V_C)$, $A_0 = A \cap \operatorname{End} V$. Then

$$A \cap \operatorname{End}(V_R) \supset (A_0)_R = A_0 \otimes_Q \mathbb{R}.$$

If for some $u \in \operatorname{End} V$, $uA \subset A$ then $uA_0 \subset A_0$. If A is a complex Lie subalgebra of $\operatorname{End}(V_C)$ then A_0 is a Lie subalgebra of $\operatorname{End} V$.

$A = B_C = B \otimes_Q \mathbb{C}$ for some $\mathbb{Q}$-subspace B of $\operatorname{End} V$ iff $\sigma A \subset A$ for all $\sigma \in \operatorname{Aut} \mathbb{C}$. If so, $B = A_0$ and $A \cap \operatorname{End}(V_R) = (A_0)_R = A_0 \otimes_Q \mathbb{R}$. If $A = (A_0)_C$ is an algebraic Lie algebra then A_0 is an algebraic Lie algebra (Chevalley [1], Ch. 2, § 14). Notice that if A is an arbitrary algebraic Lie algebra, then for all $\sigma \in \operatorname{Aut} \mathbb{C}$, σA is also an algebraic Lie algebra (if A is the Lie algebra of an algebraic group G then σA is the Lie algebra of the algebraic group σG).

0. 1. 0 Lemma. *Let $f \in \operatorname{End}(V_R)$ be a semisimple endomorphism such that the commutative Lie subalgebra $\mathbb{C}f \subset \operatorname{End}(V_C)$ is algebraic. Let $g_0 \subset \operatorname{End} V$ be a minimal algebraic Lie subalgebra such that $(g_0)_R = g_0 \otimes_Q \mathbb{R} \ni f$. Then*:

1) *The complex Lie subalgebra $(g_0)_C = g_0 \otimes_Q \mathbb{C} \subset \operatorname{End}(V_C)$ is generated by the endomorphisms σf ($\sigma \in \operatorname{Aut} \mathbb{C}$).*

2) *If $u \in \operatorname{End} V$ is an endomorphism such that*

$$[u, f] = 0 \quad \text{and} \quad uf \in Rf$$

then u commutes with g_0 and $ug_0 \subset g_0$.

0. 1. 1 Remark. The lemma was inspired by ideas of Serre and Sen [13], pp. 168—169, who treated Hodge-Tate modules.

0. 1. 2 *Proof.* The complex Lie algebra $g = (g_0)_C \subset \operatorname{End}(V_C)$ is a minimal algebraic complex Lie subalgebra such that $\sigma g \subset g$ for all $\sigma \in \operatorname{Aut} \mathbb{C}$ and g contains f. Let us denote the complex Lie subalgebra generated by all σf by $g' \subset \operatorname{End}(V_C)$. By definition, g' is a minimal complex Lie subalgebra of $\operatorname{End}(V_C)$ such that $\sigma g' \subset g'$ for all $\sigma \in \operatorname{Aut} \mathbb{C}$ and g' contains f. Clearly,

$$[u, \sigma f] = \sigma[u, f] = 0 \quad \text{and} \quad u(\sigma f) = \sigma(uf) \subset \sigma(\mathbb{C}f) = \mathbb{C}(\sigma f) \subset g'.$$

This implies that u commutes with g' and $ug' \subset g'$ (if for some $v_1, v_2 \in g'$

$$[u, v_1] = [u, v_2] = 0 \quad \text{and} \quad uv_1, uv_2 \in g' \quad \text{then} \quad [u, [v_1, v_2]] = 0$$

and $u[v_1, v_2] = [uv_1, v_2] = [v_1, uv_2] \in g'$). Therefore, one only has to prove the equality $g' = g$. Since $g' \subset g$, it would be sufficient to prove that g' is algebraic (compare the explicit descriptions of g' and g given above).

Since $\mathbb{C}f$ is an algebraic Lie subalgebra, $\mathbb{C}(\sigma f) = \sigma(\mathbb{C}f)$ is also an algebraic Lie subalgebra. This means that g' is generated by algebraic Lie subalgebras. It follows (Chevalley [1], Ch. 2, § 14) that g' is algebraic.

0. 2. Every non-degenerate symmetric bilinear form $\psi: V \times V \to \mathbb{Q}$ can be extended by $\mathbb{R}$ and $\mathbb{C}$-linearity to non-degenerate symmetric bilinear forms $V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ and $V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$. We shall denote these forms by the same symbol ψ.

Let G be an algebraic subgroup of $GL(V)$ defined over $\mathbb{Q}$ and $g \subset \operatorname{End} V$ its Lie algebra. Then g is an algebraic Lie algebra, $g_{\mathbb{R}} = g \otimes_{\mathbb{Q}} \mathbb{R} \subset \operatorname{End}(V_{\mathbb{R}})$ is the Lie algebra of the real Lie group $G(\mathbb{R})$,

$$g_{\mathbb{C}} = g \otimes_{\mathbb{Q}} \mathbb{C} = g_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \subset \operatorname{End}(V_{\mathbb{C}})$$

the Lie algebra of the complex Lie group $G(\mathbb{C})$. Notice that $g_{\mathbb{R}}$ is a real algebraic Lie algebra and $g_{\mathbb{C}}$ is a complex algebraic Lie algebra. We have $G(\mathbb{R}) = G(\mathbb{C}) \cap \operatorname{Aut} V_{\mathbb{R}}$, $G(\mathbb{Q}) = G(\mathbb{R}) \cap \operatorname{Aut} V = G(\mathbb{C}) \cap \operatorname{Aut} V$.

0. 3. We write $\mathbb{S}$ for the real algebraic group obtained by the Weil restriction of scalars of the multiplicative group G_m from $\mathbb{C}$ to $\mathbb{R}$ (Deligne [3], Sect. 2. 1):

$$\mathbb{S} = \prod_{\mathbb{C}/\mathbb{R}} G_m, \quad \mathbb{S}_{\mathbb{C}} = G_{m,\mathbb{C}} \times G_{m,\mathbb{C}}, \quad \mathbb{S}(\mathbb{R}) = \mathbb{C}^*.$$

The natural homomorphism of real algebraic groups

$$N: \mathbb{S} \to G_m$$

([3], Sect. 2. 1) induces the norm map

$$N_{\mathbb{C}/\mathbb{R}}: \mathbb{C}^* \to \mathbb{R}^*, \quad z \mapsto z\bar{z}$$

on $\mathbb{R}$-points of the groups. Let us put

$$U^1 = \{z \in \mathbb{C}^* \mid z\bar{z} = 1\} \subset \mathbb{C}^* = \mathbb{S}(\mathbb{R}).$$

Then $U^1 = (\operatorname{Ker} N)(\mathbb{R})$ is a connected compact group.

Recall [2], [3] that a rational Hodge structure on V is a homomorphism of real algebraic groups

$$h_V: \mathbb{S} \to GL(V_{\mathbb{R}}).$$

Let us put

$$V_{\mathbb{C}}^{p,q} = \{x \in V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \mid h_V(z)\, x = z^p \bar{z}^q x \quad \text{for all} \quad z \in \mathbb{C}^* = \mathbb{S}(\mathbb{R})\}.$$

We have the Hodge decomposition

$$V_{\mathbb{C}} = \bigoplus_{p,q} V_{\mathbb{C}}^{p,q}, \quad \overline{V_{\mathbb{C}}^{p,q}} = V_{\mathbb{C}}^{q,p}.$$

One says that (V, h_V) is a rational Hodge structure of weight n if $V^{p,q} = 0$ for $p+q \neq n$.

0. 3. 0 Example ([3], Section 2. 1). $V = \mathbb{Q}(m) = (2\pi i)^m \mathbb{Q}$ has a rational Hodge structure of weight $(-2m)$. $\mathbb{S}$ acts on $V_{\mathbb{R}} = (2\pi i)^m \mathbb{R}$ via the character N^{-m}. This means that $z \in \mathbb{C}^* = \mathbb{S}(\mathbb{R})$ acts on $\mathbb{R}(m)$ by multiplication with $(z\bar{z})^{-m}$.

0. 3. 1. Assume now that (V, h_V) is a rational Hodge structure of weight n (for some n). Then for any $z \in \mathbb{R}^* \subset \mathbb{C}^* = \mathbb{S}(\mathbb{R})$, $h_V(z)$ acts on $V_{\mathbb{R}}$ by multiplications with z^n.

The smallest algebraic subgroup $MT=MT(V,h_V)$ of $GL(V)$ defined over $\mathbb{Q}$ for which

$$h_V(\mathbb{C}^*)=h_V(\mathbb{S}(\mathbb{R}))\subset MT(\mathbb{R})$$

is called the Mumford-Tate group of the rational Hodge structure [5].

The smallest algebraic subgroup $Hdg=Hdg(V,h_V)$ of $GL(V)$ defined over $\mathbb{Q}$ for which $h_V(U^1)\subset Hdg(\mathbb{R})$ is called the Hodge group or the special Mumford-Tate group of the rational Hodge structure (compare with [9], [12]).

Evidently, MT and Hdg are connected algebraic groups, Hdg is a normal subgroup of MT, and MT contains the homothety group G_m of the $\mathbb{Q}$-space V.

Let $\det_{\mathbb{R}}: GL(V_{\mathbb{R}})\to G_m$ be the determinant homomorphism. Since $\det_{\mathbb{R}} h_V(U^1)$ is a connected compact subgroup of

$$\mathbb{R}^*=G_m(\mathbb{R}),\quad \det_{\mathbb{R}}(h_V(U^1))=1 \quad\text{and}\quad h_V(U^1)\subset SL(V_{\mathbb{R}}).$$

Then the definition of Hdg implies that $Hdg\subset SL(V)$. Consider the natural multiplication map

$$\mathrm{mult}\colon Hdg\times G_m\to MT,\quad u,v\mapsto uv\in MT\subset GL(V),$$

where G_m is considered as the homothety group in $GL(V)$. The homomorphism mult is surjective because the image of $\mathbb{R}$-points contains $h_V(\mathbb{C}^*)$. Since $Hdg\subset SL(V)$, the kernel of mult is finite. It follows that mult is an isogeny of algebraic groups and $\dim MT=\dim Hdg+1$. If Hdg is reductive then MT is also reductive. Since $Hdg\subset SL(V)$ and MT contains the homothety group, simple dimension arguments imply that Hdg is the connected identity component of $MT\cap SL(V)$. In particular, $Hdg(\mathbb{C})$ is a subgroup of a finite index in $MT(\mathbb{C})\cap SL(V_{\mathbb{C}})$.

0. 3. 1. 1. The definition of Hdg implies the following statements:

Let W be a $\mathbb{Q}$-subspace of V. W is Hdg-invariant iff $W_{\mathbb{R}}$ is U^1-invariant. We have

$$V^{Hdg}=V_{\mathbb{R}}^{U^1}\cap V.$$

If for some subspace $W\subset V$, $W_{\mathbb{R}}$ is $\mathbb{S}$-invariant then $W_{\mathbb{R}}$ is U^1-invariant and consequently, W is Hdg-invariant.

Let u be an endomorphism of V. Then u commutes with Hdg iff u commutes with $h_V(U^1)$. In particular, if u commutes with Hdg then u preserves the Hodge decomposition. Moreover, for any $V_{\mathbb{C}}^{p,q}\neq\{0\}$ there is a nontrivial homomorphism

$$\mathrm{End}_{Hdg}V\to \mathrm{End}_{\mathbb{C}}V_{\mathbb{C}}^{p,q}.$$

If $\psi\colon V\times V\to\mathbb{Q}$ is a bilinear form such that the corresponding $\mathbb{R}$-form $V_{\mathbb{R}}\times V_{\mathbb{R}}\to\mathbb{R}$ is U^1-invariant then ψ is Hdg-invariant.

