Invent. math. 79, 309–321 (1985)

Inventiones mathematicae

A finiteness theorem for unpolarized Abelian varieties over number fields with prescribed places of bad reduction

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Let K be a number field and S a finite set of non-Archimedean places of K. Let us fix natural numbers g and d. In [4], G. Faltings proved that the set of isomorphism classes of Abelian varieties over K, with good reduction outside S, and with a polarization of degree d, is finite. (The truth of such a statement had been suggested by A.N. Parshin [8].)

In what follows, we improve slightly on Faltings' result by omitting the assumption about polarization. Our proof is based on the quaternion trick [10, 11].

I am very grateful to A.N. Parshin, P. Deligne and V.G. Berkovich for helpful discussions.

1. **Theorem.** *The set of isomorphism classes of g-dimensional Abelian varieties over K, with good reduction outside S, is finite.*

2. **Lemma.** *Let $\alpha: X \to Z$ be an isogeny of Abelian varieties over a field F of characteristic* 0. *Then there exist an isogeny of Abelian varieties (over F) $\pi: X^8 \to Y$, an embedding $\tilde{I}: X \to X^8$, and a commutative square*

$$\begin{array}{ccc} X & \xrightarrow{\tilde{I}} & X^8 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\pi} \\ Z & \longrightarrow & Y \end{array}$$

where Y has a polarization of degree dividing $2^{16\dim X}$ and the kernel of $Z \to Y$ is a finite group scheme of order dividing $2^{2\dim X}$.

3. Proof of Theorem 1 (modulo Lemma 2)

3.1. **Proposition.** *Let X be an Abelian variety over K. The Abelian varieties over K, which are K-isogenous to X, are in finite number (up to K-isomorphism).*

3.2. *Proof.* Let T be the finite set of places of bad reduction of X. By [7], the set of places of bad reduction of any Abelian variety isogenous to X^8 coincides with T. Therefore, by Faltings' finiteness theorem ([4], Satz 6, p. 363) the set of Abelian varieties over K, isogenous to X^8 and having a polarization whose degree divides $2^{16 \dim X}$, is finite (up to isomorphism). We choose a finite set $Y_1, \ldots, Y_r$ of Abelian varieties over K such that any Abelian variety, isogenous to X^8 and having a polarization whose degree divides $2^{16 \dim X}$, is isomorphic to some Y_i. For each i $(1 \leqq i \leqq r)$ we choose an isogeny ψ_i: $Y_i \to X^8$ and put

$$D = 2^{2 \dim X} \prod_{i=1}^{r} \deg \psi_i.$$

Now, by Lemma 2 for any Abelian variety Z over K, isogenous to X^8, there exists (for some i) a homomorphism $Z \to Y_i$ whose kernel is a finite group scheme of order dividing $2^{2 \dim X}$. Consider the composition

$$\beta \colon Z \longrightarrow Y_i \xrightarrow{\psi_i} X^8.$$

$\operatorname{Ker} \beta$ is a finite group scheme of order dividing $2^{2 \dim X} \deg \psi_i$ which divides in turn D.

We denote by B the image of β which is an Abelian subvariety in X^8. The map β induces an isogeny $Z \to B$ of degree dividing D. Therefore, B is an Abelian subvariety of X^8, isogenous to X, hence B is stably isogenous to X [11], 4.2.1. (An Abelian variety is called stably isogenous to an Abelian variety A, if it is isogenous to A and isomorphic to an Abelian subvariety in A^n for some n.) As is known ([11], 4.2.1, 4.2.2), the set of Abelian varieties, stably isogenous to X, is finite (up to isomorphism). Now, the finiteness of the set of coverings of the Abelian variety with a fixed degree implies Proposition 3, because we have constructed the isogeny $Z \to B$ of degree dividing D.

3.3. *End of the Proof of Theorem 1.* We have shown that each isogeny class of Abelian varieties over K contains only a finite number of elements (up to an isomorphisms). One only has to use that the set of isogeny classes of Abelian varieties over K with a given dimension and good reduction everywhere outside S is finite ([4], Satz 5, S. 362).

3.4. *Remark.* The finiteness of the set of Abelian varieties, stably isogenous to a given Abelian variety, is an easy corollary of the following well-known statement: for any order in a semi-simple finite-dimensional $\mathbf{Q}$-algebra Λ, the set of ideal classes of Λ is finite.

