J. ALGEBRAIC GEOMETRY
**17** (2008) 481–502
S 1056-3911(07)00471-7
Article electronically published on December 10, 2007

# A FINITENESS THEOREM FOR THE BRAUER GROUP OF ABELIAN VARIETIES AND $K3$ SURFACES

ALEXEI N. SKOROBOGATOV AND YURI G. ZARHIN

**Abstract**

Let $k$ be a field finitely generated over the field of rational numbers, and $\mathrm{Br}(k)$ the Brauer group of $k$. For an algebraic variety $X$ over $k$ we consider the cohomological Brauer–Grothendieck group $\mathrm{Br}(X)$. We prove that the quotient of $\mathrm{Br}(X)$ by the image of $\mathrm{Br}(k)$ is finite if $X$ is a $K3$ surface. When $X$ is an abelian variety over $k$, and $\overline{X}$ is the variety over an algebraic closure $\overline{k}$ of $k$ obtained from $X$ by the extension of the ground field, we prove that the image of $\mathrm{Br}(X)$ in $\mathrm{Br}(\overline{X})$ is finite.

## 1. Introduction

Let $X$ be a geometrically integral smooth projective variety over a field $k$. The Tate conjecture for divisors on $X$ [30, 32, 34] is well known to be closely related to the finiteness properties of the cohomological Brauer–Grothendieck group $\mathrm{Br}\,(X) = \mathrm{H}^2_{\acute{e}t}(X, \mathbf{G}_m)$. This fact was first discovered in the case of a finite field $k$ by Artin and Tate ([31], see also Milne [18]) who studied the Brauer group of a surface. In particular, the order of $\mathrm{Br}\,(X)$ appears in the formula for the leading term of the zeta function of $X$. A stronger variant of the Tate conjecture for divisors concerns the order of the pole of the zeta function of $X$ at $s = 1$; see [30, (12) on p. 101]. It implies the finiteness of the prime-to-$p$ component of $\mathrm{Br}\,(X)$, where $X$ is a variety of arbitrary dimension, and $k$ is a finite field of characteristic $p$, as proved in [40, Sect. 2.1.2 and Remark 2.3.11].

Since Manin observed that the Brauer group of a variety over a number field provides an obstruction to the Hasse principle [17], the Brauer groups of varieties over fields of characteristic 0 have been intensively studied. Most of the existing literature is devoted to the so-called algebraic part $\mathrm{Br}_1(X)$ of $\mathrm{Br}\,(X)$, defined as the kernel of the natural map $\mathrm{Br}\,(X) \to \mathrm{Br}\,(\overline{X})$, where $\overline{X} = X \times_k \overline{k}$, and $\overline{k}$ is a separable closure of $k$. Meanwhile, if $k$ is a number

Received May 13, 2006 and, in revised form, October 12, 2006.

field, the classes surviving in $\mathrm{Br}\,(\overline{X})$ can produce a non-trivial obstruction to the Hasse principle and weak approximation (see [12] and [36] for explicit examples). Therefore, such arithmetic applications require the knowledge of the whole Brauer group $\mathrm{Br}\,(X)$.

To state and discuss our results we introduce some notation and conventions. In this paper the expression 'almost all' means 'all but finitely many'. If $B$ is an abelian group, we denote by $B_{\mathrm{tors}}$ the torsion subgroup of $B$, and write $B/_{\mathrm{tors}} := B/B_{\mathrm{tors}}$. For a prime $\ell$ let $B(\ell)$ be the subgroup of $B_{\mathrm{tors}}$ consisting of the elements whose order is a power of $\ell$, and $B(\mathrm{non}{-}\ell)$ the subgroup of $B_{\mathrm{tors}}$ consisting of the elements whose order is *not* divisible by $\ell$. If $m$ is a positive integer, we write $B_m$ for the kernel of the multiplication by $m$ in $B$.

Let $\mathrm{Br}_0(X)$ be the image of the natural map $\mathrm{Br}\,(k) \to \mathrm{Br}\,(X)$. Recall that both $\mathrm{Br}\,(X)$ and $\mathrm{Br}\,(\overline{X})$ are torsion abelian groups whenever $X$ is smooth; see [11, II, Prop. 1.4]. There are at least three reasons why the Brauer group $\mathrm{Br}\,(X)$ can be infinite: $\mathrm{Br}_0(X)$ may well be infinite; the quotient $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ injects into, and is often equal to, $\mathrm{H}^1(k, \mathrm{Pic}\,(\overline{X}))$, which may be infinite if the divisible part of $\mathrm{Pic}\,(\overline{X})$ is non-zero, or if there is torsion in the Néron–Severi group $\mathrm{NS}\,(\overline{X})$; finally, $\mathrm{Br}\,(\overline{X})$ may be infinite. This prompts the following question.

**Question 1.** *Is* $\mathrm{Br}\,(X)/\mathrm{Br}_1(X)$ *finite if* $k$ *is finitely generated over its prime subfield?*

Let $\Gamma = \mathrm{Gal}\,(\overline{k}/k)$, and let $\mathrm{Br}\,(\overline{X})^\Gamma$ be the subgroup of Galois invariants of $\mathrm{Br}\,(\overline{X})$; then $\mathrm{Br}\,(X)/\mathrm{Br}_1(X)$ naturally embeds into $\mathrm{Br}\,(\overline{X})^\Gamma$. A positive answer to Question 1 would follow from a positive answer to the following question.

**Question 2.** *Is* $\mathrm{Br}\,(\overline{X})^\Gamma$ *finite if* $k$ *is finitely generated over its prime subfield?*

In this note we prove the following two theorems.

**Theorem 1.1.** *Let* $k$ *be a field finitely generated over its prime subfield. Let* $X$ *be a principal homogeneous space of an abelian variety over* $k$*.*

(i) *If the characteristic of* $k$ *is* $0$*, then* $\mathrm{Br}\,(\overline{X})^\Gamma$ *and* $\mathrm{Br}\,(X)/\mathrm{Br}_1(X)$ *are finite.*

(ii) *If the characteristic of* $k$ *is a prime* $p \neq 2$*, then* $\mathrm{Br}\,(\overline{X})^\Gamma(\mathrm{non}{-}p)$ *and* $(\mathrm{Br}\,(X)/\mathrm{Br}_1(X))(\mathrm{non}{-}p)$ *are finite.*

**Theorem 1.2.** *Let* $k$ *be a field finitely generated over* $\mathbf{Q}$*. If* $X$ *is a* $K3$ *surface over* $k$*, then the groups* $\mathrm{Br}\,(\overline{X})^\Gamma$ *and* $\mathrm{Br}\,(X)/\mathrm{Br}_0(X)$ *are finite.*

**Remark 1.3.** The injective maps

$$\mathrm{Br}\,(X)/\mathrm{Br}_1(X) \hookrightarrow \mathrm{Br}\,(\overline{X})^\Gamma \quad \text{and} \quad \mathrm{Br}_1(X)/\mathrm{Br}_0(X) \hookrightarrow \mathrm{H}^1(k, \mathrm{Pic}\,(\overline{X}))$$

can be computed via the Hochschild–Serre spectral sequence

$$\mathrm{H}^p(k, \mathrm{H}^q_{\acute{e}t}(\overline{X}, \mathbf{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\acute{e}t}(X, \mathbf{G}_m).$$

(A description of some of its differentials can be found in [26].) Recall that in characteristic zero the Picard group $\mathrm{Pic}\,(\overline{X})$ of a $K3$ surface $X$ is a free abelian group of rank at most 20. The Galois group $\Gamma$ acts on $\mathrm{Pic}\,(\overline{X})$ via a finite quotient, so that $\mathrm{H}^1(k, \mathrm{Pic}\,(\overline{X}))$ is finite. Thus in order to prove Theorem 1.2 it suffices to establish the finiteness of $\mathrm{Br}\,(\overline{X})^\Gamma$.

In the case when the rank of $\mathrm{Pic}\,(\overline{X})$ equals 20, Theorem 1.2 was proved by Raskind and Scharaschkin [23]. In an unpublished note, J.-L. Colliot-Thélène proved that $\mathrm{Br}\,(\overline{X})^\Gamma(\ell)$ is finite for every prime $\ell$, where $X$ is a smooth projective variety over a field finitely generated over $\mathbf{Q}$, assuming the Tate conjecture for divisors on $X$. (When $\dim(X) > 2$, he assumed additionally the semisimplicity of the Galois action on $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_\ell)$.) See also [29] for some related results.

When $X$ is an abelian variety over a field finitely generated over its prime subfield, the Tate conjecture for divisors on $X$ (and the semisimplicity of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_\ell)$ for $\ell \neq p$) was proved by the second named author in characteristic $p > 2$ [37, 38], and by Faltings in characteristic zero [8, 9]. This result of Faltings combined with the construction of Kuga–Satake elaborated by Deligne [3], implies the Tate conjecture for divisors on $K3$ surfaces in characteristic zero [34, p. 80].

The novelty of our approach is due to the usage of a variant of the Tate conjecture for divisors on $X$ [39, 41] which concerns the Galois invariants of the (twisted) second étale cohomology group with coefficients in $\mathbf{Z}/\ell$ (instead of $\mathbf{Q}_\ell$), for almost all primes $\ell$. Using this variant we prove that under the conditions of Theorems 1.1 and 1.2 we have $\mathrm{Br}\,(\overline{X})^\Gamma_\ell = \{0\}$ for almost all primes $\ell$.

Let $k$ be a number field, $X(\mathbb{A}_k)$ the space of adelic points of $X$, and $X(\mathbb{A}_k)^{\mathrm{Br}}$ the subset of adelic points orthogonal to $\mathrm{Br}\,(X)$ with respect to the Brauer–Manin pairing (given by the sum of local invariants of an element of $\mathrm{Br}\,(X)$ evaluated at the local points; see [17]). We point out the following corollary to Theorem 1.2.