0. 3. 1. 2 Remark. Let $\psi\colon V\times V\to\mathbb{Q}$ be a bilinear form such that the $\mathbb{R}$-form $\psi\colon V_{\mathbb{R}}\times V_{\mathbb{R}}\to\mathbb{R}$ is U^1-invariant. Recall that $\mathbb{S}(\mathbb{R})=\mathbb{C}^*=\mathbb{R}^*U^1$ and any $z\in R^*\subset\mathbb{S}(\mathbb{R})$ acts on $V_{\mathbb{R}}$ by multiplication with z^n. It follows that

$$\psi(h_V(z)x,h_V(z)y)=(z\bar{z})^n(x,y)\quad\text{for}\quad z\in\mathbb{C}^*=\mathbb{S}(\mathbb{R});\,x,y\in V_{\mathbb{R}}.$$

This means that if we consider the form

$$(2\pi i)^{-n}\psi\colon V\times V\to(2\pi i)^{-n}\mathbb{Q}=\mathbb{Q}(-n),$$

the corresponding $\mathbb{R}$-form

$$(2\pi i)^{-n}\psi\colon V_{\mathbb{R}}\times V_{\mathbb{R}}\to\mathbb{R}(-n)$$

is $\mathbb{C}^*=\mathbb{S}(\mathbb{R})$-equivariant, and, consequently, is $\mathbb{S}$-equivariant.

Conversely, if $\varphi\colon V\times V\to\mathbb{Q}(-n)$ is a bilinear form such that the corresponding $\mathbb{R}$-form $\varphi\colon V_{\mathbb{R}}\times V_{\mathbb{R}}\to\mathbb{R}(-n)$ is $\mathbb{S}$-equivariant, then the $\mathbb{R}$-form

$$(2\pi i)^n\varphi\colon V_{\mathbb{R}}\times V_{\mathbb{R}}\to\mathbb{R}$$

is U^1-invariant.

0. 3. 2. Recall [2], [3] that the automorphism

$$C=h_V(i)\in h_V(U^1)\subset Hdg(\mathbb{R})\subset \operatorname{Aut} V_{\mathbb{R}}$$

is called the Weil operator. We have

$$Cx=i^{p-q}x \quad \text{for} \quad x\in V_{\mathbb{C}}^{p,q}.$$

Let $\psi\colon V\times V\to\mathbb{Q}$ be a bilinear form such that the $\mathbb{R}$-form $\psi\colon V_{\mathbb{R}}\times V_{\mathbb{R}}\to\mathbb{R}$ is U^1-invariant. Then ψ is *Hdg*-invariant (0. 3. 1. 1) and the $\mathbb{R}$-form

$$\psi'=(2\pi i)^{-n}\psi\colon V_{\mathbb{R}}\times V_{\mathbb{R}}\to\mathbb{R}(-n)$$

is $\mathbb{S}$-equivariant (0. 3. 1. 2).

Let us assume that

$$V_{\mathbb{R}}\times V_{\mathbb{R}}\to\mathbb{R},\quad x,y\mapsto\psi(x,Cy)=(2\pi i)^n\psi'(x,y)$$

is a positive-definite symmetric form. Then the real algebraic group $Hdg_{\mathbb{R}}$ is reductive ([2], Lemma 5. 2; [5], Lemma 2. 8). It follows that *Hdg* is reductive and, consequently, *MT* is also reductive. Such a form ψ (or a form $\psi'=(2\pi i)^{-n}\psi\colon V\times V\to\mathbb{Q}(-n)$) is called a polarization of the rational Hodge structure and a triple (V,h_V,ψ) (or (V,h_V,ψ')) is called a rational polarized Hodge structure of weight n.

Let T be a $\mathbb{Z}$-lattice in V such that the restriction of ψ to $T\times T$ takes values in $\mathbb{Z}$ (i.e. the restriction of ψ' to $T\times T$ takes values in $\mathbb{Z}(-n)=(2\pi i)^{-n}\mathbb{Z}$). Then the triple (T,h_V,ψ) (or (T,h_V,ψ')) is called a polarized Hodge structure of weight n [2], [3], [5].

0. 3. 3. Let f_V be a semisimple endomorphism of $V_{\mathbb{R}}$ defined as follows:

$$f_Vx=(p-q)\,ix=(2p-n)\,ix \quad \text{for} \quad x\in V_{\mathbb{C}}^{p,q}.$$

$\mathbb{R}f_V$ is the real Lie algebra of the real Lie group $h_V(U^1)\subset Hdg(\mathbb{R})\subset \operatorname{Aut} V_{\mathbb{R}}$. It is easy to see that $\mathbb{C}f_V$ is an algebraic subalgebra of $\operatorname{End}(V_{\mathbb{C}})$.

Since f_V generates the Lie algebra of the connected Lie group $h_V(U^1)$, it follows from (0. 3. 1. 1) that:

a) $V^{Hdg} = V_R^{h_V(U^1)} \cap V = \{x \subset V_R \mid f_V x = 0\} \cap V$.

b) Let W be a $\mathbb{Q}$-subspace of V. W is *Hdg*-invariant iff $f_V W_R \subset W_R$.

c) An endomorphism u of V commutes with *Hdg* iff u commutes with f_V; u commutes with f_V iff u preserves the Hodge decomposition.

Let $hdg_V \subset \mathrm{End}\, V$ be the Lie algebra of the algebraic group *Hdg*. Since $h_V(U^1) \subset Hdg(\mathbb{R})$ one has

$$\mathbb{R} f_V \subset (hdg_V)_R = hdg_V \otimes_{\mathbb{Q}} \mathbb{R} \subset \mathrm{End}\, V_R.$$

By definition, hdg_V is an algebraic Lie algebra. The biunique correspondence between connected algebraic groups and algebraic Lie algebras implies the following description of hdg_V.

The Lie algebra hdg_V is a minimal algebraic subalgebra of $\mathrm{End}\, V$ such that $f_V \in (hdg_V)_R$.

Applying Lemma 0. 1. 0 we obtain the following statement.

0. 3. 3. 1 Lemma. *The complex Lie algebra* $(hdg_V)_C = hdg_V \otimes_{\mathbb{Q}} \mathbb{C} \subset \mathrm{End}\, V_C$ *is generated by* σf_V, *where* σ *runs through* $\mathrm{Aut}\, \mathbb{C}$.

Let u be an endomorphism of V such that $[u, f_V] = 0$ and $u f_V = \alpha f_V \in \mathbb{R} f_V$ for some $\alpha \in \mathbb{R}$. Then u commutes with hdg_V and

$$u\, hdg_V \subset hdg_V.$$

0. 3. 4 Lemma. *Let* Γ *be a subgroup of* $\mathrm{Aut}\, V$. *Let us denote by* G *the Zariski-closure of* Γ *in* $\mathrm{Aut}\, V_C$ *and let* G^0 *be the connected identity component of* G. *Let us assume that*:

a) G^0 *is semisimple*;

b) Γ *contains a subgroup of finite index lying in* $MT(\mathbb{Q})$.

Then Γ *contains a subgroup of finite index lying in* $Hdg(\mathbb{Q})$.

0. 3. 5 *Proof.* Since G^0 is connected and semisimple, $G^0 \subset SL(V_C)$. It follows that Γ contains a subgroup of finite index lying in $SL(V_C)$. This means that $\Gamma_1 = \Gamma \cap SL(V_C) \cap MT(\mathbb{C})$ is a subgroup of a finite index in Γ. Since $Hdg(\mathbb{C})$ is a subgroup of finite index in $SL(V_C) \cap MT(\mathbb{C})$ (end of (0. 3. 1)), $\Gamma_2 = \Gamma_1 \cap Hdg(\mathbb{C})$ is a subgroup of finite index in Γ_1 and, consequently, Γ_2 is a subgroup of finite index in Γ. By definition Γ_2 lies in the group

$$Hdg(\mathbb{C}) \cap \Gamma \subset Hdg(\mathbb{C}) \cap \mathrm{Aut}\, V = Hdg(\mathbb{Q}).$$

This ends the proof.

0. 3. 6 Lemma. *Let us assume that $V = V^{Hdg} \oplus V_0$, where V_0 is a simple Hdg-module. Let $W \subset V$ be a Hdg-submodule. Then either $W \subset V^{Hdg}$ or $W \supset V_0$.*

0. 3. 7 *Proof.* We have

$$V^{Hdg} = \{x \in V \mid hdg_V x = 0\}, \quad hdg_V V \subset V_0.$$

Since W is a *Hdg*-submodule, $hdg_V W \subset W$.

$$\text{If} \quad W \not\subset V^{Hdg} \quad \text{then} \quad V_0 \supset hdg_V W \neq \{0\}, \quad \text{i.e.} \quad V_0 \cap W \supset V_0 \cap hdg_V W \neq \{0\}.$$

Since V_0 is a simple *Hdg*-module, the Hodge submodule W contains V_0.

0. 4. Everywhere in this paper Y stands for a smooth irreducible projective surface over $\mathbb{C}$, $H^2(Y)$ is the image of the natural map $H^2(Y, \mathbb{Z}) \to H^2(Y, \mathbb{Q})$. $H^2(Y)$ is a free $\mathbb{Z}$-module, whose rank is equal to

$$b_2(Y) = \dim_{\mathbb{Q}} H^2(Y, \mathbb{Q}).$$

The natural map $H^2(Y) \otimes \mathbb{Q} \to H^2(Y, \mathbb{Q})$ is an isomorphism. If we write $V = H^2(Y, \mathbb{Q})$ then

$$V_{\mathbb{R}} = H^2(Y, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} = H^2(Y, \mathbb{R}),$$
$$V_{\mathbb{C}} = H^2(Y, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = H^2(Y, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = H^2(Y, \mathbb{C}).$$

Let

$$\langle\ ,\ \rangle : H^2(Y, \mathbb{Q}) \times H^2(Y, \mathbb{Q}) \longrightarrow H^4(Y, \mathbb{Q}) = \mathbb{Q}(-2) \xrightarrow[\sim]{(2\pi i)^2} \mathbb{Q}$$

be the intersection form [3] (Y has a canonical orientation). It is a non-degenerate symmetric bilinear form, whose restriction to $H^2(Y)$ takes values in $\mathbb{Z}$. We also denote the corresponding nondegenerate symmetric bilinear intersection forms by

$$\langle\ ,\ \rangle : H^2(Y, \mathbb{R}) \times H^2(Y, \mathbb{R}) \to \mathbb{R},$$
$$\langle\ ,\ \rangle : H^2(Y, \mathbb{C}) \times H^2(Y, \mathbb{C}) \to \mathbb{C}.$$

1. Automorphisms of surfaces and endomorphisms of Hodge structures

1. 1. Let

$$H^2(Y, \mathbb{C}) = H^{2,0}(Y) \oplus H^{1,1}(Y) \oplus H^{0,2}(Y)$$

be the canonical Hodge decomposition [6], [2]. Then

$$H^{pq}(Y) = \overline{H^{qp}(Y)} \quad \text{and} \quad h^{pq}(Y) = \dim_{\mathbb{C}} H^{pq}(Y) = \dim_{\mathbb{C}} H^q(Y, \Omega_Y^p)$$
$$= \dim_{\mathbb{C}} H^p(Y, \Omega_Y^q) = \dim_{\mathbb{C}} H^{qp}(Y) = h^{qp}(Y).$$

Let

$$H^{1,1}(Y, \mathbb{R}) = H^{1,1}(Y) \cap H^2(Y, \mathbb{R}),$$
$$V_0(Y, \mathbb{R}) \;\; = [H^{2,0}(Y) \oplus H^{0,2}(Y)] \cap H^2(Y, \mathbb{R}).$$

Then

$$H^{1,1}(Y)=H^{1,1}(Y,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C},$$
$$H^{2,0}(Y)\oplus H^{0,2}(Y)=V_0(Y,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C},\quad \dim_{\mathbb{R}} V_0(Y,\mathbb{R})=2h^{2,0}(Y),$$
$$H^2(Y,\mathbb{R})=H^{1,1}(Y,\mathbb{R})\oplus V_0(Y,\mathbb{R}).$$

The $\mathbb{C}$-subspaces $H^{2,0}(Y)$ and $H^{0,2}(Y)$ are isotropic with respect to the intersection form $\langle\ ,\ \rangle$. $H^{1,1}(Y)$ is orthogonal to $H^{2,0}(Y)\oplus H^{0,2}(Y)$ with respect to $\langle\ ,\ \rangle$. It follows that the restrictions of $\langle\ ,\ \rangle$ to $H^{1,1}(Y)$ and $H^{2,0}(Y)\oplus H^{0,2}(Y)$ are non-degenerate, since $\langle\ ,\ \rangle$ is non-degenerate. We also have

$$\langle x,\bar{x}\rangle>0 \quad \text{for} \quad x\in H^{2,0}(Y)\oplus H^{0,2}(Y),\quad x\neq 0.$$

Notice that $H^{1,1}(Y,\mathbb{R})$ and $V_0(Y,\mathbb{R})$ are orthogonal with respect to $\langle\ ,\ \rangle$ and

$$H^2(Y,\mathbb{Q})\cap H^{1,1}(Y)=H^2(Y,\mathbb{Q})\cap H^{1,1}(Y,\mathbb{R}).$$

Let

$$A(Y)=H^2(Y)\cap H^{1,1}(Y)=H^2(Y)\cap H^{1,1}(Y,\mathbb{R}),$$
$$A(Y)_{\mathbb{Q}}=H^2(Y,\mathbb{Q})\cap H^{1,1}(Y).$$

$A(Y)$ is a $\mathbb{Z}$-lattice in the $\mathbb{Q}$-space $A(Y)_{\mathbb{Q}}$.