3.5. **Corollary** (to Statement 3.1). *Let A and C be Abelian varieties over K. Then there exists only finitely many Abelian varieties B over K (up to isomorphism) which can be inserted into a short exact sequence*

$$0 \to A \to B \to C \to 0.$$

Indeed, such a B is isogenous to $A \times C$.

3.6. **Corollary** (to Statement 3.1). *Let A be an Abelian variety over K. Then there exists a natural number r such that for any Abelian variety B, isogenous to*

A, an isogeny $A \to B$ exists which can be inserted into a commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ & {\scriptstyle r}\searrow \quad \swarrow & \\ & A. & \end{array}$$

Here r is considered to be multiplication by r in A.

4. *Proof of Lemma 2.* Let us fix a polarization [5, 6] $\nu: X \to X^t$. Here X^t is the dual of X. Let $\alpha: X \to Z$ be an isogeny, $W = \operatorname{Ker} \alpha$ and m be the order of the finite group scheme W. We set $\lambda = m\nu: X \to X^t$. Then W lies in the kernel $\operatorname{Ker} \lambda$ of the polarization λ. We denote by n the order of the finite group scheme $\operatorname{Ker} \lambda$ and choose a quadruple of integers $a, b, c, d \in \mathbf{Z}$ such that

$$0 \neq s = a^2 + b^2 + c^2 + d^2 \equiv -1 \bmod n.$$

We denote by I the "quaternion"

$$I = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \in M_4(\mathbf{Z}) \in M_4(\operatorname{End} X) = \operatorname{End} X^4.$$

We have:

$$I^t I = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \bmod n.$$

The polarization λ defines a polarization

$$\lambda^8: X^8 \to X^{t8} = (X^8)^t$$

of X^8, and we have:

$$\operatorname{Ker} \lambda^8 = (\operatorname{Ker} \lambda)^8.$$

4.1. The group subscheme $\tilde{W} \subset \operatorname{Ker} \lambda^8$ and the Abelian variety $Y = Y(W, \lambda) = X^8/\tilde{W}$ are constructed in ([11], §2) from the set of data $\{\lambda, W \subset \operatorname{Ker} \lambda, a, b, c, d\}$, see below. They enjoy the following properties:

The group subscheme $\tilde{W}$ is isotropic with respect to the Riemann form on $\operatorname{Ker} \lambda^8$ and is "almost maximal". This implies that the polarization $\lambda^8: X^8 \to (X^8)^t$ descends to a polarization $\mu: Y \to Y^t$ and $\deg \mu$ divides $2^{16 \dim X}$. We denote by $\pi: X^8 \to X^8/\tilde{W} = Y$ the natural isogeny and by

$$\tilde{I}: X \to X^4 \times X^4 = X^8$$

the embedding defined as follows

$$x \mapsto ((x, x, x, x), I(x, x, x, x)).$$

Then $W \subset \operatorname{Ker}(\pi\tilde{I}: X \to X^8 \to Y)$ and Ker $\pi\tilde{I}/W$ is a finite group scheme of order dividing $2^{2\dim X}$. Thus, the isogeny π and the embedding $\tilde{I}$ can be inserted into the commutative square

$$\begin{array}{ccc} X & \xrightarrow{\tilde{I}} & X^8 \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ X/W & \longrightarrow & Y \end{array}$$

where $\operatorname{Ker}(X/W \to Y)$ is a finite group scheme of order dividing $2^{2\dim X}$. To prove Lemma 2, we only have to note that $Z \approx X/W$, as W is the kernel of the isogeny $\alpha: X \to Z$.