**Corollary 1.4.** *Let $X$ be a $K3$ surface over a number field $k$. Then $X(\mathbb{A}_k)^{\mathrm{Br}}$ is an open subset of $X(\mathbb{A}_k)$.*

*Proof.* The sum of local invariants of a given element of $\mathrm{Br}\,(X)$ is a continuous function on $X(\mathbb{A}_k)$ with finitely many values. Thus the corollary is a consequence of Theorem 1.2. □

Let us mention here some open problems regarding rational points on $K3$ surfaces. Previous work on surfaces fibred into curves of genus 1 [2, 28, 27]

indicates that it is not unreasonable to expect the Manin obstruction to be the only obstruction to the Hasse principle on $K3$ surfaces. One could raise a more daring question: is the set of $k$-points dense in the Brauer–Manin set $X(\mathbb{A}_k)^{\mathrm{Br}}$? By Corollary 1.4, this would imply that the set of $k$-points on any $K3$ surface over a number field is either empty or Zariski dense. Moreover, this would also imply the weak-weak approximation for $X(k)$, whenever this set is non-empty. (This means that $k$ has a finite set of places $S$ such that for any finite set of places $T$ disjoint from $S$ the diagonal image of $X(k)$ in $\prod_T X(k_v)$ is dense.)

The paper is organized as follows. In Section 2 we recall the basic facts about the interrelations between the Brauer group, the Picard group and the Néron–Severi group (mostly due to Grothendieck [11]). We also discuss some linear algebra constructions arising from $\ell$-adic cohomology. In Section 3 we recall the finite coefficients variant of the Tate conjecture for abelian varieties and prove Theorem 1.1. Finally, Theorem 1.2 is proved in Section 4.

We are grateful to the referee whose comments helped to improve the exposition. This work was done during the special semester "Rational and integral points on higher-dimensional varieties" at the MSRI, and we would like to thank the MSRI and the organizers of the program. The second named author is grateful to Dr. Boris Veytsman for his help with TEXnical problems.

## 2. The Néron–Severi group, $\mathrm{H}^2$ and the Brauer group

We start with an easy lemma from linear algebra.

**Lemma 2.1.** *Let $\Lambda$ be a principal ideal domain, $H$ a non-zero $\Lambda$-module, $N \subset H$ a non-zero free submodule of finite rank. Let*

$$\psi : H \times H \to \Lambda$$

*be a symmetric bilinear form. Let $N^\perp$ be the orthogonal complement to $N$ in $H$ with respect to $\psi$, and let $\delta$ be the discriminant of the restriction of $\psi$ to $N$. If $\delta \neq 0$, then $N \cap N^\perp = \{0\}$ and*

$$\delta^2 H \subset N \oplus N^\perp \subset H.$$

*In particular, if $\delta$ is a unit in $\Lambda$, then $H = N \oplus N^\perp$.*

*Proof.* Let us put $N^* = \mathrm{Hom}_\Lambda(N, \Lambda)$. The form $\psi$ gives rise to a natural homomorphism of $\Lambda$-modules $e_\psi : H \to N^*$ with $N^\perp = \ker(e_\psi)$ and

$$\delta \cdot N^* \subset e_\psi(N) \subset N^*.$$

In particular, the restriction of $e_\psi$ to $N$ is injective; therefore $N \cap N^\perp = \{0\}$, and $e_\psi : N \to e_\psi(N)$ is an isomorphism. Let $u : e_\psi(N) \cong N$ be its inverse,

i.e., $ue_\psi : N \to N$ is the identity map. Let us consider the homomorphism of $\Lambda$-modules

$$P : H \to N, \quad h \mapsto \delta u(e_\psi(h)).$$

This definition makes sense since $\delta e_\psi(h) \in \delta N^* \subset e_\psi(N)$. It is clear that $\delta \cdot \ker P \subset N^\perp \subset \ker(P)$, and $P$ acts on $N$ as the multiplication by $\delta$. For any $h \in H$ we have $z = P(x) \in N$ and $P(z) = \delta z$, which implies that $P(\delta h) = P(z)$. Hence $\delta h - z \in \ker(P)$, and therefore $\delta(\delta h - z) \in N^\perp$. It follows that $\delta^2 h \in \delta z + N^\perp \subset N \oplus N^\perp$. □

**2.2.** Let us recall some useful elementary statements, which are due to Tate [31, 33]. Let $B$ be an abelian group. The projective limit of the groups $B_{\ell^n}$ (where the transition maps are the multiplications by $\ell$) is called the $\ell$-adic Tate module of $B$ and is denoted by $T_\ell(B)$. This limit carries a natural structure of a $\mathbf{Z}_\ell$-module; there is a natural injective map $T_\ell(B)/\ell \hookrightarrow B_\ell$. One may easily check that $T_\ell(B)_\ell = \{0\}$, and therefore $T_\ell(B)$ is torsion-free. Let us assume that $B_\ell$ is finite. Then all the $B_{\ell^n}$ are obviously finite, and $T_\ell(B)$ is finitely generated by Nakayama's lemma. Therefore, $T_\ell(B)$ is isomorphic to $\mathbf{Z}_\ell^r$ for some non-negative integer $r \le \dim_{\mathbf{F}_\ell}(B_\ell)$. Moreover, $T_\ell(B) = \{0\}$ if and only if $B(\ell)$ is finite.

For a field $k$ with separable closure $\overline{k}$ we denote by $\Gamma$ the Galois group $\mathrm{Gal}\,(\overline{k}/k)$. Let $X$ be a geometrically integral smooth projective variety over $k$, and let $\overline{X} = X \times_k \overline{k}$.

Let $\ell \neq \mathrm{char}(k)$ be a prime. Following [11, II, Sect. 3] we recall that the exact Kummer sequence of sheaves in the étale topology,

$$1 \to \mu_{\ell^n} \to \mathbf{G}_m \to \mathbf{G}_m \to 1,$$

gives rise to the (cohomological) exact sequence of Galois modules

$$0 \to \mathrm{Pic}\,(\overline{X})/\ell^n \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_{\ell^n}) \to \mathrm{Br}\,(\overline{X})_{\ell^n} \to 0.$$

Since $\mathrm{Pic}\,(\overline{X})$ is an extension of the Néron–Severi group $\mathrm{NS}\,(\overline{X})$ by a divisible group, we have $\mathrm{Pic}\,(\overline{X})/\ell^n = \mathrm{NS}\,(\overline{X})/\ell^n$. We thus obtain the exact sequence of Galois modules

$$0 \to \mathrm{NS}\,(\overline{X})/\ell^n \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_{\ell^n}) \to \mathrm{Br}\,(\overline{X})_{\ell^n} \to 0. \tag{1}$$

Since the groups $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_{\ell^n})$ are finite, the groups $\mathrm{Br}\,(\overline{X})_{\ell^n}$ are finite as well [11, II, Cor. 3.4]. On passing to the projective limit we get an exact sequence of $\Gamma$-modules

$$0 \to \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) \to T_\ell(\mathrm{Br}\,(\overline{X})) \to 0. \tag{2}$$

Since $T_\ell(\mathrm{Br}\,(\overline{X}))$ is a free $\mathbf{Z}_\ell$-module, this sequence shows that the torsion subgroup of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$ is contained in $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell$; that is, the torsion subgroups of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$ and $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell$ coincide, and so are both equal

to $\mathrm{NS}\,(\overline{X})(\ell)$. Tensoring the sequence (2) with $\mathbf{Q}_\ell$ (over $\mathbf{Z}_\ell$), we get the exact sequence of $\Gamma$-modules

$$0 \to \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_\ell \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_\ell(1)) \to V_\ell(\mathrm{Br}\,(\overline{X})) \to 0, \tag{3}$$

where $V_\ell(\mathrm{Br}\,(\overline{X})) = T_\ell(\mathrm{Br}\,(\overline{X})) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$. The Tate conjecture for divisors [30, 32, 34] asserts that if $k$ is finitely generated over its prime subfield, then

$$\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_\ell(1))^\Gamma = \mathrm{NS}\,(\overline{X})^\Gamma \otimes \mathbf{Q}_\ell. \tag{4}$$

Note also that (1) gives rise to the exact sequence of abelian groups

$$\begin{aligned} 0 \to (\mathrm{NS}\,(\overline{X})/\ell^n)^\Gamma \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_{\ell^n})^\Gamma \to \mathrm{Br}\,(\overline{X})^\Gamma_{\ell^n} \\ \to \mathrm{H}^1(k, \mathrm{NS}\,(\overline{X})/\ell^n) \to \mathrm{H}^1(k, \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_{\ell^n})). \end{aligned} \tag{5}$$

The lemma that follows is probably well known; cf. [14, Sect. 5, pp. 16–17] and [7, pp. 198–199].

**Lemma 2.3.** *Let $L \in \mathrm{NS}\,(\overline{X})^\Gamma$ be a Galois invariant hyperplane section class. Assume that $d = \dim(X) \geq 2$. If $\mathrm{char}(k) = 0$, then the kernel of the symmetric intersection pairing*

$$\psi_0 : \mathrm{NS}\,(\overline{X}) \times \mathrm{NS}\,(\overline{X}) \to \mathbf{Z}, \quad x, y \mapsto x \cdot y \cdot L^{d-2},$$

*is $\mathrm{NS}\,(\overline{X})_{\mathrm{tors}}$.*

*In any characteristic the same conclusion holds under the following condition:*

> *there exist a finite extension $k'/k$ with $k' \subset \overline{k}$, and a prime $q \neq \mathrm{char}(k)$ such that $\mathrm{Gal}\,(\overline{k}/k')$ acts trivially on $\mathrm{NS}\,(\overline{X})$, the $\mathrm{Gal}\,(\overline{k}/k')$-module $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))$ is semisimple, and $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))^{\mathrm{Gal}\,(\overline{k}/k')} = \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q$.*

*Proof.* We start with the case of characteristic zero. If $K$ is an algebraically closed field containing $k$, then the Néron–Severi group $\mathrm{NS}\,(X \otimes_k K)$ is identified with the group of connected components of the Picard scheme of $X$ [15, Cor. 4.18.3, Prop. 5.3, Prop. 5.10], and so does not depend on $K$. Let $k_0 \subset k$ be a subfield finitely generated over $\mathbf{Q}$, over which $X$ and $L$ are defined. Then there exists a smooth projective variety $X_0$ over the algebraic closure $\overline{k_0}$ of $k_0$ in $\overline{k}$, such that $\overline{X} = X_0 \times_{\overline{k_0}} \overline{k}$. The natural map $\mathrm{NS}\,(X_0) \to \mathrm{NS}\,(\overline{X})$ is bijective and therefore a group isomorphism.

For generalities on twisted classical cohomology groups we refer the reader to see [6, Sect. 1] or [4, Sect. 2.1].