We have $A(Y)=H^2(Y)$ and $A(Y)_{\mathbb{Q}}=H^2(Y,\mathbb{Q})$ if and only if

$$P_g=h^{2,0}(Y)=0.$$

$A(Y)$ is called the lattice of algebraic cycles and $A(Y)_{\mathbb{Q}}$ is called the $\mathbb{Q}$-lattice of algebraic cycles. According to the Lefschetz theorem $A(Y)$ consists of the first Chern classes of invertible sheaves (divisors) on Y. Let us denote the rank of $A(Y)$ by $\rho(Y)$. $\rho(Y)$ is called the Picard number of Y.

Let $V(Y)$ be the orthogonal complement of $A(Y)$ in $H^2(Y,\mathbb{Q})$ with respect to $\langle\ ,\ \rangle$. The restriction of $\langle\ ,\ \rangle$ to $V(Y)$ is non-degenerate. This implies that

$$H^2(Y,\mathbb{Q})=A(Y)_{\mathbb{Q}}\oplus V(Y)$$

and

$$\dim_{\mathbb{Q}} V(Y)=b_2(Y)-\rho(Y).$$

In particular,

$$V(Y)=0\Leftrightarrow A(Y)_{\mathbb{Q}}=H^2(Y,\mathbb{Q})\Leftrightarrow h^{2,0}(Y)=0.$$

Let $T(Y)=H^2(Y)\cap V(Y)$. The intersection $T(Y)\cap A(Y)$ is not necessarily $\{0\}$ and $A(Y)+T(Y)$ is a subgroup of finite index of $H^2(Y)$.

Notice that $V(Y)_{\mathbb{R}}$ is an orthogonal complement of $A(Y)_{\mathbb{R}}$ and $V(Y)_{\mathbb{C}}$ is an orthogonal complement of $A(Y)_{\mathbb{C}}$ with respect to $\langle\ ,\ \rangle$. It follows that

$$V(Y)_{\mathbb{C}}\supset H^{2,0}(Y)\oplus H^{0,2}(Y),\quad V(Y)_{\mathbb{R}}\supset V_0(Y,\mathbb{R}).$$

Moreover,

$$V(Y)_{\mathbb{R}} = V_0(Y, \mathbb{R}) \oplus V^{1,1}(Y, \mathbb{R}),$$
$$V(Y)_{\mathbb{C}} = H^{2,0}(Y) \oplus H^{0,2}(Y) \oplus V^{1,1}(Y, \mathbb{R}) \oplus_{\mathbb{Q}} \mathbb{C},$$

where $V^{1,1}(Y, \mathbb{R}) = H^{1,1}(Y, \mathbb{R}) \cap V(Y)_{\mathbb{R}}$.

$T(Y)$ is called the lattice of transcendental cycles and $V(Y)$ is called the $\mathbb{Q}$-lattice of transcendental cycles.

Let $GL(V(Y))$ act trivially on $A(Y)_{\mathbb{Q}}$ and consider $GL(V(Y))$ as an algebraic subgroup of $GL(A(Y)_{\mathbb{Q}} \oplus V(Y)) = GL[H^2(Y, \mathbb{Q})]$. Similarly, we consider End $V(Y)$ as a subalgebra of $\mathrm{End}(A(Y)_{\mathbb{Q}} \oplus V(Y)) = \mathrm{End}\, H^2(Y, \mathbb{Q})$. (End $V(Y)$ annihilates $A(Y)_{\mathbb{Q}}$). The Lie subalgebra End $V(Y)$ is a Lie algebra of the algebraic subgroup

$$GL(V(Y)) \subset GL[H^2(Y, \mathbb{Q})].$$

Let us denote the special orthogonal subgroup corresponding to the restriction of $\langle\ ,\ \rangle$ to $V(Y)$ by

$$SO(V(Y)) \subset GL(V(Y)) \subset GL[H^2(Y, \mathbb{Q})].$$

Further, let us denote the Lie algebra of the algebraic group $SO(V(Y))$ by $so(V(Y))$. We have

$$so(V(Y)) \subset \mathrm{End}\, V(Y) \subset \mathrm{End}\, H^2(Y, \mathbb{Q}).$$

1. 1. 0. The automorphism group Aut Y preserves the intersection form $\langle\ ,\ \rangle$ and the Hodge decomposition. This implies that Aut Y preserves $A(Y)$, $V(Y)$, $T(Y)$. This gives us the natural homomorphism

$$\chi\colon \mathrm{Aut}\, Y \to \mathrm{Aut}\, T(Y) \subset \mathrm{Aut}\, V(Y).$$

Let us denote the image of χ by $U(Y)$. Clearly,

$$U(Y) \subset SO(V(Y))(\mathbb{Q}) \subset \mathrm{Aut}\, V(Y).$$

1. 1. 1 Theorem. *$U(Y)$ is a finite group.*

1. 1. 2 Theorem. *Assume that $h^{2,0}(Y) = 1$. Then $U(Y)$ is a finite cyclic group. Let n be the order of $U(Y)$ and φ the Euler function. Then $\varphi(n) \mid b_2(Y) - \rho(Y)$.*

1. 1. 3 Corollary ([10], Theorem 10. 1. 2, pp. 98—99). *Let Y be a K3 surface. Then $U(Y)$ is a finite cyclic group of order n and $\varphi(n) \mid 22 - \rho(Y)$.*

The theorems will be proved in subsections 1. 3, 1. 6.

1. 2. Recall ([2], [3]) that there is a canonical rational Hodge structure of weight 2 on $H^2(Y, \mathbb{Q})$

$$h\colon \mathbb{S} \to GL[H^2(Y, \mathbb{R})] = GL[H^2(Y, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}]$$

such that

$$h(z)x = z^p \bar{z}^q x \quad \text{for} \quad z \in \mathbb{C}^* = \mathbb{S}(\mathbb{R}),$$
$$x \in H^{p,q}(y) \subset H^2(Y, \mathbb{C}) = H^2(Y, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

It can be easily shown that $h(U^1)$ preserves the intersection form $\langle\ ,\ \rangle$ and $H^2(Y,\mathbb{R})^{h(U^1)} = H^{1,1}(Y) \cap H^2(Y,\mathbb{R}) = H^{1,1}(Y,\mathbb{R})$.

In the notations of (0. 3. 1)

$$H^2(Y,\mathbb{Q})_C^{p,q} = H^{p,q}(Y).$$

The Weil operator $C \in \operatorname{Aut} H^2(Y,\mathbb{R})$ acts as follows (0. 3. 1. 0):

$$Cx = i^2 x = (-i)^2 x = -x \quad \text{for} \quad x \in H^{2,0}(Y) \oplus H^{0,2}(Y)$$
$$Cx = x \quad \text{for} \quad x \in H^{1,1}(Y).$$

It follows that

$$Cx = -x \quad \text{for} \quad x \in V_0(Y,\mathbb{R}), \quad Cx = x \quad \text{for} \quad x \in H^{1,1}(Y,\mathbb{R}).$$

It can be easily shown that the bilinear form

$$H^2(Y,\mathbb{R}) \times H^2(Y,\mathbb{R}) \to \mathbb{R}, \quad x, y \mapsto \langle x, Cy \rangle$$

is symmetric ($V_0(Y,\mathbb{R})$ is orthogonal to $H^{1,1}(Y,\mathbb{R})$ with respect to $\langle\ ,\ \rangle$ (1. 1)).

The generator $f_H = f_{H^2}(Y,\mathbb{Q})$ of the Lie algebra of $h(U^1)$ (0. 3. 3) acts as follows:

$$f_H x = 0 \quad \text{for} \quad x \in H^{1,1}(Y),$$
$$f_H x = (2i)x \text{ for } x \in H^{2,0}(Y), \quad f_H x = (-2i)x \text{ for } x \in H^{0,2}(Y).$$

Notice that all non-zero eigenvalues of f_H are non-real; they are $(2i)$ and $(-2i)$ with multiplicity $h^{2,0}(Y)$. f_H is a semisimple endomorphism of rank $2h^{2,0}(Y)$. Since

$$H^2(Y,\mathbb{R})^{h(U^1)} = H^{1,1}(Y,\mathbb{R}),$$
$$A(Y) = H^2(Y,\mathbb{R})^{h(U^1)} \cap H^2(Y), \quad A(Y)_{\mathbb{Q}} = H^2(Y,\mathbb{R})^{h(U^1)} \cap H^2(Y,\mathbb{Q}).$$

This implies that $h(U^1)$ acts trivially on $A(Y)_{\mathbb{R}}$, and $V(Y)_{\mathbb{R}}$ is $h(U^1)$-invariant, since $h(U^1)$ preserves $\langle\ ,\ \rangle$. So we have an inclusion

$$h(U^1) \subset SO(V(Y)_{\mathbb{R}}) = SO(V(Y))(\mathbb{R}) \subset GL[H^2(Y,\mathbb{R})].$$

In fact, we have the inclusion

$$h(U^1) \subset \operatorname{Aut}(V(Y)_{\mathbb{R}}) = GL(V(Y))(\mathbb{R}).$$

The $h(U^1)$-invariance of $\langle\ ,\ \rangle$ and the connectedness of $h(U^1)$ imply that the image lies in $SO(V(Y))(\mathbb{R})$. Since $\mathbb{R}f_H$ is the Lie algebra of $h(U^1)$,

$$f_H \in so(V(Y))_{\mathbb{R}} = so(V(Y)) \otimes_{\mathbb{Q}} \mathbb{R} \subset \operatorname{End} V(Y)_{\mathbb{R}} \subset \operatorname{End} H^2(Y,\mathbb{R}).$$

In particular, f_H annihilates $A(Y)_{\mathbb{R}}$ and $f_H V(Y)_{\mathbb{R}} \subset V(Y)_{\mathbb{R}}$. If we consider f_H as an endomorphism of $V(Y)_{\mathbb{R}}$, then its rank is equal to $2h^{2,0}(Y)$. The action of Aut Y commutes with the action of $\mathbb{S}$ on $H^2(Y,\mathbb{R})$. In particular, Aut Y commutes with $h(U^1)$ and the Weil operator C.

1. 2. 1. Let $Hdg = Hdg_{2,Y} \subset [GLH^2(Y,\mathbb{Q})]$, $MT = MT_{2,Y} \subset GL[H^2(Y,\mathbb{Q})]$ be the Hodge group and the Mumford-Tate group of the rational Hodge structure on $H^2(Y,\mathbb{Q})$ respectively.

Since $\langle\ ,\ \rangle$ is $h(U^1)$-invariant, the intersection form

$$\langle\ ,\ \rangle: H^2(Y, \mathbb{Q}) \times H^2(Y, \mathbb{Q}) \to \mathbb{Q}$$

is *Hdg*-invariant (0. 3. 1. 1).

Since $h(U^1) \subset SO(V(Y))(\mathbb{R})$, the definition of *Hdg* (0. 3. 1) implies that

$$Hdg \subset SO(V(Y)) \subset GL(V(Y)) \subset GL[H^2(Y, \mathbb{Q})].$$

We have (0. 3. 2, 0. 3. 3)

$$H^2(Y, \mathbb{Q})^{Hdg} = H^2(Y, \mathbb{R})^{h(U^1)} \cap H^2(Y, \mathbb{Q}) = H^{1,1}(Y, \mathbb{R}) \cap H^2(Y, \mathbb{Q}) = A(Y)_{\mathbb{Q}}$$

and

$$\{x \in H^2(Y, \mathbb{R}), f_H x = 0\} \cap H^2(Y, \mathbb{Q}) = H^2(Y, \mathbb{Q})^{Hdg} = A(Y)_{\mathbb{Q}}.$$

Let $hdg \subset \operatorname{End} H^2(Y, \mathbb{Q})$ be the Lie algebra of the connected algebraic group *Hdg* (0. 3. 3). Since $Hdg \subset SO(V(Y))$ we have $hdg \subset so(V(Y)) \subset \operatorname{End} V(Y)$.