5. *The quaternion trick.* For the convenience of the reader we recall the construction of the group subscheme $\tilde{W} \subset \operatorname{Ker} \lambda^8$ from ([11], § 2). Since char $F=0$, the finite commutative group schemes over F are finite Abelian groups provided with an action of the Galois group $G = \operatorname{Gal}(\bar{F}/F)$ of F, i.e. they are finite Galois modules. ($\bar{F}$ is an algebraic closure of F.) Then Ker λ is a finite Galois submodule of order n in the group $X(\bar{F})$ of $\bar{F}$-points of X, and W is a submodule in Ker λ. We note that $I^t I v = -v$ for any $v \in (\operatorname{Ker} \lambda)^4$; in particular $Iv = 0 \Leftrightarrow v = 0$ in $(\operatorname{Ker} \lambda)^4$. We have the Riemann form

$$e: \operatorname{Ker} \lambda \times \operatorname{Ker} \lambda \to \bar{F}^*$$

which is a non-degenerate skew-symmetric bilinear pairing, equivariant with respect to the action of the Galois group [5, 6]. The Galois module Ker λ^8 is a direct sum of eight copies of the Galois module Ker λ, and the Riemann form

$$e_8: \operatorname{Ker} \lambda^8 \times \operatorname{Ker} \lambda^8 \to \bar{F}^*$$

corresponding to the polarization λ^8 is a direct orthogonal sum of eight copies of e. We denote by

$$W^{\perp} \subset \operatorname{Ker} \lambda \subset X(\bar{F})$$

the orthogonal complement of W in Ker λ with respect to the Riemann form e; it is a Galois submodule of Ker λ. The product of the orders of the groups W and $W^{\perp}$ is equal to the order n of Ker λ, i.e. to the degree of the polarization λ. We define the Galois submodule

$$\tilde{W} \subset \operatorname{Ker} \lambda^8 = (\operatorname{Ker} \lambda)^8 = (\operatorname{Ker} \lambda)^4 \times (\operatorname{Ker} \lambda)^4$$

as the sum

$$\tilde{W} = W_1 + W_2$$

of two Galois submodules.

Here W_1 and W_2 are the Galois modules isomorphic to W^4 and $(W^{\perp})^4$, respectively, defined as follows:

$$W_1 = \{(x, Ix) \mid x \in W^4 \subset (\operatorname{Ker} \lambda)^4\} \subset (\operatorname{Ker} \lambda)^4 \times (\operatorname{Ker} \lambda)^4$$
$$= (\operatorname{Ker} \lambda)^8 = \operatorname{Ker} \lambda^8,$$

$$W_2 = \{(y, -Iy) | y \in (W^{\perp})^4 \subset (\operatorname{Ker}\lambda)^4\} \subset (\operatorname{Ker}\lambda)^4 \times (\operatorname{Ker}\lambda)^4 = (\operatorname{Ker}\lambda)^8 = \operatorname{Ker}\lambda^8.$$

It is clear that W_1 and W_2 are isotropic and mutually orthogonal with respect to the Riemann form e_8. Therefore, $\tilde{W} = W_1 + W_2$ is an isotropic Galois submodule in $\operatorname{Ker}\lambda^8$ with respect to the Riemann form. This implies that the polarization λ^8 descends to a polarization μ on $Y = X^8/\tilde{W}$. Let us investigate the degree of μ.

5.1. The product of the orders of the groups W_1 and W_2 is equal to the square-root of the order of the group $(\operatorname{Ker}\lambda)^8 = \operatorname{Ker}\lambda^8$. Hence, if $W_1 \cap W_2 = \{0\}$, then $\tilde{W}$ is a maximal isotropic subgroup of $\operatorname{Ker}\lambda^8$, and the polarization λ^8 descends to a principal polarization on $X^8/\tilde{W}$. (This condition is satisfied if $W = \{0\}$, $W = \operatorname{Ker}\lambda$, or if $\deg\lambda$ is odd. See 5.3.)

In general, the intersection $W_1 \cap W_2$ is isomorphic to the group W_0^4, where W_0 is a subgroup of elements of period $\leqq 2$ in $W \cap W^{\perp}$. In particular, W_0 belongs to the group elements of period $\leqq 2$ in $X(\bar{F})$, hence its order divides $2^{2\dim X}$, and the order of the group $W_1 \cap W_2 \approx W_0^4$ divides $2^{8\dim X}$. This implies that the order of the isotropic Galois submodule

$$\tilde{W} = W_1 + W_2 \subset \operatorname{Ker}\lambda^8$$

is the square-root of the order of $\operatorname{Ker}\lambda^8$ divided by a power of 2. This power is not greater than $8\dim X$. This implies in turn that λ^8 descends to a polarization of degree dividing $2^{16\dim X}$.