Fix an embedding $\overline{k_0} \hookrightarrow \mathbf{C}$ and consider the complex variety $X_{\mathbf{C}} = X_0 \times_{\overline{k_0}} \mathbf{C}$. The natural map $\mathrm{NS}\,(X_0) \to \mathrm{NS}\,(X_{\mathbf{C}})$ is an isomorphism. Since the intersection indices do not depend on the choice of an algebraically closed ground

field, it suffices to check the non-degeneracy of $\psi_0$ for the complex variety $X_{\mathbf{C}}$. In order to do so, consider the canonical embedding

$$\mathrm{NS}\,(X_{\mathbf{C}}) \otimes \mathbf{Q} \hookrightarrow \mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1)),$$

and the symmetric bilinear form

$$\rho:\ \mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1)) \times\ \mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1)) \to\ \mathbf{Q}, \quad x, y \mapsto x \cup y \cup L^{d-2}.$$

The hard Lefschetz theorem says that the map

$$\mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1)) \longrightarrow \mathrm{H}^{2d-2}(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(d-1)), \quad x \mapsto x \cup L^{d-2},$$

is an isomorphism of vector spaces over $\mathbf{Q}$. Poincaré duality now implies that $\rho$ is non-degenerate. Let us show that the restriction of $\rho$ to $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q} \subset \mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1))$ is also non-degenerate. Indeed, let $P \subset \mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1))$ be the kernel of the multiplication by $L^{d-1}$. The group $\mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1))$ is the orthogonal direct sum $\mathbf{Q}L \oplus P$. On the one hand, the form $\rho$ is positive definite on $\mathbf{Q}L$ since $L$ is ample. On the other hand, the restriction of $\rho$ to $P$ is negative definite, due to the Hodge–Riemann bilinear relations [35, Ch. V, Sect. 5, Thm. 5.3]. This implies the non-degeneracy of $\rho$ on $\mathrm{NS}\,(X_{\mathbf{C}}) \otimes \mathbf{Q}$, because this space is the direct sum of $\mathbf{Q}L$ and $(\mathrm{NS}\,(X_{\mathbf{C}}) \otimes \mathbf{Q}) \cap P$. To finish the proof, we note that the form induced by $\rho$ on the Néron–Severi group coincides with $\psi_0$, whereas the kernel of $\mathrm{NS}\,(X_{\mathbf{C}}) \to \mathrm{NS}\,(X_{\mathbf{C}}) \otimes \mathbf{Q}$ is the torsion subgroup $\mathrm{NS}\,(X_{\mathbf{C}})_{\mathrm{tors}} = \mathrm{NS}\,(\overline{X})_{\mathrm{tors}}$.

Now let us prove the lemma in the case of arbitrary characteristic, assuming the condition on the Galois module $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))$.

Let us replace $k$ by $k'$. Consider the symmetric Galois-invariant $\mathbf{Q}_q$-bilinear form

$$\rho_q:\ \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1)) \ \times\ \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1)) \to\ \mathbf{Q}_q, \quad x, y \mapsto x \cup y \cup L^{d-2}.$$

The hard Lefschetz theorem, proved by Deligne [5] in all characteristics, says that the map

$$h_L : \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1)) \longrightarrow \mathrm{H}^{2d-2}_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(d-1)), \quad x \mapsto x \cup L^{d-2},$$

is an isomorphism of vector spaces over $\mathbf{Q}_q$. Thus $h_L$ is an isomorphism of Galois modules. Poincaré duality now implies that $\rho_q$ is non-degenerate.

Since $h_L$ is an isomorphism of Galois modules, we have

$$\mathrm{H}^{2d-2}_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(d-1))^{\Gamma} = h_L(\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))^{\Gamma}) = (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q) \cup L^{d-2}.$$

By the semisimplicity of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))$ there is a unique $\Gamma$-invariant vector subspace $W$ that is also a semisimple $\Gamma$-submodule such that

$$\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1)) = (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q) \oplus W.$$

Our condition implies that $W^\Gamma = \{0\}$. If $M \subset \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))$ is a vector subspace that is also a simple $\Gamma$-submodule, and if $M \to \mathbf{Q}_q$ is a non-zero $\Gamma$-invariant linear form, then $M$ is the trivial $\Gamma$-module $\mathbf{Q}_q$. It follows that the trivial $\Gamma$-module $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q$ is orthogonal to $W$ with respect to $\rho_q$. Now the non-degeneracy of $\rho_q$ implies that its restriction

$$\psi_q : \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q \ \times \ \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q \to \mathbf{Q}_q, \quad x, y \mapsto x \cdot y \cdot L^{d-2},$$

is also non-degenerate. By the compatibility of the cohomology class of the intersection of algebraic cycles and the cup-product of their cohomology classes [19, Ch. VI, Prop. 9.5 and Sect. 10], the bilinear form $\psi_q$ is obtained from $\psi_0$ by tensoring it with $\mathbf{Q}_q$. To finish the proof we note that the kernel of $\mathrm{NS}\,(\overline{X}) \to \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q$ is $\mathrm{NS}\,(\overline{X})_{\mathrm{tors}}$. □

**Remark 2.4.** (i) Since $\mathrm{NS}\,(\overline{X})$ is a finitely generated abelian group, there exists a finite extension $k'/k$ with $k' \subset \overline{k}$, such that $\mathrm{Gal}\,(\overline{k}/k')$ acts trivially on $\mathrm{NS}\,(\overline{X})$.

(ii) Recall that $V := \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_\ell(1))$ is a finite-dimensional $\mathbf{Q}_\ell$-vector space. Let $G_{\ell,k}$ be the image of $\Gamma = \mathrm{Gal}\,(\overline{k}/k)$ in $\mathrm{Aut}_{\mathbf{Q}_\ell}(V)$; it is a compact subgroup of $\mathrm{Aut}_{\mathbf{Q}_\ell}(V)$ and, by the $\ell$-adic version of Cartan's theorem [24], is an $\ell$-adic Lie subgroup of $\mathrm{Aut}_{\mathbf{Q}_\ell}(V)$. If $k'/k$ is a finite extension with $k' \subset \overline{k}$, then $\Gamma' = \mathrm{Gal}\,(\overline{k}/k')$ is an open subgroup of finite index in $\Gamma$; hence the image $G_{\ell,k'}$ of $\Gamma'$ is an open subgroup of finite index in $G_{\ell,k}$. In particular, $G_{\ell,k}$ and $G_{\ell,k'}$ have the same Lie algebra, which is a $\mathbf{Q}_\ell$-Lie subalgebra of $\mathrm{End}_{\mathbf{Q}_\ell}(V)$. Applying Prop. 1 of [25], we conclude that $V$ is semisimple as a $G_{\ell,k'}$-module if and only if it is semisimple as a $G_{\ell,k}$-module. It follows that $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_\ell(1))$ is semisimple as a $\Gamma'$-module if and only if it is semisimple as a $\Gamma$-module.

The following statement was inspired by [11, III, Sect. 8, pp. 143–147] and [31, Sect. 5].

**Proposition 2.5.** *Let $X$ be a smooth projective geometrically integral variety over a field $k$. Assume that one of the following conditions holds.*

(a) *$X$ is a curve or a surface.*
(b) $\mathrm{char}(k) = 0$.
(c) *There exist a finite extension $k'/k$ with $k' \subset \overline{k}$ and a prime $q \neq \mathrm{char}(k)$ such that $\mathrm{Gal}\,(\overline{k}/k')$ acts trivially on $\mathrm{NS}\,(\overline{X})$, the $\mathrm{Gal}\,(\overline{k}/k')$-module $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))$ is semisimple, and $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))^{\mathrm{Gal}\,(\overline{k}/k')} = \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q$.*

*Then for almost all primes $\ell$ the $\Gamma$-module $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell$ is a direct summand of the $\Gamma$-module $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$. If* (c) *is satisfied, then $\mathrm{Br}\,(\overline{X})^\Gamma(q)$ is finite.*

*Proof.* (a) If $X$ is a curve, then $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) = \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell \cong \mathbf{Z}_\ell$, and there is nothing to prove. Note that in this case $\mathrm{Br}\,(\overline{X}) = 0$ [11, III, Cor. 5.8]. Thus from now on we assume that $\dim(X) \geq 2$.

Let $X$ be a surface, $n = |\mathrm{NS}\,(\overline{X})_{\mathrm{tors}}|$. The cycle map defines the commutative diagram of pairings given by the intersection index and the cup-product:

$$\begin{array}{ccccc} \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) & \times & \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) & \to & \mathbf{Z}_\ell \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{NS}\,(\overline{X}) & \times & \mathrm{NS}\,(\overline{X}) & \to & \mathbf{Z} \end{array} \tag{6}$$

The diagram commutes by the compatibility of the cohomology class of the intersection of algebraic cycles and the cup-product of their cohomology classes [19, Ch. VI, Prop. 9.5 and Sect. 10]. The kernel of the pairing on the Néron–Severi group is its torsion subgroup. Let $\delta$ be the discriminant of the induced bilinear form on $\mathrm{NS}\,(\overline{X})/_{\mathrm{tors}}$. Let $H$ be the $\Gamma$-module $\psi^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))/_{\mathrm{tors}}$, and let $\psi$ be the Galois-invariant $\mathbf{Z}_\ell$-bilinear form on $H$ coming from the top pairing of (6). Let $N$ be the $\Gamma$-submodule $\mathrm{NS}\,(\overline{X})/_{\mathrm{tors}} \otimes \mathbf{Z}_\ell \subset H$. It is clear that $N$ is a free $\mathbf{Z}_\ell$-submodule of $H$, and $\delta$ is the discriminant of the restriction of $\psi$ to $N$. Let $N^\perp$ be the orthogonal complement to $N$ in $H$ with respect to $\psi$; $N^\perp$ is obviously a $\Gamma$-submodule of $H$.

Applying Lemma 2.1 (with $\Lambda = \mathbf{Z}_\ell$) we conclude that

$$N \cap N^\perp = \{0\} \quad \text{and} \quad \delta^2 H \subset N \oplus N^\perp.$$

Now let $\tilde{M}$ be the preimage of $N^\perp$ in $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$. Clearly, $\tilde{M}$ is a Galois submodule, and $\tilde{M} \cap (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell)$ is the torsion subgroup of $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell$ and therefore coincides with $\mathrm{NS}\,(\overline{X})(\ell)$. It is also clear that

$$\delta^2\, \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) \subset (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell) + \tilde{M}.$$

Let us put $M = n\tilde{M} \subset \tilde{M}$. We have

$$M \cap (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell) = \{0\} \quad \text{and} \quad n\delta^2\, \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) \subset (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell) \oplus M.$$

Since $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$ is a finitely generated $\mathbf{Z}_\ell$-module, $(\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell) \oplus M$ is a subgroup of finite index in $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$. This index is 1 if $\ell$ does not divide $n\delta$.