Recall (0. 3. 3) that

$$f_H \in hdg_{\mathbb{R}} = hdg \otimes_{\mathbb{Q}} \mathbb{R} \subset so(V(Y))_{\mathbb{R}} \subset \operatorname{End} V(Y)_{\mathbb{R}}.$$

1. 2. 1. 0 Remark. Let $Hdg\,Y \subset GL[H^*(Y, \mathbb{Q})]$ be the Hodge group of Y [12]. Then $Hdg = Hdg_{2,Y}$ is the image of $Hdg\,Y$ in $GL[H^2(Y, \mathbb{Q})]$. If Y is a $K3$ surface then

$$Hdg = Hdg\ Y.$$

1. 2. 2. The goal of this subsection is to present a polarization of the rational Hodge structure on $H^2(Y, \mathbb{Q})$. Recall (0. 3. 2) that the existence of the polarization implies the reductiveness of *Hdg* and *MT*.

Let $l \in A(Y) \subset A(Y)_{\mathbb{Q}}$ be a class of hyperplane sections of Y and let $P(Y)_{\mathbb{Q}} \subset H^2(Y, \mathbb{Q})$ be the orthogonal complement of l in $H^2(Y, \mathbb{Q})$ with respect to $\langle\ ,\ \rangle$. Since $\langle l, l \rangle > 0$,

$$P(Y)_{\mathbb{Q}} \cap \mathbb{Q}l = \{0\} \quad \text{and} \quad H^2(Y, \mathbb{Q}) = \mathbb{Q}l \oplus P(Y)_{\mathbb{Q}}.$$

Since l and $\langle\,,\rangle$ are *Hdg*-invariant, $P(Y)_{\mathbb{Q}}$ is also *Hdg*-invariant. Clearly, $V(Y) \subset P(Y)_{\mathbb{Q}}$. Define

$$P(Y) = P(Y)_{\mathbb{Q}} \cap H^2(Y).$$

Then $P(Y)$ is the orthogonal of l in $H^2(Y)$ and $P(Y)_{\mathbb{R}} = P(Y) \otimes \mathbb{R} = P(Y)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ is the orthogonal complement of l in $H^2(Y, \mathbb{R})$ with respect to $\langle\ ,\ \rangle$. $P(Y)$ is called the lattice of primitive cycles.

$$T(Y) = V(Y) \cap H^2(Y), \quad P(Y)_{\mathbb{Q}} \cap H^2(Y) = P(Y),$$
$$V(Y)_{\mathbb{R}} = V(Y) \otimes_{\mathbb{Q}} \mathbb{R} \subset P(Y)_{\mathbb{R}}.$$

Clearly, $\mathbb{Z}l+P(Y)$ is a $\mathbb{Z}$-lattice in $H^2(Y,\mathbb{Q})$. Recall ([2], [6]) that the symmetric bilinear form

$$P(Y)_{\mathbb{R}} \times P(Y)_{\mathbb{R}} \to \mathbb{R}, \quad x, y \mapsto -\langle x, Cy\rangle$$

is positive-definite (the Hodge-Riemann relations). Since $V(Y)_{\mathbb{R}} \subset P(Y)_{\mathbb{R}}$, the symmetric bilinear form

$$\psi\colon V(Y)\times V(Y) \to \mathbb{R}, \quad \psi(x,y) = -\langle x, Cy\rangle$$

is positive-definite. Clearly, Aut Y preserves ψ. Now, let

$$\langle\ ,\ \rangle'\colon H^2(Y,\mathbb{Q})\times H^2(Y,\mathbb{Q}) \to \mathbb{Q}$$

be the symmetric bilinear form defined as follows:

a) The restriction of $\langle\ ,\ \rangle'$ to $\mathbb{Q}l$ coincides with $\langle\ ,\ \rangle$.

b) The restriction of $\langle\ ,\ \rangle'$ to $P(Y)_{\mathbb{Q}}$ coincides with $-\langle\ ,\ \rangle$.

c) $\mathbb{Q}l$ and $P(Y)_{\mathbb{Q}}$ are orthogonal with respect to $\langle\ ,\ \rangle'$.

Clearly, the form

$$H^2(Y,\mathbb{R})\times H^2(Y,\mathbb{R}) \to \mathbb{R}, \quad x, y \mapsto \langle x, Cy\rangle'$$

is a positive-defined symmetric bilinear form. This means (0. 3. 2) that $\langle\ ,\ \rangle'$ is a polarization of the rational Hodge structure on $H^2(Y,\mathbb{Q})$. In particular, the groups *Hdg* and *MT* are reductive.

Notice that the polarization $\langle\ ,\ \rangle'$ takes integral values on the lattive $\mathbb{Z}l+P(Y)$. It follows (0. 3. 2) that the triple $(\mathbb{Z}l+P(Y), \mathrm{h}, \langle\ ,\ \rangle')$ is a polarized Hodge structure of weight 2.

1. 3 *Proof* of Theorem 1. 1. 1. The automorphism group Aut Y preserves the form ψ. It follows that the image $U(Y)$ lies in the orthogonal group $O(V(Y)_{\mathbb{R}}, \psi)$ of the positive-definite form ψ.

1. 3. 0 Lemma. *All elements of $U(Y)$ are of finite order.*

The lemma will be proved in subsection 1. 3. 2.

1. 3. 1. End of the proof of the Theorem 1. 1. 1 (modulo Lemma 1. 3. 0).

The kernel of the natural map

$$\mathrm{Aut}\, T(Y) \to \mathrm{Aut}\,(T(Y)/3\,T(Y))$$

does not contain elements of finite order (Serre [14], Part 2, Ch. 4, Appendix 3). It follows that the composition

$$U(Y) \to \mathrm{Aut}\, T(Y) \to \mathrm{Aut}\,(T(Y)/3\,T(Y))$$

is an embedding. Since $\mathrm{Aut}\,(T(Y)/3\,T(Y))$ is a finite group, $U(Y)$ is also finite.

1. 3. 2 *Proof* of Lemma 1. 3. 0. Let $u \in U(Y)$. Then

$$u \in \operatorname{Aut} T(Y) \quad \text{and} \quad u \in O(V(Y)_{\mathbb{R}}, \psi).$$

Since u belongs to the orthogonal group of the positive-definite form, u is a semisimple endomorphism of $V(Y)_{\mathbb{R}}$ and for all eigen values α of u, $|\alpha| = 1$. Since u preserves the $\mathbb{Z}$-lattice in $V(Y)$, an eigen value α of u is an algebraic integer. Moreover, for each automorphism σ of the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ $\sigma\alpha$ is also an eigen value of u, and, consequently, $|\sigma\alpha| = 1$. It follows that α is a root of unity ([17], Ch. 4, § 4, Th. 8; [2], Lemma 6. 4). This means that u is a semisimple endomorphism, whose eigen values are all roots of unity. It follows that u is a periodic automorphism.

1. 4. Let us decompose $V(Y)$ into a direct sum of simple *Hdg*-modules

$$V(Y) = W_1 \oplus \cdots \oplus W_{r(Y)}$$

The number $r(Y)$ does not depend on the choice of the decomposition.

1. 4. 1 Theorem. $r(Y) \leqq h^{2,0}(Y)$. *In particular, if* $h^{2,0}(Y) = 1$ *then the Hdg-module* $V(Y)$ *is simple.*

1. 4. 2 *Proof.* f_H does not annihilate $W_{j,\mathbb{R}} = W_j \otimes_{\mathbb{Q}} \mathbb{R}$ for any j because

$$W_j \not\subset H^2(Y, \mathbb{Q})^{Hdg}$$

(1. 2. 1). Therefore, f_H has non-zero eigen-values in $W_{j,\mathbb{R}}$. Since all non-zero eigen values of f_H are non-real, f_H has at least one pair of non-zero complex conjugate eigen values in $W_{j,\mathbb{R}}$, namely $2i$ and $-2i$ (1. 2). It follows that the multiplicity of the eigen value $2i$ of f_H is greater or equal to $r(Y)$. But the multiplicity of $2i$ is equal to $h^{2,0}(Y)$ (1. 2). It follows that $h^{2,0}(Y) \geqq r(Y)$.

1. 5. Let

$$E = E_Y = \operatorname{End}_{Hdg} V(Y) \subset \operatorname{End} V(Y) \subset \operatorname{End} H^2(Y, \mathbb{Q}).$$

We have

$$U(Y) \subset E^*, \quad E \subset \operatorname{End}_{Hdg} H^2(Y, \mathbb{Q}), \quad E = \operatorname{End}_{hdg} V(Y),$$

E commutes with f_H. In particular, E preserves the Hodge decomposition. The reductiveness of *Hdg* implies that E is a finite-dimensional semisimple algebra over $\mathbb{Q}$. The form $\langle \ , \ \rangle$ defines the involution $a \mapsto a'$ of E as follows:

$$\langle ax, y \rangle = \langle x, a'y \rangle \quad \text{for} \quad a \in E;\ x, y \in V(Y).$$

Let us extend the involution $a \mapsto a'$ by $\mathbb{R}$-linearity to the $\mathbb{R}$-algebra

$$E_{\mathbb{R}} = E \otimes_{\mathbb{Q}} \mathbb{R} \subset \operatorname{End} V(Y)_{\mathbb{R}}.$$

We have

$$\langle ax, y \rangle = \langle x, a'y \rangle, \quad \langle ax, ay \rangle = \langle a'ax, y \rangle \quad \text{for} \quad a \in R_{\mathbb{R}};\ x, y \in V(Y)_{\mathbb{R}}.$$

In particular, an element $a \in E_R$ preserves $\langle\ ,\ \rangle$ iff $a'a=1$. We have

$$\psi(ax, y) = -\langle ax, Cy\rangle = -\langle x, a'Cy\rangle = -\langle x, Ca'y\rangle = \psi(x, a'y)$$

(E_R commutes with $C \in Hdg\ (R)$). Since ψ is a positive-definite symmetric form,

$$\psi(a'ax, y) = \psi(x, a'ay) \quad \text{for} \quad a \in E_R;\ x, y \in V_R,$$
$$\psi(a'ax, x) = \psi(ax, ax) > 0 \quad \text{for} \quad a \in E_R, \quad x \in V(Y)_R$$

such that $ax \neq 0$.

This means that $a'a$ is a positive symmetric operator with respect to ψ for a non-zero $a \in E_R$. In particular, all eigen values of $a'a$ are real and non-negative. Let $\mathrm{Tr}_V\colon E \to Q$ be the trace map attached to the action of E on the Q-space $V(Y)$. Extending Tr_V by R-linearity to E_R we get the trace map $\mathrm{Tr}_V\colon E_R \to R$ attached to the action of E_R on the R-space $V(Y)_R$. We have $\mathrm{Tr}_V(a'a) > 0$ for a non-zero $a \in E_R$, since all eigen values of $a'a$ are non-negative. In order to study the algebra E with the involution "′" we shall use the Albert classification ([8], § 21), which is usually applied to the endomorphism algebra of an Abelian variety with a Rosati involution. For the sake of simplicity we restrict ourselves to the case of a commutative field E.

1. 5. 1 Theorem. *Let $E = E_Y$ be a commutative field. Then one of the following statements holds*:

I) *E is a totally real number field and the involution "′" is the identity map.*

II) *The subfield $E_0 = \{u \in E \mid u' = u\}$ is a totally real number field and E is an imaginary quadratic extension of E_0. The involution "′" is the complex conjugation.*

1. 5. 2 *Proof.* Clearly, E is a number field. Notice that $V(Y)$ is provided with the natural structure of a finite-dimensional vector space over E. Let $\mathrm{tr} = \mathrm{tr}_{E/Q}\colon E \to Q$ be the canonical trace map attached to the number field E. If $m = \dim_E V(Y)$ then $\mathrm{Tr}_V = m\,\mathrm{tr}$. This means that $\mathrm{tr}(a'a) = \frac{1}{m}\mathrm{Tr}_V(a'a) > 0$ for a non-zero $a \in E_R$. One only has to apply the first step of the Albert classification ([8], § 21, Appendix 1).