5.2. Let us find for what $x \in X(\bar{F})$

$$\tilde{I}x = ((x, x, x, x), I(x, x, x, x)) \in \tilde{W}.$$

First of all, for any $x \in W$

$$\tilde{I}x \in W_1 \subset W_1 + W_2 = \tilde{W}.$$

Moreover,

$$\tilde{I}x \in W_1 \Leftrightarrow x \in W.$$

I claim that

$$\tilde{I}x \in \tilde{W} \Rightarrow 2x \in W.$$

It is enough to prove that $\tilde{I}x \in \tilde{W} \Rightarrow \tilde{I}(2x) \in W_1$. Let us put

$$y = (x, x, x, x) \in X(\bar{F})^4 = X^4(\bar{F})$$

and let

$$\tilde{I}x = (y, Iy) \in \tilde{W} = W_1 + W_2.$$

Then, by the definitions of W_1 and W_2, we can find

$$u \in W^4 \subset (\operatorname{Ker}\lambda)^4 \subset X(\bar{F})^4 = X^4(\bar{F}),$$
$$v \in (W^{\perp})^4 \subset (\operatorname{Ker}\lambda)^4 \subset X(\bar{F})^4 = X^4(\bar{F})$$

such that

$$\tilde{I}x = (y, Iy) = (u, Iu) + (v, -Iv),$$

i.e.

$$u+v=y, \quad I(u-v)=Iy.$$

Hence,

$$I(u-v)=I(u+v)$$

and

$$I(2v)=0.$$

Since $v\in(\operatorname{Ker}\lambda)^4$, then $2v=0$ (Sect. 5).

This implies that

$$\tilde{I}(2x)=2\tilde{I}x=2(u,Iu)\in W_1, \quad 2x\in W.$$

Thus,

$$W\subset \operatorname{Ker}(\pi\tilde{I}\colon X\xrightarrow{\tilde{I}} X^8\longrightarrow X^8/\tilde{W})\subset\tfrac{1}{2}W\subset X(\bar{F}).$$

It follows that the kernel of

$$X/W\to X^8/\tilde{W}$$

is killed by multiplication by 2, hence its order divides

$$2^{2\dim(X/W)}=2^{2\dim X}.$$

5.3 Let us examine in more detail the case where $W=\operatorname{Ker}\lambda$. Then

$$X/W=X/\operatorname{Ker}\lambda=X^t, \quad W^\perp=\{0\}.$$

It is clear that

$$\tilde{W}=\{(x,Ix)\,|\,x\in(\operatorname{Ker}\lambda)^4\}\subset(\operatorname{Ker}\lambda)^4\times(\operatorname{Ker}\lambda)^4=(\operatorname{Ker}\lambda)^8=\operatorname{Ker}\lambda^8$$

is a maximal isotropic subgroup in $\operatorname{Ker}\lambda^8$ and $Y(W,\lambda)=X^8/\tilde{W}$ is an Abelian variety with a principal polarization. The natural isomorphism

$$\varphi\colon X^4\times X^4\to X^4\times X^4=X^8$$
$$(u,v)\mapsto(u,Iu)+(0,v)=(u,Iu+v)$$

induces isomorphisms ($\tilde{W}=\varphi((\operatorname{Ker}\lambda)^4\times\{0\})\subset\varphi(X^4\times\{0\})$)

$$X^8/\tilde{W}\simeq X^4/(\operatorname{Ker}\lambda)^4\times X^4=(X/\operatorname{Ker}\lambda)^4\times X^4$$
$$=(X^t)^4\times X^4=(X\times X^t)^4.$$

In particular, *the Abelian variety* $(X\times X^t)^4$ *has always a principal polarization.* This statement is true over a field of any characteristic; the proof is the same, except that we have to use finite group schemes (over fields) instead of finite groups.

According to Deligne ([2], 1.22, 1.27) the existence of a principal polarization on $(X\times X^t)^4$ makes it possible to *omit the assumption about polarizations* in the statement of the finiteness theorem for Faltings height ([4], § 3, Satz 1).