(b) and (c). Let us choose a $\Gamma$-invariant hyperplane section class $L \in \mathrm{NS}\,(\overline{X})^\Gamma$. By Lemma 2.3 the symmetric bilinear form on $\mathrm{NS}\,(\overline{X})/_{\mathrm{tors}}$ induced by $\psi_0$ is non-degenerate. Let $\delta \in \mathbf{Z}$ be the discriminant of this form, $\delta \neq 0$. Let us consider the Galois-invariant symmetric $\mathbf{Z}_\ell$-bilinear form

$$\psi_1 : \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) \times \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) \to \mathbf{Z}_\ell, \quad x, y \mapsto x \cup y \cup L^{d-2}.$$

The compatibility of (the cohomology class of) the intersection of algebraic cycles and the cup-product of their cohomology classes [19, Ch. VI, Prop. 9.5 and Sect. 10] implies that the restriction of $\psi_1$ to $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell$ coincides with the form induced by $\psi_0$. It follows from the hard Lefschetz theorem and Poincaré duality that $\ker(\psi_1) = \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))_{\mathrm{tors}}$.

Let $H$ be the $\Gamma$-module $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))/_{\mathrm{tors}}$, and let $\psi$ be the Galois-invariant $\mathbf{Z}_\ell$-bilinear form on $H$ defined by $\psi_1$. Let $N$ be the $\Gamma$-submodule $\mathrm{NS}\,(\overline{X})/_{\mathrm{tors}} \otimes \mathbf{Z}_\ell \subset H$. It is clear that $N$ is a free $\mathbf{Z}_\ell$-submodule of $H$, and the discriminant of the restriction of $\psi$ to $N$ is $\delta$. The rest of the proof is the same as in case (a).

Now suppose that under the condition of (c) the group $\mathrm{Br}\,(\overline{X})^\Gamma(q)$ is infinite. Since $\mathrm{Br}\,(\overline{X})^\Gamma \subset \mathrm{Br}\,(\overline{X})^{\mathrm{Gal}(\overline{k}/k')}$, we can extend the ground field from $k$ to $k'$. For any $n$ the group $\mathrm{Br}\,(\overline{X})_{q^n}$ is finite; thus there is an element of order $q^n$ in $\mathrm{Br}\,(\overline{X})^\Gamma_{q^n}$ for every $n$; i.e., the set $S(n)$ of elements of order $q^n$ in $\mathrm{Br}\,(\overline{X})^\Gamma_{q^n}$ is non-empty for all $n$. Since the projective limit of non-empty finite sets $S(n)$ is a non-empty subset of $T_q(\mathrm{Br}\,(\overline{X})^\Gamma) \setminus \{0\}$, we conclude that

$$T_q(\mathrm{Br}\,(\overline{X}))^\Gamma = T_q(\mathrm{Br}\,(\overline{X})^\Gamma) \neq \{0\}.$$

It follows that $V_q(\mathrm{Br}\,(\overline{X}))^\Gamma \neq \{0\}$. However, the semisimplicity of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))$ implies that the exact sequence of Galois modules (3) splits; that is,

$$\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1)) \cong (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q) \oplus V_q(\mathrm{Br}\,(\overline{X})).$$

By condition (c) we have $V_q(\mathrm{Br}\,(\overline{X}))^\Gamma = \{0\}$. This contradiction proves the finiteness of $\mathrm{Br}\,(\overline{X})^\Gamma(q)$. □

**Corollary 2.6.** *Let $X$ be a smooth projective geometrically integral variety over a field $k$. Assume that $X/k$ satisfies one of the conditions* (a), (b), (c) *of Proposition* 2.5. *Then the map* $\mathrm{H}^1(k, \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}/\ell) \to \mathrm{H}^1(k, \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell))$ *in* (5) *is injective for almost all $\ell$.*

*Proof.* By Proposition 2.5, the $\Gamma$-module $\mathrm{NS}\,(\overline{X})/\ell = (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell)/\ell$ is a direct summand of the $\Gamma$-module $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))/\ell$ for almost all $\ell$. We have an exact sequence

$$0 \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))/\ell \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell) \to \mathrm{H}^3_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))_\ell \to 0.$$

By a theorem of Gabber [10], for almost all $\ell$ the $\mathbf{Z}_\ell$-module $\mathrm{H}^3_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell)$ has no torsion. Since $\mathrm{H}^3_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell)$ and $\mathrm{H}^3_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$ are isomorphic as abelian groups, for almost all $\ell$ we have $\mathrm{H}^3_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))_\ell = \{0\}$; hence $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell) = \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))/\ell$. Thus $\mathrm{NS}\,(\overline{X})/\ell$ is a direct summand of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell)$. This proves the corollary. □

**Corollary 2.7.** *Suppose that $k$ is finitely generated over its prime subfield, and* $\mathrm{char}(k) \neq 2$. *Let $A$ be an abelian variety over $k$. Then for all $\ell \neq \mathrm{char}(k)$ the subgroup* $\mathrm{Br}\,(\overline{A})^\Gamma(\ell)$ *is finite.*

*Proof.* Let $\ell$ be a prime different from $\mathrm{char}(k)$. The Tate conjecture for divisors (4) is true for any abelian variety $A$ over such a field; in addition, the natural Galois action on the $\ell$-adic cohomology groups of $\overline{A}$ is semisimple. (These assertions were proved by the second named author [37, 38] in finite

characteristic not equal to 2, and by Faltings [8, 9] in characteristic zero.) This implies that $A$ satisfies condition (c) of Proposition 2.5 for every prime $q \neq \mathrm{char}(k)$. Now the result follows from the last assertion of Proposition 2.5. $\square$

## 3. Proof of Theorem 1.1

Let $A$ and $A'$ be abelian varieties over an *arbitrary* field $k$. We write $\mathrm{Hom}(A, A')$ for the group of homomorphisms $A \to A'$. We have

$$\mathrm{Hom}(A, A') = \mathrm{Hom}_\Gamma(\overline{A}, \overline{A}') = \mathrm{Hom}(\overline{A}, \overline{A}')^\Gamma.$$

Since $\mathrm{Hom}(\overline{A}, \overline{A}')$ has no torsion, we have that $\mathrm{Hom}(A, A')/n$ is a subgroup of $\mathrm{Hom}(\overline{A}, \overline{A}')/n$.

Let $A^t$ be the dual abelian variety of $A$. We have $(A^t)^t = A$ ([16, Ch. V, Sect. 2, Prop. 9], [21, p. 132]). Every divisor $D$ on $\overline{A}$ defines the homomorphism $\overline{A} \to \overline{A}^t$ sending $a \in A(\overline{k})$ to the linear equivalence class of $T_a^*(D) - D$ in $\mathrm{Pic}^0(\overline{A})$, where $T_a$ is the translation by $a$ in $A$. If $L$ is the algebraic equivalence class of $D$ in $\mathrm{NS}(\overline{A})$, then this map depends only on $L$, and is denoted by $\phi_L : \overline{A} \to \overline{A}^t$ [21, Sect. 8]. For $\alpha \in \mathrm{Hom}(\overline{A}, \overline{A}^t)$ we denote by $\alpha^t \in \mathrm{Hom}(\overline{A}, \overline{A}^t)$ the transpose of $\alpha$. Note that $\phi_L^t = \phi_L$. Moreover, if we set

$$\mathrm{Hom}(\overline{A}, \overline{A}^t)_{\mathrm{sym}} := \{u \in \mathrm{Hom}(\overline{A}, \overline{A}^t) \mid u = u^t\},$$

then the group homomorphism

$$\mathrm{NS}(\overline{A}) \to \mathrm{Hom}(\overline{A}, \overline{A}^t)_{\mathrm{sym}}, \quad L \mapsto \phi_L,$$

is an isomorphism [16], [21, Sect. 20, formula (I) and Thm. 1 on p. 186, Thm. 2 on p. 188 and Remark on p. 189]. For any $\alpha \in \mathrm{Hom}(\overline{A}, \overline{A}^t)$ we have $(\alpha^t)^t = \alpha$, and thus

$$\alpha + \alpha^t \in \mathrm{Hom}(\overline{A}, \overline{A}^t)_{\mathrm{sym}}. \tag{7}$$

**3.1.** Let $\ell$ be a prime different from the characteristic of $k$, $i$ a positive integer, and $n = \ell^i$. The kernel $A_n$ of the multiplication by $n$ in $A(\overline{k})$ is a Galois submodule, isomorphic to $(\mathbf{Z}/n)^{2\dim(A)}$ as an abelian group.

The natural map $\mathrm{Hom}(\overline{A}, \overline{A}')/n \to \mathrm{Hom}(A_n, A'_n)$ is *injective* [20, p. 124]. It commutes with the Galois action on both sides; in particular, the image of $\mathrm{Hom}(A, A')/n \subset \mathrm{Hom}(\overline{A}, \overline{A}')/n$ lies in $\mathrm{Hom}_\Gamma(A_n, A'_n)$.