1. 5. 3 Remarks. a) We have

$$\mathrm{End}_{Hdg(R)} V(Y)_R = \mathrm{End}_{hdg(R)} V(Y)_R = E_R.$$

In particular, the center $Z(R)$ of $Hdg(R)$ lies in E_R^*. Since the elements of $Hdg(R)$ preserve $\langle\ ,\ \rangle$, $Z(R) \subset \{a \in E_R^* \mid a \text{ preserves } \langle\ ,\ \rangle\} = \{a \in E_R \mid a'a = 1\}$.

b) Let us assume that E is a totally real number field. Then the involution "′" is the identity map and the set $\{a \in E_R \mid a'a = 1\} = \{a \in E_R \mid a^2 = 1\}$ is finite. It follows that the center $Z(R)$ of the reductive group $Hdg\ (R)$ is finite. This means that Hdg is semisimple.

c) Let E be a commutative field and $[E : Q] = \dim_Q V(Y)$, i.e. $\dim_E V(Y) = 1$. Then Hdg is commutative and, therefore, E is an imaginary quadratic extension of a totally real number field.

d) If E is a commutative field then $[E:\mathbb{Q}] \mid \dim_{\mathbb{Q}} V(Y)$, i.e. $[E:\mathbb{Q}] \mid b_2(Y)-\rho(Y)$. In particular, if $b_2(Y)-\rho(Y)$ is odd then $[E:\mathbb{Q}]$ is also odd and E is a totally real number field.

1. 6 Theorem. *Let* $P_g = h^{2,0}(Y) = 1$ (for example Y is a $K3$ surface). *Then*:

a) *The Hdg-module* $V(Y)$ *is simple and the algebra* $E = E_Y$ *is a commutative field*;

b) $U(Y) \subset \mu_E \subset E^*$ *where* μ_E *is the group of roots of unity in E.*

1. 6. 1 *Proof.* a) By (1. 5. 1) the *Hdg*-module $V(Y)$ is simple. It follows that E is a division algebra. In order to prove that E is commutative let us consider the natural non-trivial homomorphism

$$\varepsilon\colon E \to \operatorname{End}_{\mathbb{C}} H^{2,0}(Y) = \mathbb{C}$$

(since $H^{2,0}(Y) \subset V(Y)_{\mathbb{C}}$, 1 goes to 1). Since E is a division algebra, $\varepsilon\colon E \to \mathbb{C}$ is an embedding, and, consequently, E is a commutative field.

In order to prove the statement b) it is sufficient to recall that $U(Y)$ is a finite group (1. 1. 1) lying in E^* (1. 5).

1. 6. 2 *Proof* of Theorem 1. 1. 2. Let us denote the order of μ_E by m. Then E contains the cyclotomic field $\mathbb{Q}(\mu_E)$, whose degree over $\mathbb{Q}$ is equal to $\varphi(m)$. It follows that $\varphi(m)$ divides $[E:\mathbb{Q}]$. According to Theorem 1.6, $U(Y)$ is a subgroup of μ_E. If n is the order of $U(Y)$ then $n \mid m$ and $U(Y)$ is cyclic of order n because μ_E is a cyclic group. Then we have $\varphi(n) \mid \varphi(m)$. Since

$$\varphi(m) \mid [E:\mathbb{Q}] \quad \text{and} \quad [E:\mathbb{Q}] \mid b_2(Y)-\rho(Y)\ (1.\ 5.\ 3d), \quad \varphi(n) \mid b_2(Y)-\rho(Y).$$

1. 7 Theorem. *Let us assume that* $h^{2,0}(Y) = 1$. *Let* Γ *be a subgroup of* $\operatorname{Aut} H^2(Y, \mathbb{Q})$ *satisfying the following conditions.*

a) $H^2(Y, \mathbb{Q})^\Gamma$ *is a rational Hodge substructure of* $H^2(Y, \mathbb{Q})$, i.e. $H^2(Y, \mathbb{Q})^\Gamma \otimes_{\mathbb{Q}} \mathbb{R}$ *is an* $\mathbb{S}$ *invariant subspace of* $H^2(Y, \mathbb{R})$:

b) Γ *contains a subgroup of finite index lying in* $MT(\mathbb{Q})$.

c) *Let us denote the Zariski-closure of* Γ *in* $\operatorname{Aut} H^2(Y, \mathbb{C})$ *by* G *and let* G^0 *be the connected identity component G. Then* G^0 *is semisimple.*

Then one of the following statements holds.

1) Γ *is a finite group*;

2) $H^2(Y, \mathbb{Q})^\Gamma \subset H^2(Y, \mathbb{Q})^{Hdg} = A(Y)_{\mathbb{Q}}$. *If* $\Gamma \subset \operatorname{Aut} H^2(Y)$ *then*

$$H^2(Y)^\Gamma = H^2(Y) \cap H^2(Y, \mathbb{Q}) \subset A(Y).$$

1. 7. 1 *Proof.* The conditions b) and c) imply that Γ contains a subgroup of finite index, lying in $Hdg(\mathbb{Q})$ (0. 3. 4). The condition a) implies that $H^2(Y, \mathbb{Q})^\Gamma$ is a *Hdg*-submodule (0. 3. 2). The simplicity of the *Hdg*-module $V(Y)$ and the decomposition $H^2(Y, \mathbb{Q}) = H^2(Y, \mathbb{Q})^{Hdg} \oplus V(Y)$ imply that either

$$H^2(Y, \mathbb{Q})^\Gamma \subset H^2(Y, \mathbb{Q})^{Hdg} = A(Y)_{\mathbb{Q}} \quad \text{or} \quad H^2(Y, \mathbb{Q})^\Gamma \supset V(Y) \quad (0.\ 3.\ 6).$$

If $H^2(Y, \mathbb{Q})^\Gamma \subset H^2(Y, \mathbb{Q})^{Hdg}$ then we have proven the theorem. Therefore, we have to assume that $H^2(Y, \mathbb{Q})^\Gamma$ contains $V(Y)$. Let

$\Gamma' = \Gamma \cap Hdg(\mathbb{Q})$.

Γ' is a subgroup of finite index in Γ.

I claim that $\Gamma' = \{1\}$ and, consequently, Γ is a finite group. In fact,

$$H^2(Y, \mathbb{Q})^{\Gamma'} \supset H^2(Y, \mathbb{Q})^{\Gamma'} \supset V(Y), \quad H^2(Y, \mathbb{Q})^{\Gamma'} \supset H^2(Y, \mathbb{Q})^{Hdg}.$$

It follows that

$$H^2(Y, \mathbb{Q}) \supset H^2(Y, \mathbb{Q})^{\Gamma'} \supset H^2(Y, \mathbb{Q})^{Hdg} \oplus V(Y) = H^2(Y, \mathbb{Q}).$$

This means that $\Gamma' = \{1\}$.

1. 8. Let

$F = \{u \in E \mid \text{there exists an } \alpha \in \mathbb{R} \text{ such that } ux = \alpha x \text{ for } x \in H^{2,0}(Y)\}$.

Clearly, F is a subalgebra of E.

1. 8. 1 Theorem. *F hdg $\subset$ hdg.*

1. 8. 2 *Proof.* Choose an $u \in F$. We have to prove that $u\, hdg \subset hdg$. Since $u \in F \subset E$, u commutes with f_H. Since $H^{0,2}(Y) = \bar{H}^{2,0}(Y)$ and α is real,

$$ux = \alpha x \quad \text{for} \quad x \in H^{0,2}(Y).$$

This means that

$$ux = \alpha x \quad \text{for} \quad x \in H^{2,0}(Y) \oplus H^{0,2}(Y).$$

Recall that

$$H^2(Y, \mathbb{C}) = [H^{2,0}(Y) \oplus H^{0,2}(Y)] \oplus H^{1,1}(Y)$$

is a f_H-invariant decomposition and f_H annihilates $H^{1,1}(Y)$. It follows that

$$uf_H = \alpha f_H \in \mathbb{R} f_H.$$

Since u commutes with f_H, one only has to apply (0. 3. 3. 1).

1. 9 Corollary. *Assume that $h^{2,0}(Y) = 1$. Let $E_0 = \{u \in E \mid u' = u\}$. Then $E_0\, hdg \subset hdg$, i.e. hdg is an E_0-Lie algebra. In particular, hdg is an E-Lie algebra if E is a totally real number field.*

1. 9. 1 *Proof.* E acts on $H^{2,0}(Y)$ via the homomorphism

$$\varepsilon\colon E \to \operatorname{End}_{\mathbb{C}} H^{2,0}(Y) = \mathbb{C} \quad (1. 6. 1).$$

This means that

$$ax = \varepsilon(a)x \text{ for } a \in E_0,\ x \in H^{2,0}(Y).$$

It is sufficient to prove that $\varepsilon(a) \in \mathbb{R}$ for $a \in E_0$ and to apply (1. 8. 1). Recall (1. 5. 1; 1. 6. 1) that E_0 is a totally real number field. This means that the image of any homomorphism $E_0 \to \mathbb{C}$ lies in $\mathbb{R}$. In particular, $\varepsilon(E_0) \subset \mathbb{R}$ and $\alpha = \varepsilon(a) \in \mathbb{R}$ for all $a \in E_0$. This ends the proof.

1. 9. 2 Remark. Let us decompose *hdg* into a direct sum (*hdg* is reductive)

$$hdg = hdg^s \oplus c,$$

where hdg^s is semisimple and c is the center of *hdg*. Clearly, the center c and

$$hdg^s = [hdg, hdg] = [hdg^s, hdg^s]$$

are also E_0-Lie algebras.

In particular, the E_0-Lie algebra *hdg* is a reductive subalgebra of $\mathrm{End}_{E_0} V(Y)$ Moreover, its centralizer is the field E and *hdg* is an irreducible subalgebra of $\mathrm{End}_{E_0} V(Y)$.

1. 9. 3 Remark. Let us put

$$V_\varepsilon(Y) = \{x \in V(Y)_{\mathbb{R}} \mid ax = \varepsilon(a)x \quad \text{for} \quad a \in E_0\} \subset V(Y)_{\mathbb{R}},$$

$$hdg_\varepsilon = \{u \in hdg_{\mathbb{R}} \mid au = \varepsilon(a)u \quad \text{for} \quad a \in E_0\} \subset hdg_{\mathbb{R}}.$$

Clearly, $V_\varepsilon(Y)$ is $hdg_{\mathbb{R}}$-invariant and E-invariant, hdg_ε is an ideal in $hdg_{\mathbb{R}}$,

$$f_H \in hdg_\varepsilon \subset hdg_{\mathbb{R}}, \quad hdg_\varepsilon V(Y)_{\mathbb{R}} \subset V_\varepsilon(Y).$$

If

$$f \in hdg_\varepsilon \subset hdg_{\mathbb{R}} \subset \mathrm{End}\, V(Y)_{\mathbb{R}}$$

is a semisimple endomorphism of $V(Y)_{\mathbb{R}}$ then the restriction of f to $V_\varepsilon(Y)$ is a semisimple endomorphism of $V_\varepsilon(Y)$ of the same rank ($f V(Y)_{\mathbb{R}} \subset hdg_\varepsilon V(Y)_{\mathbb{R}} \subset V_\varepsilon(Y)$).

In particular, $V_\varepsilon(Y)$ is f_H-invariant and the restriction of f_H to $V_\varepsilon(Y)$ is a semisimple endomorphism, whose rank is equal to $2h^{2,0}(Y) = 2$ (1. 2).