5.3.1. *Remark.* Let r be a natural number such that $\operatorname{Ker}\lambda$ is killed by multiplication by r. Let us assume that there is an integer $a\in\mathbf{Z}$ with

$$a^2 \equiv -1 \pmod{r}.$$

The polarization λ defines a polarization

$$\lambda^2: X^2 \to (X^t)^2 = (X^2)^t$$

of the Abelian variety $X^2 = X \times X$. Here

$$\operatorname{Ker} \lambda^2 = (\operatorname{Ker} \lambda)^2,$$

and the Riemann form

$$e_2: \operatorname{Ker} \lambda^2 \times \operatorname{Ker} \lambda^2 \to \bar{F}^*$$

is a direct orthogonal sum of two copies of the Riemann form e.

Let us define a finite Galois submodule V, isomorphic to $\operatorname{Ker}\lambda$, as follows (compare with [9], §4, Proof of Statement 4):

$$V = \{(x, ax) \mid x \in \operatorname{Ker} \lambda\} \subset (\operatorname{Ker} \lambda)^2 = \operatorname{Ker} \lambda^2.$$

Clearly, V is isotropic with respect to e_2 and its order is equal to the square-root of the order of $\operatorname{Ker} \lambda^2$, i.e. V is a maximal isotropic subgroup in $\operatorname{Ker} \lambda^2$.

This implies that X^2/V has a principal polarization. On the other hand, we have an isomorphism

$$\begin{aligned} f\colon X \times X &\to X \times X = X^2 \\ (x, y) &\mapsto (x, ax) + (0, y) = (x, ax + y) \end{aligned}$$

and

$$V = f(\operatorname{Ker} \lambda \times \{0\}) \subset f(X \times \{0\}).$$

Thus, we obtain isomorphisms

$$X^2/V \simeq (X/\operatorname{Ker} \lambda) \times X = X^t \times X = X \times X^t.$$

In particular, $X \times X^t$ *has a principal polarization if* $-1 \pmod{r}$ *is a square.* Similarly, if $-1 \bmod r$ is not a square but can be represented as a sum of two squares, then $(X \times X^t)^2$ has a principal polarization.

5.3.2. *Example.* Let X be an Abelian surface in a projective space $\mathbf{P}^4$ over the field of complex numbers [14]. Then $O(1)$ induces a polarization

$$\lambda: X \to X^t$$

with

$$\operatorname{Ker} \lambda \approx (\mathbf{Z}/5\mathbf{Z})^2$$

([14], Th. 5.2, p. 76; Th. 6.1, p. 78). Since

$$\left(\frac{-1}{5}\right) = 1,$$

the four-dimensional Abelian variety $X \times X^t$ has a principal polarization. [Concerning polarizations on Abelian surfaces see [13]. In particular, there is a natural isomorphism of the Néron-Severi groups

$$\mathrm{NS}(X) \simeq \mathrm{NS}(X'),$$

preserving intersection indices and Euler characteristics ([13], Th. 1).]

5.4. Let A be an Abelian variety over the number field K. Let A_n be the kernel of the multiplication by n in $A(\bar{K})$ for any $n \geqq 1$. It is well known that A_n is a finite Galois submodule of $A(\bar{K})$. There is a natural embedding

$$\operatorname{End} A \otimes \mathbf{Z}/n\mathbf{Z} \hookrightarrow \operatorname{End} A_n.$$

5.4.1. **Corollary** (to Statement 3.1). *There exists a natural number r such that, for each $n \geqq 1$ and Galois submodule $W \subset A_n$, an isogeny u: $A \to A$ exists such that*

$$r W \subset u A_n \subset W.$$

In particular, if n is relatively prime to r, then $u A_n = W$.

5.4.2. *Proof.* Let r be as in (3.6). Let us put $B = A/W$ and consider the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A/W = B \\ & {\scriptstyle n}\searrow \quad \swarrow{\scriptstyle v} & \\ & A/A_n = A. & \end{array}$$

Since the preimage of B_n in $A(\bar{K})$ is equal to $n^{-1} W$, we have $v B_n = W$. By (3.6) there exists a homomorphism β: $A \to B$ with $\operatorname{Ker} \beta \subset A_r$ and $\beta A_n \supset r B_n$. Let us put

$$u = v \beta \in \operatorname{End} A.$$

We have

$$B_n \supset \beta A_n \supset r B_n.$$

This implies that

$$W = v B_n \supset v(\beta A_n) = u A_n,$$
$$u A_n = v(\beta A_n) \supset v(r B_n) = r(v B_n) = W,$$
$$W \supset u A_n \supset r W.$$

Replacing u by $u + mn$ for sufficiently large m, we can assume that u is an isogeny.