For any $\alpha \in \mathrm{Hom}(\overline{A}, \overline{A}^t)$ and any $x, y \in A_n$ we have ([16, Ch. VII, Sect. 2, Thm. 4], [21, p. 186])

$$e_{n,A^t}(\alpha x, y) = e_{n,A}(x, \alpha^t y).$$

Thus $\mathrm{Hom}(\overline{A}, \overline{A}^t)_{\mathrm{sym}}/n$ is a subgroup of

$$\mathrm{Hom}(A_n, A_n^t)_{\mathrm{sym}} := \{u \in \mathrm{Hom}(A_n, A_n^t) \mid e_{n,A^t}(ux, y) = e_{n,A}(x, uy), \quad \forall x, y \in A_n\}.$$

Moreover, if $\ell$ is odd, then, by (7), we have

$$\mathrm{Hom}(\overline{A}, \overline{A}^t)_{\mathrm{sym}}/n = \mathrm{Hom}(\overline{A}, \overline{A}^t)/n \cap \mathrm{Hom}(A_n, A_n^t)_{\mathrm{sym}}. \tag{8}$$

**Remark 3.2.** The two (non-degenerate, Galois-equivariant) Weil pairings

$$e_{n,A} : A_n \times A_n^t \to \mu_n \quad \text{and} \quad e_{n,A^t} : A_n^t \times A_n \to \mu_n$$

differ by $-1$ [16, Ch. VII, Sect. 2, Thm. 5(iii) on p. 193]; that is,

$$e_{n,A^t}(y, x) = -e_{n,A}(x, y)$$

for all $x \in A_n$, $y \in A_n^t$. Since for each $u \in \mathrm{Hom}(A_n, A_n^t)$ we have

$$e_{n,A}(x, uy) = -e_{n,A^t}(uy, x) = -e_{n,A}(y, u^t x),$$

we conclude that $u$ lies in $\mathrm{Hom}(A_n, A_n^t)_{\mathrm{sym}}$ if and only if the bilinear form $e_{n,A}(x, uy)$ is *skew-symmetric*; that is, for any $x, y \in A_n$ we have

$$e_{n,A}(x, uy) = -e_{n,A}(y, ux).$$

**3.3.** For a module $M$ over a commutative ring $\Lambda$ we denote by $\mathrm{S}^2_\Lambda M$ the submodule of $M \otimes_\Lambda M$ generated by $x \otimes x$ for all $x \in M$. Let $\wedge^2_\Lambda M = (M \otimes_\Lambda M)/\mathrm{S}^2_\Lambda M$. We have $x \otimes y + y \otimes x \in \mathrm{S}^2_\Lambda M$; these elements generate $\mathrm{S}^2_\Lambda M$ if 2 is invertible in $\Lambda$.

From the Kummer sequence one obtains the well-known canonical isomorphism $\mathrm{H}^1_{\acute{e}t}(\overline{A}, \mu_n) = \mathrm{Pic}\,(\overline{A})_n = A_n^t$. Thus we have canonical isomorphisms of Galois modules (cf. [1, Sect. 2], [19], [20]):

$$\mathrm{H}^2_{\acute{e}t}(\overline{A}, \mu_n) = \wedge^2_{\mathbf{Z}/n} A_n^t(-1) = \mathrm{Hom}(\wedge^2_{\mathbf{Z}/n} A_n, \mu_n).$$

Clearly, there is a canonical embedding of Galois modules

$$\mathrm{Hom}(\wedge^2_{\mathbf{Z}/n} A_n, \mu_n) \hookrightarrow \mathrm{Hom}(A_n, A_n^t),$$

whose image coincides with the set of $u : A_n \to A_n^t$ such that the bilinear form $e_{n,A}(x, uy)$ is *alternating*, i.e., $e_{n,A}(x, ux) = 0$ for all $x \in A_n$. Combining it with Remark 3.2, we conclude that if $\ell$ is *odd*, then there is a canonical isomorphism of Galois modules

$$\mathrm{H}^2_{\acute{e}t}(\overline{A}, \mu_n) \cong \mathrm{Hom}(A_n, A_n^t)_{\mathrm{sym}}. \tag{9}$$

Let us recall a variant of the Tate conjecture on homomorphisms that first appeared in [39].

**Proposition 3.4.** *Let $k$ be a field finitely generated over its prime subfield,* $\mathrm{char}(k) \neq 2$. *If $A$ and $A'$ are abelian varieties over $k$, then the natural injection*

$$\mathrm{Hom}(A, A')/\ell \hookrightarrow \mathrm{Hom}_\Gamma(A_\ell, A'_\ell) \tag{10}$$

*is an isomorphism for almost all $\ell$.*

*Proof.* In the finite characteristic case this is proved in [39, Thm. 1.1]. When $A = A'$ and $k$ is a number field, Cor. 5.4.5 of [41] (based on the results of Faltings [8]) says that for almost all $\ell$ we have

$$\mathrm{End}(A)/\ell = \mathrm{End}_\Gamma(A_\ell). \tag{11}$$

The same proof works over arbitrary fields that are finitely generated over $\mathbf{Q}$, provided one replaces the reference to Prop. 3.1 of [41] by the reference to the corollary on p. 211 of Faltings [9]. Applying (11) to the abelian variety $A \times A'$, we deduce that (10) is a bijection. □

**Lemma 3.5.** *Let $k$ be a field finitely generated over its prime subfield,* $\mathrm{char}(k) \neq 2$, *and let $A$ be an abelian variety over $k$. Then for almost all $\ell$ we have the following statements:*

(i) *the injective map* $(\mathrm{NS}\,(\overline{A})/\ell)^\Gamma \hookrightarrow \mathrm{H}^2_{\acute{e}t}(\overline{A}, \mu_\ell)^\Gamma$ *in* (5) *is an isomorphism;*
(ii) $\mathrm{Br}\,(\overline{A})^\Gamma(\ell) = \{0\}$.

*Proof.* Suppose that $\ell$ is odd. By (8) we have

$$\mathrm{Hom}(\overline{A}, \overline{A}^t)_{\mathrm{sym}}/\ell = \mathrm{Hom}(\overline{A}, \overline{A}^t)/\ell \cap \mathrm{Hom}(A_\ell, A^t_\ell)_{\mathrm{sym}}.$$

Proposition 3.4 implies that for almost all $\ell$ we have

$$\mathrm{Hom}(A, A^t)/\ell = \mathrm{Hom}(A_\ell, A^t_\ell)^\Gamma = \mathrm{Hom}_\Gamma(A_\ell, A^t_\ell).$$

We thus obtain an isomorphism

$$\mathrm{Hom}(A, A^t)_{\mathrm{sym}}/\ell = \mathrm{Hom}_\Gamma(A_\ell, A^t_\ell)_{\mathrm{sym}}. \tag{12}$$

The left hand side of (12) is $\mathrm{Hom}_\Gamma(\overline{A}, \overline{A}^t)_{\mathrm{sym}}/\ell \cong \mathrm{NS}\,(\overline{A})^\Gamma/\ell$; see the beginning of this section. The right hand side of (12) is isomorphic to $\mathrm{H}^2_{\acute{e}t}(\overline{A}, \mu_\ell)^\Gamma$ by (9). It follows that $\mathrm{NS}\,(\overline{A})^\Gamma/\ell$ and $\mathrm{H}^2_{\acute{e}t}(\overline{A}, \mu_\ell)^\Gamma$ have the same number of elements. Since $\mathrm{NS}\,(\overline{A})$ has no torsion, $\mathrm{NS}\,(\overline{A})^\Gamma/\ell$ is a subgroup of $(\mathrm{NS}\,(\overline{A})/\ell)^\Gamma$, and hence the injective map in (i) is bijective. Statement (ii) follows from (i), Corollary 2.6 and the exact sequence (5). □

*End of proof of Theorem* 1.1. Let $A$ be an abelian variety over $k$, and $X$ a principal homogeneous space of $A$. In characteristic 0 (resp. in characteristic $p$) it suffices to show that $\mathrm{Br}\,(\overline{X})^{\Gamma}$ (resp. $\mathrm{Br}\,(\overline{X})^{\Gamma}(\mathrm{non}{-}p)$) is finite. For this we can go over to a finite extension $k'/k$ such that $X \times_k k' \simeq A \times_k k'$, and so assume that $X = A$. The theorem now follows from Lemma 3.5 (ii) and Corollary 2.7. $\square$

## 4. Proof of Theorem 1.2

**4.1.** In this subsection we recall some well-known results which will be used later in this section.

Let $A$ be an abelian variety over a field $k$, $\ell$ a prime different from $\mathrm{char}(k)$, $n = \ell^i$. Let $\pi_1^{\acute{e}t}(\overline{A}, 0)^{(\ell)}$ be the maximal abelian $\ell$-quotient of the Grothendieck étale fundamental group $\pi_1^{\acute{e}t}(\overline{A}, 0)$. Let us consider the Tate $\ell$-module $T_\ell(A) := T_\ell(A(\overline{k}))$. It is well known [16, 21] that $T_\ell(A)$ is a free $\mathbf{Z}_\ell$-module of rank $2\dim(A)$ equipped with a natural structure of a $\Gamma$-module, and the natural map $T_\ell(A)/n \to A_n$ is an isomorphism of Galois modules. Recall [20, pp. 129–130] that the isogeny $\overline{A} \stackrel{n}{\to} \overline{A}$ is a Galois étale covering with the Galois group $A_n$ acting by translations. This defines a canonical surjection $f_n : \pi_1^{\acute{e}t}(\overline{A}, 0)^{(\ell)} \twoheadrightarrow A_n$. The $f_n$ glue together into a canonical isomorphism of Galois modules $\pi_1^{\acute{e}t}(\overline{A}, 0)^{(\ell)} \to T_\ell(A)$, which induces the canonical isomorphisms of Galois modules

$$\mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}_\ell) = \mathrm{Hom}_{\mathbf{Z}_\ell}(\pi_1^{\acute{e}t}(\overline{A}, 0)^{(\ell)}, \mathbf{Z}_\ell) = \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(A), \mathbf{Z}_\ell).$$

Since $\mathrm{H}^j_{\acute{e}t}(\overline{A}, \mathbf{Z}_\ell)$ is torsion-free for any $j$ [20, Thm. 15.1(b) on p. 129], the reduction modulo $n$ gives rise to natural isomorphisms of Galois modules

$$\mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}/n) = \mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}_\ell)/n = \mathrm{Hom}(A_n, \mathbf{Z}/n).$$