1. 9. 4 Remark. Since E_0 is a totally real field, the algebra $E_{0,\mathbb{R}} = E_0 \otimes_{\mathbb{Q}} \mathbb{R}$, in the usual way, becomes the direct sum of fields $R_\sigma = R$ indexed by the embeddings $\sigma\colon E_0 \hookrightarrow \mathbb{R}$, where

$$\mathbb{R}_\sigma = \{a \in E_{0,\mathbb{R}} \mid ea = \sigma(e)a \quad \text{for} \quad e \in E_0\} = E_0 \otimes_{E_0,\sigma} \mathbb{R}.$$

This implies that the $E_{0,\mathbb{R}}$-module $V(Y)_{\mathbb{R}} = V(Y) \otimes_{\mathbb{Q}} \mathbb{R} = V(Y) \otimes_{E_0} (E_0 \otimes_{\mathbb{Q}} \mathbb{R})$, in the usual way, becomes the direct sum of $\mathbb{R}(=\mathbb{R}_\sigma)$-vector space $V_\sigma(Y)$ such that

$$V_\sigma(Y) = \mathbb{R}_\sigma V(Y)_{\mathbb{R}} = \{x \in V(Y)_{\mathbb{R}} \mid ex = \sigma(e)x \quad \text{for} \quad e \in E_0\} = V(Y) \otimes_{E_0,\sigma} \mathbb{R}$$

and $\dim_{\mathbb{R}} V_\sigma(Y) = \dim_{E_0} V(Y)$ for each σ.

Similarly, the $E_{0,\mathbb{R}}$-Lie algebra $hdg_{\mathbb{R}} = hdg \otimes_{\mathbb{Q}} \mathbb{R} \subset \mathrm{End}_{E_0} V(Y) \otimes_{\mathbb{Q}} \mathbb{R}$ becomes the direct sum of $\mathbb{R}(=\mathbb{R}_\sigma)$-Lie algebras $hdg_\sigma \subset \mathrm{End}_{\mathbb{R}} V_\sigma(Y)$ such that

$$hdg_\sigma = \mathbb{R}_\sigma hdg_{\mathbb{R}} = \{u \subset hdg_{\mathbb{R}} \mid eu = \sigma(e)u \quad \text{for} \quad e \in E_0\}$$

$$= hdg \otimes_{E_0,\sigma} \mathbb{R} \quad \text{and} \quad \dim_{\mathbb{R}} hdg_\sigma = \dim_{E_0} hdg.$$

Moreover, the $hdg_{\mathbb{R}}$-module $V(Y)_{\mathbb{R}}$ is the direct sum of the faithful hdg_σ-modules $V_\sigma(Y)$ (hdg is the direct sum of hdg_σ) and (here we consider $V(Y)$ as an E_0-space)

$$\begin{aligned}\mathrm{End}_{hdg_\sigma} V_\sigma(Y) &= \mathrm{End}_{hdg_\sigma} V(Y) \otimes_{E_0,\sigma} \mathbb{R} = E \otimes_{E_0,\sigma} \mathbb{R} \\ &= \begin{cases} \mathbb{R} & \text{if } E = E_0, \\ \mathbb{C} & \text{if } E \neq E_0 \end{cases}\end{aligned}$$

(if $E \neq E_0$ then E is an imaginary quadratic extension of E_0). In particular, hdg_σ is an irreducible subalgebra of $\mathrm{End}_{\mathbb{R}} V_\sigma(Y)$.

Since $\langle ax, y\rangle = \langle x, ay\rangle$ for $a \in E_0$, $V_\sigma(Y)$ and $V_\tau(Y)$ are orthogonal with respect to $\langle\ ,\ \rangle$ if $\sigma \neq \tau$. The non-degeneracy of $\langle\ ,\ \rangle$ implies that the restriction F_σ of $\langle\ ,\ \rangle$, to $V_\sigma(Y)$ is non-degenerate for each σ. Since $hdg_{\mathbb{R}}$ preserves $\langle\ ,\ \rangle$, hdg_σ preserves F_σ.

1. 9. 5. Recall that we have the distinguished embedding $\varepsilon\colon E_0 \hookrightarrow \mathbb{R}$ (1. 9. 1, 1. 9. 3). The statements above (1. 9. 4) give us an $\mathbb{R}$-vector space $V_\varepsilon(Y)$, a non-degenerate symmetric bilinear form $F_\varepsilon\colon V_\varepsilon(Y) \times V_\varepsilon(Y) \to \mathbb{R}$, an irreducible Lie subalgebra hdg_ε of $\mathrm{End}_{\mathbb{R}} V_\varepsilon(Y)$ satisfying the following conditions:

a) $\dim_{\mathbb{R}} V_\varepsilon(Y) = \dim_{E_0} V(Y)$, $\dim_{\mathbb{R}} hdg_\varepsilon = \dim_{E_0} hdg$.

b) $\mathrm{End}_{hdg_\sigma} V_\varepsilon(Y) = \begin{cases} \mathbb{R} \text{ if } & E = E_0, \\ \mathbb{C} \text{ if } & E \neq E_0. \end{cases}$

c) hdg_ε contains a semisimple endomorphism f_H of rank 2 (1. 9. 3).

2. Surfaces with $P_g = 1$

2. 0. Everywhere in this section we assume that $P_g = h^{2,0}(Y) = 1$. The main purpose of this section is to compute explicitly the Hodge group Hdg in terms of the field $E = E_Y$. We prove that Hdg is "as maximal as possible".

2. 1. Let

$$\Phi\colon V(Y) \times V(Y) \to E, \quad x, y \mapsto \alpha$$

be the pairing defined by the formula

$$\langle ex, y\rangle = \mathrm{tr}_{E/\mathbb{Q}}(e\alpha) \quad \text{for all} \quad e \in E.$$

The existence and uniqueness of such an α follows from the non-degeneracy of the pairing $E \times E \to \mathbb{Q}$, $e_1, e_2 \mapsto \mathrm{tr}_{E/\mathbb{Q}}(e_1 e_2)$. It is easily checked that Φ is non-degenerate, $\Phi(ex, y) = e\Phi(x, y)$ and $\Phi(x, y) = [\Phi(y, x)]'$ for all $e \in E$.

This means the following (1. 5. 1, 1. 6). If E is a totally real number field then Φ is a non-degenerate symmetric E-bilinear form. If E is an imaginary quadratic extension of the totally real number field E_0 then Φ is a non-degenerate Hermitian E-sesquilinear form with respect to the complex conjugation "′". Since Hdg commutes with E and preserves $\langle\ ,\ \rangle$, Hdg preserves Φ.

Recall (1. 9) that *hdg* is an E_0-Lie algebra. Let us put

$$m = \dim_{E_0} V(Y).$$

2. 2. Assume that $E = E_0$. This means that Φ is a non-degenerate symmetric bilinear form. Consider the special orthogonal group

$$SO(V(Y), \Phi) \subset GL(V(Y))$$

of the E-vector space $V(Y)$ relative to the symmetric form Φ. *Apriori* $SO(V(Y), \Phi)$ is an algebraic group over E, but we regard it as an algebraic $\mathbb{Q}$-rational subgroup of $GL(V(Y))$. Let

$$so(V(Y), \Phi) \subset \operatorname{End} V(Y)$$

be the Lie algebra of $SO(V(Y), \Phi)$. Clearly, $so(V(Y), \Phi)$ is an E_0-Lie algebra and

$$\dim_{E_0} so(V(Y), \Phi) = \frac{m^2 - m}{2}.$$

Since *Hdg* is a connected algebraic group, which commutes with E and preserves Φ, $Hdg \subset SO(V(Y), \Phi)$ and $hdg \subset so(V(Y), \Phi)$.

2. 2. 1 Theorem. $Hdg = SO(V(Y), \Phi)$.

2. 2. 2 Remark. It is sufficient to prove that

$$\dim_{E_0} hdg = \frac{m^2 - \mathrm{m}}{2}.$$

2. 2. 3 Example. If $E = \mathbb{Q}$ then $\Phi = \langle\ ,\ \rangle$,

$$SO(V(Y), \Phi) = SO(V(Y)) \quad \text{and} \quad Hdg = SO(V(Y)).$$

2. 3. Assume that E is an imaginary quadratic extension of a totally real number field E_0. This means that Φ is a nondegenerate Hermitian E-sesquilinear form with respect to the complex conjugation "$'$". Consider the unitary group

$$U(V(Y), \Phi) \subset GL(V(Y))$$

of the E-vector space $V(Y)$ relative to the Hermitian form Φ. *A priori* $U(V(Y), \Phi)$ is an algebraic group over E_0, but we regard it as an algebraic $\mathbb{Q}$-rational subgroup of $GL(V(Y))$. Let

$$u(V(Y), \Phi) \subset \operatorname{End} V(Y)$$

be the Lie algebra of $U(V(Y), \Phi)$. Clearly, $u(V(Y), \Phi)$ is an E_0-Lie algebra and

$$\dim_{E_0} u(V(Y), \Phi) = \frac{m^2}{4}.$$

Since Hdg is an algebraic group, which commutes with E and preserves Φ, $Hdg \subset U(V(Y), \Phi)$ and $hdg \subset u(V(Y), \Phi)$.

2.3.1 Theorem. $Hdg = U(V(Y), \Phi)$.

2.3.2 Remark. It is sufficient to prove that

$$\dim_{E_0} hdg = \frac{m^2}{4}.$$

2.4. In the course of the proof we shall use the following result [7].

2.4.1 Theorem (of Kostant). *Let W be an m-dimensional $\mathbb{R}$-vector space, $\varphi\colon W \times W \to \mathbb{R}$ a nondegenerate symmetric bilinear form, $g \subset \operatorname{End} W$ an irreducible Lie algebra, which preserves φ. Assume that g contains a semisimple endomorphism f of rank 2.*

Then one of the following statements is true.

I. $\operatorname{End}_g W = \mathbb{R}$ *and g is the Lie algebra $so(W, \varphi)$ of the special orthogonal group of W relative to the symmetric form φ.*

II. $\operatorname{End}_g W = \mathbb{C}$ *and one can define on the $\mathbb{C}$-space W a non-degenerate Hermitian form*

$$\varphi'\colon W \times W \to \mathbb{C}, \quad x, y \mapsto a$$

by the formula

$$\varphi(bx, y) = \operatorname{tr}_{C/\mathbb{R}}(ba) = ba + \overline{ba} \text{ for all } b \in C.$$

Then g is the Lie algebra $u(W, \varphi')$ of the unitary group of W relative to the Hermitian form φ'.

2.4.2 Corollary. 1) *If* $\operatorname{End}_g W = \mathbb{R}$ *then*

$$\dim_{\mathbb{R}} g = \frac{m^2 - m}{2};$$

2) *If* $\operatorname{End}_g W = \mathbb{C}$ *then*

$$\dim_{\mathbb{R}} g = \frac{m^2}{4}.$$

2.5 *Proof* of the theorems 2.2.1 and 2.3.1. According to (2.2.2, 2.3.2) it is sufficient to prove that

$$\dim_{E_0} hdg = \begin{cases} \dfrac{m^2 - m}{2} & \text{if } E = E_0, \\[2mm] \dfrac{m^2}{4} & \text{if } E \neq E_0. \end{cases}$$

Applying (2. 4. 1; 2. 4. 2) to $W = V_\varepsilon(Y)$, $\varphi = F_\varepsilon$, $g = hgd_\varepsilon$, $f = f_H$ (1. 9. 5) we obtain the following (2. 5; 2. 5. 1).

$$\dim_{\mathbb{R}} hdg_\varepsilon = \begin{cases} \dfrac{m^2 - m}{2} & \text{if } E = E_0, \\ \dfrac{m^2}{4} & \text{if } E \neq E_0, \end{cases}$$

where $m = \dim_{E_0} V(Y) = \dim_{\mathbb{R}} V_\varepsilon(Y)$ (1. 9. 5a).

But $\dim_{E_0} hdg = \dim_{\mathbb{R}} hdg_\varepsilon$ (1. 9. 5a). This ends the proof.

2. 6 Remarks. a) One may generalize both theorems to hold for any rational polarized Hodge structure of even weight $n = 2r$ with level 2, whose Hodge number $h^{r+1, r-1} = 1$. In particular, the statements of the theorems hold for the Hodge group of the rational Hodge structure attached to the fourth rational cohomology group of a cubic fourfold.

b) Let Y be a $K3$ surface. Then $Hdg = Hdg\, Y$. Using our theorems and classical invariant theory one may explicitly define all Hodge classes on the products $Y \times \cdots \times Y$. In particular, if $E = \mathbb{Q}$ then all Hodge classes are algebraic.

c) Analogues of our theorems are true for Hodge-Tate modules. For example, one may apply our methods to $K3$ surfaces and cubic fourfolds over local fields, if the cohomology of the variety in question admits the Hodge-Tate decomposition. In fact, our methods were inspired by ideas of Serre and Sen [13], who treated Hodge modules of special type.