5.4.3. **Corollary.** *For all but finitely many prime l the following statements hold true.*

a) *The $\mathbf{Z}/l\mathbf{Z}$-algebra $E_l = \operatorname{End} A \otimes \mathbf{Z}/l\mathbf{Z}$ is semisimple;*

b) *If W is a Galois submodule in A_l then there exists $u \in E_l$ such that $u^2 = u$ and $u A_l = W$, i.e. $(1-u) A_l$ is a complement of W in A_l. In particular, A_l is a semisimple Galois module.*

5.4.4. *Proof.* a) is well-known (see for instance [11], 3.2.).

Let r be as in (5.4.1) and let us assume that E_l is semisimple and l is relatively prime to r. By (5.4.1) there is

$$v \in \operatorname{End} A \otimes \mathbf{Z}/l\mathbf{Z} = E_l$$

such that $vA_l = W$. Let I be the left ideal in E_l generated by v.

By semisimplicity there is an idempotent u, generating I. Clearly, $uA_l = W$.

5.4.5. **Corollary.** *For all but finitely many prime l the endomorphism algebra of the Galois module A_l is precisely E_l.*

5.4.6. *Proof.* Assuming the semisimplicity of E_l one has only to apply (5.4.3) to the Abelian variety $A \times A$ and to the graphs of endomorphisms of the Galois module A_l lying in $A_l \times A_l = (A \times A)_l$ (see [11], 3.4).

5.4.7. *Remark.* Applying (5.4.1) to $A \times A$ and to the graphs of endomorphisms of the Galois module A_n lying in $(A \times A)_n$ one easily obtains the following refinement of the Faltings' theorem on homomorphisms [4] (compare with [12]):

For any Abelian variety A over the number field K there exists a natural number r satisfying the following conditions.

If an endomorphism u_n of a Galois module A_n can be extended to an endomorphism of the Galois module A_{nr}, then there exists an endomorphism u of A coinciding with u_n on A_n.

6. The characteristic p case

Let F be a field of characteristic $p > 0$. The construction of (5.3) can be made over F. This implies the following improvement of Lemma 2 (over F).

Let $\alpha\colon X \to Z$ be an F-isogeny of Abelian varieties over F. There exist an isogeny of Abelian varieties (over F) $\delta\colon X^8 \to Y$, and embedding $i\colon X \to X^8 = X \times X^7$, $x \mapsto (x, 0)$ and a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & X^8 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\delta} \\ Z & \longrightarrow & Y \end{array}$$

where $Y = Z^4 \times (Z^t)^4$ is an Abelian variety with a principal polarization and $Z \to Y = Z \times (Z^3 \times (Z^t)^4)$, $z \mapsto (z, 0)$ is an embedding.

Indeed, let us choose an isogeny $\gamma\colon X^7 \to Z^3 \times (Z^t)^4$ and define

$$\delta\colon X^8 = X \times X^7 \to Z \times (Z^3 \times (Z^t)^4) = Y$$

by the formula $\delta(x, u) = (\alpha x, \gamma u)$.

Now, let $\lambda\colon X \to X^t$ be a polarization such that there exists an F-isogeny $\beta\colon Y \to X^t$ with $\lambda = \beta\alpha$. This implies that $W = \operatorname{Ker}\alpha$ is a finite group subscheme of $\operatorname{Ker}\lambda$. Let $n=\deg\lambda$. Then $\operatorname{Ker}\lambda$ and W are killed by multiplication by n. Let us choose a quadruple of integers a, b, c, d with