Now suppose that we are given a field embedding $\overline{k} \hookrightarrow \mathbf{C}$. Let us consider the complex abelian variety $B = A(\mathbf{C})$. The exponential map establishes a canonical isomorphism of compact Lie groups $\mathrm{Lie}(B)/\Pi \to B$ [21, Sect. 1]. Here $\mathrm{Lie}(B) \cong \mathbf{C}^{\dim(B)}$ is the tangent space to $B$ at the origin, $\Pi$ is a discrete lattice of rank $2\dim(B)$, and the natural map $\mathrm{H}_1(B, \mathbf{Z}) \otimes \mathbf{R} \to \mathrm{Lie}(B)$ is an isomorphism of real vector spaces. Clearly, $V$ is the universal covering space of $B$, and the fundamental group $\pi_1(B, 0) = \mathrm{H}_1(B, \mathbf{Z}) = \Pi$ acts on $V$ by translations. We have

$$B_n = \frac{1}{n}\Pi/\Pi \subset V/\Pi = B.$$

The isogeny $B \xrightarrow{n} B$ is an unramified Galois covering of connected spaces (in the classical topology) with the Galois group $B_n$, corresponding to the subgroup $n\Pi \subset \Pi$. It is identified with $V/n\Pi \to V/\Pi$, and the corresponding homomorphism $\varphi_n : \Pi \twoheadrightarrow B_n = \frac{1}{n}\Pi/\Pi$ sends $c$ to $\frac{1}{n}c + \Pi$. The comparison theorem for fundamental groups implies that $\varphi_n$ coincides with the composition

$$\pi_1(B,0) \to \pi_1^{\acute{e}t}(B,0) \to \pi_1^{\acute{e}t}(B,0)^{(\ell)} \xrightarrow{f_n} B_n.$$

We obtain the following sequence of homomorphisms:

$$\begin{aligned} &\mathrm{Hom}(B_n, \mathbf{Z}/n) \hookrightarrow \mathrm{Hom}(\pi_1^{\acute{e}t}(B,0)^{(\ell)}, \mathbf{Z}/n) \\ &= \mathrm{Hom}(\pi_1^{\acute{e}t}(B,0), \mathbf{Z}/n) \to \mathrm{Hom}(\pi_1(B,0), \mathbf{Z}/n). \end{aligned} \tag{13}$$

The same comparison theorem implies that the last map in (13) is bijective. It follows easily that all the homomorphisms in (13) are isomorphisms. Recall that

$$\mathrm{Hom}(\pi_1^{\acute{e}t}(B,0)^{(\ell)}, \mathbf{Z}/n) = \mathrm{H}^1_{\acute{e}t}(B, \mathbf{Z}/n), \quad \mathrm{Hom}(\pi_1(B,0), \mathbf{Z}/n) = \mathrm{H}^1(B, \mathbf{Z}/n).$$

Note also that $\varphi_n$ establishes a canonical isomorphism

$$\Pi/n = \pi_1(B,0)/n \to B_n, \quad c \mapsto \frac{1}{n}c + \Pi,$$

which gives us the canonical isomorphisms

$$\mathrm{H}^1(B, \mathbf{Z}/n) = \mathrm{H}^1(B, \mathbf{Z})/n = \mathrm{Hom}(B_n, \mathbf{Z}/n) = \mathrm{H}^1_{\acute{e}t}(B, \mathbf{Z}/n).$$

Taking the projective limits with respect to $i$ (recall that $n = \ell^i$), we get the canonical isomorphisms

$$\mathrm{H}^1(B, \mathbf{Z}) \otimes \mathbf{Z}_\ell = \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(B), \mathbf{Z}_\ell) = \mathrm{H}^1_{\acute{e}t}(B, \mathbf{Z}_\ell).$$

On the other hand, taking the projective limit of the $\varphi_n$, we get the natural map [21, p. 237]

$$\mathrm{H}_1(B, \mathbf{Z}) = \Pi \to T_\ell(B), \quad x \mapsto \{x/\ell^i\}_{i=1}^\infty,$$

which extends by $\mathbf{Z}_\ell$-linearity to the natural isomorphism of $\mathbf{Z}_\ell$-modules

$$\varphi^{(\ell)} : \mathrm{H}_1(B, \mathbf{Z}) \otimes \mathbf{Z}_\ell = \Pi \otimes \mathbf{Z}_\ell \cong T_\ell(B).$$

We have

$$A_n = B_n = \mathrm{H}_1(B, \mathbf{Z})/n. \tag{14}$$

The comparison theorem for étale and classical cohomology implies that $\mathrm{H}^1_{ét}(\overline{A}, \mathbf{Z}/n) = \mathrm{H}^1(B, \mathbf{Z}/n)$; thus we obtain

$$\begin{gathered}\mathrm{H}^1_{ét}(\overline{A}, \mathbf{Z}/n) = \mathrm{Hom}(A_n, \mathbf{Z}/n) = \\ \mathrm{Hom}(B_n, \mathbf{Z}/n) = \mathrm{Hom}(\mathrm{H}_1(B, \mathbf{Z})/n, \mathbf{Z}/n), \\ T_\ell(A) = T_\ell(B) = \mathrm{H}_1(B, \mathbf{Z}) \otimes \mathbf{Z}_\ell, \\ \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(A), \mathbf{Z}_\ell) = \mathrm{H}^1_{ét}(\overline{A}, \mathbf{Z}_\ell) = \mathrm{H}^1_{ét}(B, \mathbf{Z}_\ell) = \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(B), \mathbf{Z}_\ell).\end{gathered} \tag{15}$$

**Lemma 4.2.** *Let $M$ and $N$ be subgroups of $\mathbf{Z}^n$ such that $M \cap N = 0$. Then for almost all $\ell$ the natural maps $M/\ell \to (\mathbf{Z}/\ell)^n$ and $N/\ell \to (\mathbf{Z}/\ell)^n$ are injective, and the intersection of their images is $\{0\}$.*

*Proof.* There is a subgroup $L \subset \mathbf{Z}^n$ such that $L \cap (M \oplus N) = 0$, and $L \oplus M \oplus N$ is of finite index in $\mathbf{Z}^n$. For all $\ell$ not dividing this index, the canonical map $M/\ell \to (\mathbf{Z}/\ell)^n$ and the similar map for $N$ are injective. Moreover, $(\mathbf{Z}/\ell)^n$ is the direct sum of $L/\ell$, $M/\ell$ and $N/\ell$. This proves the lemma. □

**Lemma 4.3.** *Let $X$ be a K3 surface over a field $k$ finitely generated over $\mathbf{Q}$. Then the injective map $(\mathrm{NS}\,(\overline{X})/\ell)^\Gamma \to \mathrm{H}^2_{ét}(\overline{X}, \mu_\ell)^\Gamma$ in (5) is an isomorphism for almost all primes $\ell$.*

*Proof.* It suffices to prove the lemma for a finite extension $k'/k$, $k' \subset \overline{k}$, and $\Gamma' = \mathrm{Gal}\,(\overline{k}/k') \subset \Gamma$. Indeed, for any $\Gamma$-module $M$ the composition of the natural inclusion $M^\Gamma \hookrightarrow M^{\Gamma'}$ and the norm map $M^{\Gamma'} \to M^\Gamma$ is the multiplication by the degree $[k' : k]$. Hence if $(\mathrm{NS}\,(\overline{X})/\ell)^{\Gamma'} \to \mathrm{H}^2_{ét}(\overline{X}, \mu_\ell)^{\Gamma'}$ is surjective for all primes $\ell$ not dividing a certain integer $N$, then so is the original map $(\mathrm{NS}\,(\overline{X})/\ell)^\Gamma \to \mathrm{H}^2_{ét}(\overline{X}, \mu_\ell)^\Gamma$ for all primes $\ell$ not dividing $N[k' : k]$. In particular, we can assume without loss of generality that $\Gamma$ acts trivially on $\mathrm{NS}\,(\overline{X})$.

Now let us fix an embedding $\overline{k} \hookrightarrow \mathbf{C}$ and identify $\overline{k}$ with its image in $\mathbf{C}$.

The group $\mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1)) \simeq \mathbf{Z}^{22}$ has a natural $\mathbf{Z}$-valued bilinear form $\psi$ given by the intersection index. By Poincaré duality $\psi$ is *unimodular*, i.e., the map $\mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1)) \to \mathrm{Hom}(\mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1)), \mathbf{Z})$ induced by $\psi$ is an isomorphism. Since $X(\mathbf{C})$ is simply connected we have $\mathrm{H}^1(X(\mathbf{C}), \mathbf{Z}) = \{0\}$, and by Poincaré duality this implies $\mathrm{H}^3(X(\mathbf{C}), \mathbf{Z}) = \{0\}$. Recall that $\mathrm{NS}\,(\overline{X}) = \mathrm{NS}\,(X_{\mathbf{C}})$ (see the beginning of the proof of Lemma 2.3). Since $X(\mathbf{C})$ is simply connected we have

$$\mathrm{Pic}\,(X_{\mathbf{C}}) = \mathrm{NS}\,(X_{\mathbf{C}}) = \mathrm{NS}\,(\overline{X}) = \mathrm{Pic}\,(\overline{X}).$$

We define the lattice of transcendental cycles $T_X$ as the orthogonal complement to the injective image of $\mathrm{NS}\,(\overline{X})$ in $\mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1))$. The restriction of $\psi$ to $\mathrm{NS}\,(\overline{X})$ is non-degenerate, and we write $\delta$ for the absolute value of the corresponding discriminant. Then $\mathrm{NS}\,(\overline{X}) \cap T_X = 0$, and $\mathrm{NS}\,(\overline{X}) \oplus T_X$ is a

subgroup of $\mathrm{H}^2(X(\mathbf{C}),\mathbf{Z}(1))$ of finite index $\delta$. Let $\ell$ be a prime not dividing $\delta$. Then we have

$$\mathrm{H}^2(X(\mathbf{C}),\mathbf{Z}(1))/\ell = (\mathrm{NS}\,(\overline{X})/\ell) \oplus (T_X/\ell).$$

The restriction of the $\mathbf{Z}/\ell$-valued pairing induced by $\psi$ to $\mathrm{NS}\,(\overline{X})/\ell$ is a non-degenerate $\mathbf{Z}/\ell$-bilinear form, so that $T_X/\ell$ is the orthogonal complement to $\mathrm{NS}\,(\overline{X})/\ell$. Since $\mathrm{H}^3(X(\mathbf{C}),\mathbf{Z}) = \{0\}$, we have $\mathrm{H}^2(X(\mathbf{C}),\mathbf{Z}(1))/\ell = \mathrm{H}^2(X(\mathbf{C}),\mu_\ell)$. The comparison theorem gives an isomorphism of $\mathbf{Z}_\ell$-modules

$$\mathrm{H}^2_{\acute{e}t}(\overline{X},\mathbf{Z}_\ell(1)) = \mathrm{H}^2(X(\mathbf{C}),\mathbf{Z}(1)) \otimes \mathbf{Z}_\ell, \tag{16}$$

which is compatible with cup-products [7, Prop. 6.1, p. 197], [6, Example 2.1(b), pp. 28–29]. Reducing modulo $\ell$ we get an isomorphism of $\mathbf{Z}/\ell$-vector spaces $\mathrm{H}^2_{\acute{e}t}(\overline{X},\mu_\ell) = \mathrm{H}^2(X(\mathbf{C}),\mathbf{Z}(1))/\ell$, compatible with cup-products. Thus for $\ell$ not dividing $\delta$ we have an orthogonal direct sum

$$\mathrm{H}^2_{\acute{e}t}(\overline{X},\mu_\ell) = (\mathrm{NS}\,(\overline{X})/\ell) \oplus (T_X/\ell),$$

so that for these $\ell$ the abelian group $T_X/\ell$ carries a natural $\Gamma$-(sub)module structure. (Here we use the compatibility of the cycle maps $\mathrm{Pic}\,(\overline{X}) \to \mathrm{H}^2_{\acute{e}t}(\overline{X},\mu_\ell)$ and $\mathrm{Pic}\,(\overline{X}) \to \mathrm{H}^2(X(\mathbf{C}),\mu_\ell)$; see [13, Prop. 3.8.5, pp. 296–297].)