2. 7 *Proof* of the theorem of Kostant. Since g is irreducible, $D = \operatorname{End}_g W$ is a division algebra. Since $f \in g$ commutes with D, the $\mathbb{R}$-subspace $\operatorname{Im} f$ of W is D-invariant and we have a non-trivial homomorphism $D \to \operatorname{End}_{\mathbb{R}}(\operatorname{Im} f) \approx M_2(\mathbb{R})$. This implies that $D = \mathbb{R}$ or $D = \mathbb{C}$. If $D = \mathbb{R}$ then g is an absolutely irreducible subalgebra of $\operatorname{End}_{\mathbb{R}} W$. If $D = \mathbb{C}$ then $\operatorname{Im} f$ is a one-dimensional $\mathbb{C}$-vector space.

Case I. Assume $D = \mathbb{R}$ and g to be absolutely irreducible. Then $g_{\mathbb{C}} = g \otimes_{\mathbb{R}} \mathbb{C}$ is an irreducible subalgebra of $\operatorname{End}_{\mathbb{C}} W_{\mathbb{C}}$, where $W = W \otimes_{\mathbb{R}} \mathbb{C}$. Extend φ to $W_{\mathbb{C}}$ (unique) as a complex bilinear form. Of course, $g_{\mathbb{C}}$ preserves φ and lies in the Lie algebra $so(W_{\mathbb{C}}, \varphi) = so(W, \varphi) \otimes_{\mathbb{R}} \mathbb{C}$ of the corresponding special orthogonal group. Now, $g_{\mathbb{C}}$ is an irreducible subalgebra of $\operatorname{End}_{\mathbb{C}} W_{\mathbb{C}}$, leaving invariant φ and containing a semisimple endomorphism f of rank 2 ($f \in g \subset g_{\mathbb{C}}$). Hence, by [7], p. 116 $g_{\mathbb{C}} = so(W_{\mathbb{C}}, \varphi)$ and $g = so(W, \varphi)$.

Case II. Assume $D = \mathbb{C}$ and $\operatorname{Im} f$ to be a one-dimensional $\mathbb{C}$-space. Regard W as a $\mathbb{C}$-space and define an Hermitian form φ' as in (2. 4. 1, II). Clearly, $g \subset u(W, \varphi')$. To prove (2. 4. 1, II) it suffices to show that

$$\dim_{\mathbb{R}} g \geqq \dim_{\mathbb{R}} u(W, \varphi') = \frac{m^2}{4} = \dim_{\mathbb{C}} \operatorname{End}_{\mathbb{C}} W.$$

Let us consider the complex Lie algebra

$$g' = \mathbb{C}g = \{u + iv \mid u, v \in g\} \subset \mathrm{End}_{\mathbb{C}} W.$$

Clearly, $g' \supset g \ni f$ and $\dim_{\mathbb{R}} g \geqq \dim_{\mathbb{C}} g'$. Since g' contains an irreducible subalgebra g of $\mathrm{End}_{\mathbb{R}} W$, g' is an irreducible subalgebra of $\mathrm{End}_{\mathbb{C}} W$, containing a semisimple endomorphism f of rank 1 $(\dim_{\mathbb{C}}(\mathrm{Im} f) = 1)$. Hence, by [7], p. 116

$$g' = \mathrm{End}_{\mathbb{C}} W \quad \text{and} \quad \dim_{\mathbb{R}} g \geqq \dim_{\mathbb{C}} g' = \frac{m^2}{4}.$$

3. Families of surfaces with $P_g = 1$

3. 0. In this paper a family of sufaces is a smooth projective morphism of relative dimension 2

$$f: X \to S$$

of smooth connected schemes of finite type over $\mathbb{C}$ with connected geometric fibres. For the sake of simplicity we assume that there exists a closed embedding (for some n)

$$X \hookrightarrow \mathbb{P}^n \times S$$

such that f is induced by the projection map $\mathbb{P}^n \times S \to S$. Let L be the invertible sheaf on X which is the inverse image of $O(1)$ on $\mathbb{P}^n$. Then L is relatively ample for f. Moreover, L induces a very ample invertible sheaf L_s on each geometric fibre $X_s = f^{-1}(s)$. We call such an L a polarization. Notice that all geometric fibres X_s are irreducible smooth projective surfaces.

Further, we consider only $\mathbb{C}$-points of S, i.e. the points of a corresponding complex variety S^{an}. Notice that the Hodge numbers

$$h^{p,q}(X_s) = \dim_{\mathbb{C}} H^q(X_s, \Omega^p_{X_{s/\mathbb{C}}})$$

do not depend upon the choice of s [4]. In particular, the validity of the statement "$h^{2,0}(X_s) = 1$" does not depend upon the choice of s.

There is the locally constant sheaf (the local system) $R^2 f_* \mathbb{Q}$ on S^{an} [3]. Its fibre at s is equal to $H^2(X_s, \mathbb{Q})$. The sheaf is defined by the representations of the fundamental group

$$\pi_1(S, s) \to \mathrm{Aut}\, H^2(X_s) \subset \mathrm{Aut}\, H^2(X_s, \mathbb{Q}).$$

Let us denote the image of the fundamental group in $\mathrm{Aut}\, H^2(X_s)$ by Γs. Γs is called the global monodromy group of the family. Notice that Γs preserves the lattice $H^2(X_s)$ and the intersection form $\langle\ ,\ \rangle$ on $H^2(X_s, \mathbb{Q})$.

Let t be another point of S. Then the path from s to t defines the natural isomorphisms

$$H^2(X_s) \xrightarrow{\sim} H^2(X_t), \quad H^2(X_s, \mathbb{Q}) \xrightarrow{\sim} H^2(X_t, \mathbb{Q}), \quad \Gamma s \xrightarrow{\sim} \Gamma t,$$

compatible with the intersection forms, and the diagram

$$\begin{array}{ccc} \pi_1(S, s) & \xrightarrow{\sim} & \pi_1(S, t) \\ \downarrow & & \downarrow \\ \Gamma s & \xrightarrow{\sim} & \Gamma t \\ \downarrow & & \downarrow \\ \operatorname{Aut} H^2(X_s) & \xrightarrow{\sim} & \operatorname{Aut} H^2(X_t) \\ \downarrow & & \downarrow \\ \operatorname{Aut} H^2(X_s, \mathbb{Q}) & \xrightarrow{\sim} & \operatorname{Aut} H^2(X_t, \mathbb{Q}) \end{array}$$

is commutative. The choice of another path from s to t multiplies (from the right) the isomorphism $H^2(X_s, \mathbb{Q}) \xrightarrow{\sim} H^2(X_t, \mathbb{Q})$ with the automorphism $u \in \Gamma t \subset \operatorname{Aut} H^2(X_t, \mathbb{Q})$. Here u is the image of the loop composed of the two paths. This implies the following statements.

3. 0. 1. For all points s, t of the base S the groups Γs and Γt are isomorphic. In particular, the validity of the statements "$\Gamma s = 1$", "Γs is finite" does not depend upon the choice of s. There are natural isomorphisms (3. 0)

$$H^2(X_s) \xrightarrow{\sim} H^2(X_t), \quad H^2(X_s, \mathbb{Q}) \xrightarrow{\sim} H^2(X_t, \mathbb{Q})$$

compatible with the intersection forms and depending only on the choice of the path from s to t.

3. 0. 2. Let us fix a polarization L on X and let $l_s \in H^2(X_s, \mathbb{Q})$ be the class of the very ample invertible sheaf L_s on X_s. l_s goes to l_t under the isomorphism $H^2(X_s) \xrightarrow{\sim} H^2(X_t)$ and the positive integer $d = \langle l_s, l_s \rangle$ does not depend upon the choice of s. In particular, $l_s \in H^2(X_s)^{\Gamma s} = H^2(X_s)^{\pi_1(S, s)}$. As in (1. 2. 2) let us denote the orthogonal complement of l_s in $H^2(X_s)$ with respect to $\langle\ ,\ \rangle$ by $P(X_s)$. The $\pi_1(S, s)$-invariance of l_s and $\langle\ ,\ \rangle$ implies the $\pi_1(S, s)$-invariance of $P(X_s)$. This implies the $\pi_1(S, s)$-invariance of the $\mathbb{Z}$-lattice $\mathbb{Z} l_s + P(X_s)$ in the $\mathbb{Q}$-space $H^2(X_s, \mathbb{Q})$. It follows that $\{\mathbb{Z} l_s + P(X_s)\}_{s \in S}$ is a local system on S^{an}.

3. 0. 3 Recall (1. 2. 2) that l_s gives rise to the polarization of the rational Hodge structure of weight 2 on $H^2(X_s, \mathbb{Q})$ and also on the $\mathbb{Z}$-lattice $\mathbb{Z} l_s + P(X_s)$ in $H^2(X_s, \mathbb{Q})$ (1. 2. 2). Moreover, $\{\mathbb{Z} l_s + P(X_s)\}_{s \in S}$ gives us a holomorphic family of polarized Hodge structures of weight 2, because this family is the direct sum of the holomorphic family $\{P(X_s)\}_{s \in S}$ and the constant family $\{\mathbb{Z} l_s\}_{s \in S}$.

Recall (1. 1) that

$$A_s \;= A(X_s) = H^2(X_s) \cap H^{1,1}(X_s),$$

$$A_{s,\mathbb{Q}} = A(X_s)_{\mathbb{Q}} = H^2(X_s, \mathbb{Q}) \cap H^{1,1}(X_s)$$

are the $\mathbb{Z}$ and $\mathbb{Q}$-lattices of algebraic cycles. It is known ([3], Sect. 4. 1. 2) that the $\mathbb{Q}$-subspace $H^2(X_s, \mathbb{Q})^{\pi_1(S,s)} = H^2(X_s, \mathbb{Q})^{\Gamma s}$ is a rational Hodge substructure of $H^2(X_s, \mathbb{Q})$, which does not depend on s. This means that the $\mathbb{R}$-subspace $H^2(X_s, \mathbb{Q})^{\Gamma s} \otimes \mathbb{R}$ is $\mathbb{S}$-invariant and the natural isomorphisms (3. 0. 1)

$$H^2(X_s, \mathbb{Q})^{\Gamma s} \otimes \mathbb{R} \xrightarrow{\sim} H^2(X_t, \mathbb{Q})^{\Gamma t} \otimes \mathbb{R}$$

commute with the action of $\mathbb{S}$. It follows that the validity of the statements

$$\text{“}H^2(X_s, \mathbb{Q})^{\pi_1(S,s)} = H^2(X_s, \mathbb{Q})^{\Gamma s} \subset H^{1,1}(X_s) \cap H^2(X_s, \mathbb{Q}) = A_{s,\mathbb{Q}}\text{”}$$

and

$$\text{“}H^2(X_s)^{\pi_1(S,s)} = H^2(X_s)^{\Gamma s} \subset H^{1,1}(X_s) \cap H^2(X_s) = A_s\text{”}$$

does not depend upon the choice of $s \in S$.

3. 1 Theorem (Algebraicity of invariant cycles, a geometric analogue of the Tate's conjecture). *Let s be a point of S and $h^{2,0}(X_s) = 1$. If the global monodromy group Γs is infinite then*

$$H^2(X_p)^{\pi_1(S,s)} = H^2(X_s)^{\Gamma s} \subset A_s$$

and

$$H^2(X_s, \mathbb{Q})^{\pi_1(S,s)} = H^2(X_s, \mathbb{Q})^{\Gamma s} \subset A_{s,\mathbb{Q}}.$$

3. 1. 0 Remarks. a) The case of a noncomplete family with an infinite local monodromy group was treated in [16]. Since any local monodromy group is imbedded in the global one, the theorem 3. 1 implies the corresponding results of [16].

The case of a family of Abelian Surfaces was treated in [3].

b) In the statement of theorem 3.1 one may drop the assumption that f is a family of *surfaces*. An analogue of the theorem holds for families of cubic fourfolds.

3. 2 *Proof.* Recall (3. 0, 3. 0. 1, 3. 0. 3) that the validity of the assumptions and the statement of the theorem does not depend upon the choice of a point s. Let us choose s such that Γs contains a subgroup of finite index, lying in $MT_s(\mathbb{Q})$. Here $MT_s = MT_{2,X_s}$ is the Mumford-Tate group of the rational Hodge structure on $H^2(X_s, \mathbb{Q})$ (1. 2). The existence of such a point s is a corollary of results of Déligne [5], proposition 7. 5, p. 225, applied to the holomorphic family of polarized Hodge structures $\{\mathbb{Z}l_s + P(X_s)\}_{s \in S}$ of weight 2 (3. 0. 3).

Let us denote the Zarisky-closure of Γs in $\operatorname{Aut} H^2(S_s, \mathbb{C})$ by G and let G^0 be the connected identity component of G. It is known ([3], Sect. 4. 2. 9) that G^0 is semisimple. Applying the theorem 1. 7 to the surface $Y = X_s$ and the group $\Gamma = \Gamma s$ we obtain that either Γs is finite or

$$H^2(X_s, \mathbb{Q})^{\Gamma s} \subset A(X_s)_{\mathbb{Q}} = A_{s,\mathbb{Q}}, \quad H^2(X_s)^{\Gamma s} \subset A(X_s) = A_s.$$

However by assumption Γs is infinite.

4. Families of $K3$ surfaces

4. 0. Recall [1], ch. 8 that an irreducible smooth projective surface Y over $\mathbb{C}$ is called a $K3$ surface if $H^1(Y, O_Y)=0$ and $\Omega_Y^2 \approx O_Y$ If Y is a $K3$ surface then $h^{2,0}(Y)=1$. The Hodge duality implies that there is no non-trivial vector field on Y ([1], Ch. 8). If M is a very ample invertible sheaf on a $K3$ surface Y then the Kodaira vanishing theorem and the Serre duality imply that

$$H^i(Y, M)=0 \quad \text{for} \quad i>0$$

4. 1. Theorem. *Let $f\colon X \to S$ be a family of $K3$ surfaces, i.e. for all $s \in S$ the fibres $X_s = f^{-1}(s)$ are $K3$ surfaces. Let us assume that for some $t \in S$ the global monodromy group Γt is finite.*

Then the family $f\colon X \to S$ is isotrivial, i.e. there exists an étale morphism $T \to S$ such that the T-scheme $X \times_S T$ is isomorphic to the constant family $Y \times T$, for $Y = X_t$.

4. 0. 1. The theorems 3. 1 and 4. 1 give us the following corollary.

4. 1. 1 Corollary. *Let $f\colon X \to S$ be a non-isotrivial family of $K3$ surfaces. Then for all $s \in S$.*

$$H^2(X_s, \mathbb{Q})^{\pi_1(S,s)} \subset A_{s,\mathbb{Q}}, \quad H^2(X_s)^{\pi_1(S,s)} \subset A_s.$$

4. 1. 2 Remarks. Simultaneously and independently the corollary 4. 1. 1 was also proved by G. A. Mustafin, who used different methods. Earlier, the case of a one-dimensional non-complete family with stable reduction was treated in [16].

4. 2 *Proof* of the theorem 4. 1. Replacing if necessary S by a finite etale covering S' and X by $X \times_S S'$ we may assume that $\Gamma t = 1$. This implies that $\Gamma s = 1$ for all $s \in S$ (3. 0. 1). It follows that the natural isomorphism

$$H^2(X_s, \mathbb{Q}) = H^2(X_s, \mathbb{Q})^{\Gamma s} \xrightarrow{\sim} H^2(X_t, \mathbb{Q})^{\Gamma t} = H^2(X_t, \mathbb{Q})$$

is an isomorphism of rational Hodge structures, which does not depend on the choice of a path from s to t. This gives us a natural isomorphism

$$H^2(X_s) \xrightarrow{\sim} H^2(X_t)$$

compatible with the intersection form.

4. 2. 1. Let us fix an embedding $X \hookrightarrow \mathbb{P}^n \times S$ as in (3. 0) and let the relatively ample invertible sheaf on X be the corresponding polarization. Let $l_s \in H^2(X_s, \mathbb{Q})$ be a class of the very ample invertible sheaf L_s on X_s. We know (3. 0. 2) that l_s goes to l_t under the isomorphism

$$H^2(X_s, \mathbb{Q}) = H^2(X_s, \mathbb{Q})^{\Gamma s} \xrightarrow{\sim} H^2(X_t, \mathbb{Q})^{\Gamma t} = H^2(X_t, \mathbb{Q}).$$

Applying the global Torelli theorem for polarized $K3$ surfaces [11] we obtain an isomorphism between the polarized $K3$ surfaces (X_s, L_s) and (Y, M) for $Y = X_t$ and $M = L_t$. This means that there exists an isomorphism $X_s \xrightarrow{\sim} Y$ such that L_s is isomorphic to the inverse image of M. Let us put

$$m = \dim \Gamma(Y, M) = \dim \Gamma(X_s, L_s).$$

Since $M = L_t$ is very ample,

$$H^i(Y, M) = H^i(X_s, L_s) = 0 \quad \text{for} \quad i > 0 \quad (4.0).$$

Since f is a projective morphism, the direct image $E = f_* L$ is a coherent sheaf on S. Replacing S by a non-empty affine open subscheme we may assume that $E = f_* L$ is free. Recall that

$$\Gamma(f^{-1}(V), L) = \Gamma(V, f_* L) = \Gamma(V, E)$$

for any open subset V of S. Since E is free, this implies the flatness of L because an inductive limit of free modules is a flat module.

Recall that the function

$$s \mapsto \dim \Gamma(X_s, L_s) = m$$

is constant on $\mathbb{C}$-points of S. Since $\mathbb{C}$-points are dense in any closed subset of S (Hilbert Nullstellensatz), the base change theorems for cohomology of coherent sheaves ([8], Ch. 2, § 5) imply that $R^i f_* L = 0$ for $i > 0$ and that $E = f_* L$ is a free sheaf of rank m with fibres $\Gamma(X_s, L_s)$ at s.

Since S is affine, $\Gamma(X, L) = \Gamma(S, f_* L) = \Gamma(S, E)$ is a free $\Gamma(S, O_s)$-module of rank m and the module of global sections $\Gamma(X, L) = \Gamma(S, E)$ generates $\Gamma(X_s, L_s) = (f_* L)_s = E_s$. In particular, the sections of $E = f_* L$ have no common zeroes in $X(\mathbb{C})$. It follows that they have no common zeroes in X (Hilbert Nullstellensatz).

4. 2. 2. Let us fix an isomorphism $E \xrightarrow{\sim} O_s^m$. This gives us a basis of the $\Gamma(S, O_s)$-module $\Gamma(S, E) = \Gamma(X, L)$ and a closed embedding

$$X \hookrightarrow \mathbb{P}(E) \xrightarrow{\sim} \mathbb{P}^{m-1} \times S,$$

inducing the natural closed embedding

$$X_s \hookrightarrow \mathbb{P}[\Gamma(X_s, L_s)] \xrightarrow{\sim} \mathbb{P}^{m-1}.$$

We shall identify X with its image in $\mathbb{P}^{m-1} \times S$ and view all X_s as subvarieties of $\mathbb{P}^{m-1}$.

Let us fix a basis of $\Gamma(Y, M)$. This gives us a closed embedding

$$Y \hookrightarrow \mathbb{P}[\Gamma(Y, M)] \xrightarrow{\sim} \mathbb{P}^{m-1}.$$

We shall identify Y with its image in $\mathbb{P}^{m-1}$.

4. 2. 3. Let $G = PGL(m)$ be the projective linear group. There is a natural action

$$G \times \mathbb{P}^{m-1} \longrightarrow \mathbb{P}^{m-1}.$$

Since (X_s, L_s) and (Y, M) are isomorphic, there exists an $g \in G(\mathbb{C})$ such that $gY = X_s (Y, X_s \subset \mathbb{P}^{m-1})$. Clearly, the set

$$K_s = \{g \in G(\mathbb{C}) \mid gY = X_s\}$$

is a principal homogeneous space over the automorphism group $\mathrm{Aut}(Y, M)$ of polarized $K3$ surface, which is finite ([11], § 2).

Let us put $Y_S = Y \times S$. The action of G on $\mathbb{P}^{m-1}$ gives rise to the action

$$(G \times S) \times (\mathbb{P}^{m-1} \times S) \to \mathbb{P}^{m-1} \times S, \quad (g, p) \mapsto gp$$

of the S-group scheme $G_S = G \times S$ on $\mathbb{P}^{m-1} \times S$.

Consider the functor F which assigns to every S-scheme T the set

$$\{g \in (G \times S)(T) = \mathrm{Hom}_S(T, G \times S) \mid g Y_T = X_T \subset (P^{m-1} \times S) \times_S T\},$$

where $Y_T = Y \times T = Y_S \times_S T$, $X_T = X \times_S T$.

For example, for any $\mathbb{C}$-point s: $\mathrm{Spec}\,\mathbb{C} \to S$

$$F(s) = \{g \in G(C) \mid g Y = X_s\} = K_s.$$

Clearly, if for some T, $F(T)$ is nonempty then the T-scheme X_T is isomorphic to Y_T. Moreover, for any $g_1, g_2 \in F(T)$ there exists exactly one automorphism u of Y_T such that $g_2 = g_1 u$.

Notice that $F(T) = F_1(T) \cap F_2(T)$ where F_1, F_2 are the functors which assign to every S-scheme T the sets

$$F_1(T) = \{g \in (G \times S)(T) \mid g Y_T \subset X_T\},$$
$$F_2(T) = \{g \in (G \times S)(T) \mid g^{-1} X_T \subset Y_T\}$$

respectively. Clearly, the functors F_1, F_2 are represented by the schemes T_1, T_2 respectively, which are defined by the cartesian squares

$$\begin{array}{ccc} T_1 & \longrightarrow & (G \times S) \times_S Y_S \\ \downarrow & & \downarrow \varphi \\ X & \longrightarrow & \mathbb{P}^{m-1} \times S, \end{array} \qquad \begin{array}{ccc} T_2 & \longrightarrow & (G \times S) \times_S X \\ \downarrow & & \downarrow \psi \\ Y_S & \longrightarrow & \mathbb{P}^{m-1} \times S. \end{array}$$

Here φ is the restriction of

$$(G \times S) \times_S (\mathbb{P}^{m-1} \times S) \mapsto \mathbb{P}^{m-1} \times S, \quad (g, p) \mapsto gp$$

and ψ is the restriction of

$$(G \times S) \times_S (\mathbb{P}^{m-1} \times S) \mapsto \mathbb{P}^{m-1} \times S, \quad (g, p) \mapsto g^{-1} p.$$

Obviously, F is represented by

$$T_0 = T_1 \times_{G \times S} T_2.$$

This means that the T-scheme X_T is isomorphic to Y_T for any S-scheme T such that $\mathrm{Hom}_S(T, T_0)$ is non-empty.

Let r: $T_0 \to S$ be the structure morphism. I claim that r is unramified. In order to prove this it is sufficient to check only $\mathbb{C}$-points (any morphism is unramified at an open subset).

Let s: Spec $\mathbb{C} \to S$ be a $\mathbb{C}$-point of S and s': Spec $\mathbb{C}[a]/(a^2) \to S$ be a "thickening" of s, i.e. s is a composition of s' and the natural map Spec $\mathbb{C} \to$ Spec $\mathbb{C}[a]/(a^2)$. The unramifiedness means that the natural map $F(s') \to F(s)$ is an injection. Let $g_1, g_2 \in F(s')$. Then $g_2 = g_1 u$ for some $u \in \operatorname{Aut} Y_{s'} = \operatorname{Aut}(Y \times \mathbb{C}[a]/(a^2))$. If g_1 and g_2 go into the same element of $F(s) = K_s$ then u acts as the identity map on Y and may be identified with a vector field on Y. But there is no non-trivial vector field on Y (4. 0). This implies that u is the identity map and $g_2 = g_1$.

Since $F(s) = K_s$ is non-empty for any $\mathbb{C}$-point $s \to S$, the map $T_0(\mathbb{C}) \to S(\mathbb{C})$ is surjective. This implies that for any open nonempty subset V in S the open subset $r^{-1}(V)$ is also non-empty. Let us choose a non-empty open V in S such that the restriction of r

$$r': r^{-1}(V) \to V$$

is a flat morphism. Clearly, r' is unramified and, consequently, étale. Let us put $T = r^{-1}(V)$. Clearly, $F(T) = \operatorname{Hom}_S(T, T_0)$ is non-empty and, consequently, $X_T = X \times_S T$ is isomorphic to $Y_T = Y \times T$.

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