Let $L \in \mathrm{Pic}\,(\overline{X}) = \mathrm{NS}\,(\overline{X})$ be a $\Gamma$-invariant hyperplane section class, and $P \subset \mathrm{H}^2(X(\mathbf{C}),\mathbf{Z}(1))$ the orthogonal complement to $L$ with respect to $\psi$. Then (16) implies that $P_\ell = P \otimes \mathbf{Z}_\ell$ is both a Galois and a $\mathbf{Z}_\ell$-submodule of $\mathrm{H}^2_{\acute{e}t}(\overline{X},\mathbf{Z}_\ell(1))$. It is clear that $P_\ell$ is the orthogonal complement to $L$ in $\mathrm{H}^2_{\acute{e}t}(\overline{X},\mathbf{Z}_\ell(1))$ with respect to the Galois-invariant intersection pairing

$$\psi_\ell : \mathrm{H}^2_{\acute{e}t}(\overline{X},\mathbf{Z}_\ell(1)) \times \mathrm{H}^2_{\acute{e}t}(\overline{X},\mathbf{Z}_\ell(1)) \to \mathbf{Z}_\ell.$$

Similarly, $T_X \otimes \mathbf{Z}_\ell$ is the orthogonal complement to $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell$ in $\mathrm{H}^2_{\acute{e}t}(\overline{X},\mathbf{Z}_\ell(1))$ with respect to $\psi_\ell$, and so is a Galois submodule.

Let $C^+(P)$ be the even Clifford $\mathbf{Z}$-algebra of $(P,\psi)$. The complex vector space $P_{\mathbf{C}} := P \otimes \mathbf{C}$ inherits from $\mathrm{H}^2(X(\mathbf{C}),\mathbf{C}(1))$ the Hodge decomposition of type $\{(-1,1),(0,0),(1,-1)\}$ with Hodge numbers $h^{1,-1} = h^{-1,1} = 1$. By the Lefschetz theorem, $T_X$ intersects trivially with the $(0,0)$-subspace. The $\mathbf{Z}$-algebra $C^+(P)$ naturally carries a weight zero Hodge structure of type $\{(-1,1),(0,0),(1,-1)\}$ induced by the Hodge structure on $P$ (via the identification $C^+(P) = \bigoplus_i \wedge^{2i} P$); see [3, Lemma 4.4]. On the other hand, $C^+(P) \otimes \mathbf{Z}_\ell$ coincides with the even Clifford $\mathbf{Z}_\ell$-algebra $C^+(P_\ell)$ of $(P_\ell,\psi_\ell)$. Clearly, $C^+(P_\ell)$ carries a natural $\Gamma$-module structure induced by the Galois action on $P_\ell$ (via the identification $C^+(P_\ell) = \bigoplus_i \wedge^{2i}_{\mathbf{Z}_\ell} P_\ell$). In his adaptation of the Kuga–Satake construction, Deligne ([3], pp. 219–223, in particular Prop. 5.7 and Lemma 6.5.1; see also [22] and [7], pp. 218–219) shows that after

replacing $k$ by a finite extension, there exists an abelian variety $A$ over $k$ and an injective ring homomorphism

$$u : C^+(P) \hookrightarrow \mathrm{End}(\mathrm{H}^1(A(\mathbf{C}), \mathbf{Z}))$$

satisfying the following properties.

(a) $u : C^+(P) \hookrightarrow \mathrm{End}(\mathrm{H}^1(A(\mathbf{C}), \mathbf{Z}))$ is a morphism of weight zero Hodge structures.

(b) The $\mathbf{Z}_\ell$-algebra homomorphism

$$u_\ell : C^+(P_\ell) \hookrightarrow \mathrm{End}_{\mathbf{Z}}(\mathrm{H}^1(A(\mathbf{C}), \mathbf{Z})) \otimes \mathbf{Z}_\ell = \mathrm{End}_{\mathbf{Z}_\ell}(\mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}_\ell))$$

obtained from $u$ by tensoring it with $\mathbf{Z}_\ell$, and then applying the comparison isomorphism $\mathrm{H}^1(A(\mathbf{C}), \mathbf{Z}) \otimes \mathbf{Z}_\ell = \mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}_\ell)$, is an injective homomorphism of Galois modules.

Replacing, if necessary, $k$ by a finite extension we may and will assume that all the endomorphisms of $\overline{A}$ are defined over $k$, that is, $\mathrm{End}(A) = \mathrm{End}(\overline{A})$.

Using the compatible isomorphisms (see Subsection 4.1)

$$\mathrm{H}^1(A(\mathbf{C}), \mathbf{Z}) = \mathrm{Hom}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z}), \mathbf{Z}),$$

$$\mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}_\ell) = \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(A), \mathbf{Z}_\ell), \quad T_\ell(A) = \mathrm{H}_1(A(\mathbf{C}), \mathbf{Z}) \otimes \mathbf{Z}_\ell,$$

we obtain the compatible ring anti-isomorphisms

$$t : \mathrm{End}(\mathrm{H}^1(A(\mathbf{C}), \mathbf{Z})) \cong \mathrm{End}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})),$$

$$t_\ell : \mathrm{End}_{\mathbf{Z}_\ell}(\mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}_\ell)) \cong \mathrm{End}_{\mathbf{Z}_\ell}(T_\ell(A))$$

of weight zero Hodge structures and Galois modules, respectively. Taking the compositions, we get an injective homomorphism of weight zero Hodge structures

$$t\,u : C^+(P) \hookrightarrow \mathrm{End}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})),$$

which, extended by $\mathbf{Z}_\ell$-linearity, coincides with the injective homomorphism of Galois modules

$$t_\ell\, u_\ell : C^+(P_\ell) \hookrightarrow \mathrm{End}_{\mathbf{Z}_\ell}(T_\ell(A)).$$

We shall identify $C^+(P)$ and $\mathrm{End}(A)$ with their images in $\mathrm{End}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z}))$. Note that all the elements of $\mathrm{End}(A) \subset \mathrm{End}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z}))$ have pure Hodge type $(0, 0)$.

Let us first consider the case when $\operatorname{rk} \mathrm{NS}\,(\overline{X}) \geq 2$. Then there exists a non-zero element $m \in \mathrm{NS}\,(\overline{X})^{\Gamma} \cap P$. Then

$$m \wedge T_X \subset \wedge^2 P \subset C^+(P) \subset \mathrm{End}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})).$$

Since $m \wedge T_X$ does not contain non-zero elements of type $(0,0)$, we have

$$(m \wedge T_X) \cap \mathrm{End}(A) = 0.$$

Using (14) and (15), we observe that for all but finitely many $\ell$ the $\Gamma$-module $T_X/\ell$ is isomorphic to

$$(m \wedge T_X)/\ell \subset \mathrm{End}_{\mathbf{Z}_\ell}(T_\ell(A))/\ell = \mathrm{End}_{\mathbf{F}_\ell}(A_\ell).$$

Lemma 4.2 implies that $(m \wedge T_X)/\ell$ intersects trivially with $\mathrm{End}(A)/\ell$ for almost all $\ell$. By the variant of the Tate conjecture (Proposition 3.4), for almost all $\ell$ we have $\mathrm{End}_{\mathbf{F}_\ell}(A_\ell)^{\Gamma} = \mathrm{End}_{\Gamma}(A_\ell) = \mathrm{End}(A)/\ell$; thus every $\Gamma$-invariant element of $m \wedge (T_X/\ell)$ is contained in $\mathrm{End}(A)/\ell$, and hence must be zero. It follows that $(T_X/\ell)^{\Gamma} = 0$ for almost all $\ell$. Therefore, $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell)^{\Gamma} = (\mathrm{NS}\,(\overline{X})/\ell)^{\Gamma}$ for almost all $\ell$.

It remains to consider the case $\operatorname{rk} \mathrm{NS}\,(\overline{X}) = 1$. Then $T_X = P \simeq \mathbf{Z}^{21}$, and so $\wedge^{20} T_X$ is the dual lattice of $T_X$. We have

$$\wedge^{20} T_X = \wedge^{20} P \subset C^+(P) \subset \mathrm{End}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})).$$

Since $T_X$ does not contain non-zero elements of type $(0,0)$, the same is true for the dual Hodge structure $\wedge^{20} T_X$. Thus $\wedge^{20} T_X \cap \mathrm{End}(A) = 0$, and the same arguments as before show that $(\wedge^{20} T_X/\ell)^{\Gamma} = 0$ for almost all $\ell$. The bilinear $\mathbf{Z}/\ell$-valued form induced by the cup-product on $T_X/\ell \subset \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell)$ is non-degenerate for almost all $\ell$, so that this Galois module is self-dual. Thus the Galois modules $T_X/\ell$ and $\wedge^{20} T_X/\ell$ are isomorphic, and we conclude that $(T_X/\ell)^{\Gamma} = 0$. This finishes the proof. □

**Lemma 4.4.** *Let $X$ be a $K3$ surface over a field $k$ finitely generated over $\mathbf{Q}$. Then $\mathrm{Br}\,(\overline{X})^{\Gamma}(\ell)$ is finite for all $\ell$.*

*Proof.* By Proposition 2.5, it suffices to check the validity of the Tate conjecture for divisors and the semisimplicity of the Galois module $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_\ell(1))$. Both these assertions follow from the corresponding results on abelian varieties, proved by Faltings in [8, 9]. The latter follows from the semisimplicity of the Galois action on the $\ell$-adic cohomology groups of abelian varieties combined with Proposition 6.26(d) of [7]. The former follows from the validity of the Tate conjecture for divisors on abelian varieties, as explained on p. 80 of [34]. □

*End of proof of Theorem* 1.2. By Remark 1.3, it suffices to show that $\mathrm{Br}\,(\overline{X})^{\Gamma}$ is finite. By the exact sequence (5), Corollary 2.6 and Lemma 4.3 we have $\mathrm{Br}\,(\overline{X})^{\Gamma}_{\ell} = 0$ for almost all $\ell$. Now the finiteness of $\mathrm{Br}\,(\overline{X})^{\Gamma}$ follows from Lemma 4.4. □

## References


[1] V.G. Berkovich, *The Brauer group of abelian varieties*. Funktsional. Anal. i Prilozhen. **6** (1972), no. 3, 10–15; Functional Anal. Appl. **6** (1972), no. 3, 180–184. MR0308134 (46:7249)

[2] J.-L. Colliot-Thélène, A.N. Skorobogatov and Sir Peter Swinnerton-Dyer, *Hasse principle for pencils of curves of genus one whose Jacobians have rational* 2*-division points*. Invent. Math. **134** (1998), 579–650. MR1660925 (99k:11095)

[3] P. Deligne, *La conjecture de Weil pour les surfaces* $K3$. Invent. Math. **15** (1972), 206–226. MR0296076 (45:5137)

[4] P. Deligne, *La conjecture de Weil*. I. Publ. Math. IHES **43** (1974), 273–307. MR0340258 (49:5013)

[5] P. Deligne, *La conjecture de Weil*. II. Publ. Math. IHES **52** (1980), 137–252. MR601520 (83c:14017)

[6] P. Deligne (notes by J. Milne), *Hodge cycles on abelian varieties*. In: Hodge cycles, motives and Shimura varieties. Springer Lecture Notes in Math. **900** (1982), 9–100.

[7] P. Deligne, J. Milne, *Tannakian categories*. In: Hodge cycles, motives and Shimura varieties. Springer Lecture Notes in Math. **900** (1982), 101–228. MR654325 (84m:14046)

[8] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*. Invent. Math. **73** (1983), 349–366; Erratum **75** (1984), 381. MR718935 (85g:11026a)

[9] G. Faltings, *Complements to Mordell*. In: G. Faltings, G. Wüstholz et al., Rational points, 2nd edition, F. Viehweg & Sohn, 1986. MR863887 (87m:11025)

[10] O. Gabber, *Sur la torsion dans la cohomologie* $\ell$*-adique d'une variété*. C. R. Acad. Sci. Paris Sér. I Math. **297** (1983), no. 3, 179–182. MR725400 (85f:14018)

[11] A. Grothendieck, *Le groupe de Brauer*. I, II, III. In: Dix exposés sur la cohomologie des schémas (A. Grothendieck, N.H. Kuiper, eds.), North-Holland, 1968, 46–188; available at www.grothendieckcircle.org . MR0244269 (39:5586a)

[12] D. Harari, *Obstructions de Manin transcendantes*. In: Number theory (Paris, 1993–1994), LMS Lecture Note Ser. **235** Cambridge Univ. Press, 1996, 75–87. MR1628794 (99e:14025)

[13] J. P. Jouanolou, *Cohomologie de quelques schémas classiques et théorie cohomologique des classes de Chern*. Exposé VII. SGA 5, Cohomologie $\ell$-adique et fonctions $L$ (dirigé par A. Grothendieck). Springer Lecture Notes in Math. **589** (1977).

[14] S. Kleiman, *The standard conjectures*. In: Motives (U. Jannsen, S. Kleiman, J.-P. Serre, eds.). Proc. Symp. Pure Math. **55**, Part 1 (1991), 3–20. Amer. Math. Soc., Providence, RI. MR1265519 (95k:14010)

[15] S. Kleiman, *The Picard scheme*. In: Fundamental algebraic geometry (Grothendieck's FGA explained). Mathematical Surveys and Monographs vol. **123** (2005), Amer. Math. Soc., Providence, RI; arXiv math.AG/0504020. MR2223410

[16] S. Lang, Abelian varieties, 2nd edition. Springer-Verlag, 1983. MR713430 (84g:14041)

[17] Yu.I. Manin, *Le groupe de Brauer–Grothendieck en géométrie diophantienne*. In: Actes Congrès Internat. Math. Nice I (Gauthier-Villars, 1971), 401–411. MR0427322 (55:356)

[18] J.S. Milne, *On a conjecture of Artin and Tate*. Ann. Math. **102** (1975), 517–533. MR0414558 (54:2659)

[19] J.S. Milne, Étale cohomology. Princeton University Press, Princeton, NJ, 1980. MR559531 (81j:14002)

[20] J.S. Milne, *Abelian varieties*. In: Arithmetic geometry (G. Cornell, J.H. Silverman, eds.). Springer-Verlag, 1986. MR861974

[21] D. Mumford, Abelian varieties, 2nd edition. Oxford University Press, London, 1974. MR0282985 (44:219)

[22] I.I. Piatetski-Shapiro, I.R. Shafarevich, *The arithmetic of surfaces of type $K3$*. Trudy Mat. Inst. Steklov. **132** (1973), 44–54; Proc. Steklov Institute of Mathematics, **132** (1975), 45–57. MR0335521 (49:302)

[23] W. Raskind and V. Scharaschkin, *Descent on simply connected surfaces over algebraic number fields*. In: Arithmetic of higher-dimensional algebraic varieties (Palo Alto, 2002), B. Poonen, Yu. Tschinkel, eds. Progr. Math. **226**, Birkhäuser, 2004, 185–204. MR2029870 (2005h:11133)

[24] J.-P. Serre, *Lie Algebras and Lie Groups*. Springer Lecture Notes in Math. **1500** (1992). MR1176100 (93h:17001)

[25] J.-P. Serre, *Sur les groupes de Galois attachés aux groupes p-divisibles*. In: Proc. conf. on local fields (Driebergen, 1966), Springer-Verlag, Berlin, 1967, 118–131. MR0242839 (39:4166)

[26] A.N. Skorobogatov, *On the elementary obstruction to the existence of rational points*. Mat. Zametki **81** (2007), 112–124. (Russian) Mathematical Notes 81 (2007), 97–107.

[27] A. Skorobogatov and P. Swinnerton-Dyer, 2-*descent on elliptic curves and rational points on certain Kummer surfaces*. Adv. Math. **198** (2005), 448–483. MR2183385 (2006g:11129)

[28] P. Swinnerton-Dyer, *Arithmetic of diagonal quartic surfaces*, II. Proc. London Math. Soc. **80** (2000), 513–544. MR1744774 (2001d:11069)

[29] S.G. Tankeev, *On the Brauer group of an arithmetic scheme*. II. Izv. Ross. Akad. Nauk Ser. Mat. **67** (2003), 155–176; Izv. Math. **67** (2003), 1007–1029. MR2018744 (2005a:14023)

[30] J. Tate, *Algebraic cycles and poles of zeta functions*. In: Arithmetical algebraic geometry, Harper and Row, New York, 1965, 93–110. MR0225778 (37:1371)

[31] J. Tate, *On the conjectures of Birch and Swinnerton-Dyer and a geometric analog*, Séminaire Bourbaki **306** (1965/1966); In: Dix exposés sur la cohomologie des schémas (A. Grothendieck, N.H. Kuiper, eds.), North-Holland, 1968, 189–214.

[32] J. Tate, *Endomorphisms of abelian varieties over finite fields*. Invent. Math. **2** (1966), 134–144. MR0206004 (34:5829)

[33] J. Tate, *Relations between $K_2$ and Galois cohomology*. Invent. Math. **36** (1976), 257–274. MR0429837 (55:2847)

[34] J. Tate, *Conjectures on algebraic cycles in $\ell$-adic cohomology*. In: Motives (U. Jannsen, S. Kleiman, J.-P. Serre, eds.). Proc. Symp. Pure Math. **55**, Part 1 (1991), 71–83. Amer. Math. Soc., Providence, RI. MR1265523 (95a:14010)

[35] R.O. Wells, Differential analysis on complex manifolds. Prentice-Hall, Inc., Englewood Cliffs, NJ, 1973. MR0515872 (58:24309a)

[36] O. Wittenberg, *Transcendental Brauer–Manin obstruction on a pencil of elliptic curves*. In: Arithmetic of higher-dimensional algebraic varieties (Palo Alto, 2002), B. Poonen, Yu. Tschinkel, eds. Progr. Math. **226**, Birkhäuser, 2004, 259–267. MR2029873 (2005c:11082)

[37] Yu.G. Zarhin, *Endomorphisms of Abelian varieties over fields of finite characteristic*. Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 272–277; Math. USSR Izv. **9** (1975), 255–260. MR0371897 (51:8114)

[38] Yu.G. Zarhin, *Abelian varieties in characteristic p*. Mat. Zametki **19** (1976), 393–400; Math. Notes **19** (1976), 240–244. MR0422287 (54:10278)

[39] Yu.G. Zarhin, *Endomorphisms of abelian varieties and points of finite order in characteristic p*. Mat. Zametki **21** (1977), 737–744; Math. Notes **21** (1977), 415–419. MR0485893 (58:5692)

[40] Yu.G. Zarhin, *The Brauer group of an abelian variety over a finite field*. Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 211–243; Math. USSR Izvestia **20** (1983), 203–234. MR651646 (83h:14035)

[41] Yu.G. Zarhin, *A finiteness theorem for unpolarized abelian varieties over number fields with prescribed places of bad reduction*. Invent. Math. **79** (1985), 309–321. MR778130 (86d:14041)

Department of Mathematics, South Kensington Campus, Imperial College, London, SW7 2BZ England, United Kingdom

Institute for the Information Transmission Problems, Russian Academy of Sciences, 19 Bolshoi Karetnyi, Moscow, 127994 Russia
*E-mail address*: a.skorobogatov@imperial.ac.uk

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

Institute for Mathematical Problems in Biology, Russian Academy of Sciences, Pushchino, Moscow Region, Russia
*E-mail address*: zarhin@math.psu.edu