$$s=a^2+b^2+c^2+d^2 \equiv -1 \pmod{n}.$$

In what follows we discuss the construction of Sect. 5 over F, i.e. the construction of "almost maximal isotropic" group subscheme $\tilde{W}$ of $\operatorname{Ker}\lambda^8$, canonical isogeny $\pi: X^8 \to X^8/\tilde{W}=Y(W,\lambda)$ and a polarization μ on $Y(W,\lambda)$ of small degree. When $n=\deg\lambda$ is prime to p, or F is perfect, the arguments of Sect. 5 can be applied *mutatis mutandis;* but when p divides n and F is not perfect we have to prove that the quotient $X^8/\tilde{W}$ is an Abelian variety defined over F. The general theory of Abelian varieties only implies that $X^8/\tilde{W}$ is defined over a field F' which is a finite purely inseparable extension of F. But if $X^8/\tilde{W}$ is defined over F so are π and μ as F'-homomorphisms between Abelian varieties defined over F and all is OK. We shall prove (see below) that $X^8/\tilde{W}$ is defined over F if $p \neq 2$. So, the construction of Sect. 5 can be made over F if one of the following conditions holds:

a) F is perfect;

b) n is prime to p;

c) $p \neq 2$;

d) n is odd (corollary of b)+c)).

We shall begin from the construction of $\tilde{W}$. Let us put

$$V=\operatorname{Ker}(\beta^t\colon X=X^{tt}\to Z^t)\subset X.$$

Since $\lambda=\lambda^t=\alpha^t\beta^t$, we have $V\subset\operatorname{Ker}\lambda$.

6.0. **Lemma.** *Let $\bar{F}$ be an algebraic closure of F, $\bar{X}=X\otimes\bar{F}$, $\bar{Z}=Z\otimes\bar{F}$. Let $\bar{\lambda}\colon \bar{X}\to\bar{X}^t$ be the polarization induced by λ and $\bar{W}=X\otimes\bar{F}=\operatorname{Ker}(\bar{\alpha}\colon \bar{X}\to\bar{Z})$, $\bar{V}=V\otimes\bar{F}=\operatorname{Ker}(\bar{\beta}^t\colon \bar{X}\to\bar{Z}^t)\subset\operatorname{Ker}\bar{\lambda}$ be the kernels of the isogenies $\bar{\alpha}$ and $\bar{\beta}^t$ induced by α and β^t respectively. Let L be an ample invertible sheaf on $\bar{X}$ inducing λ and $e=e^L\colon \operatorname{Ker}\bar{\lambda}\times\operatorname{Ker}\bar{\lambda}\to\mathbf{G}_m$ be the skew-symmetric non-degenerate pairing arising from the theta group of L* (Mumford [5], §23). *Then $\bar{V}$ coincides with the orthogonal complement $\bar{W}^\perp$ of W in $\operatorname{Ker}\bar{\lambda}$ with respect to e.*

Proof. At the beginning let us assume that $\bar{W}$ is isotropic, i.e. $\bar{W}\subset\bar{W}^\perp$. Then one may descend $\bar{\lambda}$ to a polarization $v\colon \bar{Z}\to\bar{Z}^t$ on $\bar{Z}=X/W$ and $\bar{\lambda}=\bar{\alpha}^t v\bar{\alpha}$. Since $\lambda=\alpha^t\beta^t$, $\bar{\lambda}=\bar{\alpha}^t\bar{\beta}^t$ and $\bar{\alpha}^t(\bar{\beta}^t-v\bar{\alpha})=0$. Since $\bar{\alpha}^t$ is an isogeny, $\bar{\beta}^t=v\bar{\alpha}$ and $\bar{V}=\operatorname{Ker}\bar{\beta}^t=\operatorname{Ker}(v\bar{\alpha}\colon \bar{X}\to\bar{X}/\bar{W}=\bar{Z}\to\bar{Z}^t)$. But $\operatorname{Ker}v=\bar{W}^\perp/\bar{W}\subset\bar{X}/\bar{W}=\bar{Z}$ ([5], §23, Lemma 2). This implies that $\bar{W}^\perp=\operatorname{Ker}\bar{\beta}^t=\bar{V}$.

In general, let us replace λ by $n\lambda$. Then β is replaced by $n\beta$, $\bar{\lambda}$ by $n\bar{\lambda}$, $\operatorname{Ker}\bar{\lambda}$ by $n^{-1}\operatorname{Ker}\bar{\lambda}\subset X$, $\bar{V}$ and $\bar{W}^\perp$ by $n^{-1}\bar{V}$ and $n^{-1}\bar{W}^\perp$ respectively and $\bar{W}$ becomes isotropic. This implies that $n^{-1}\bar{V}=n^{-1}\bar{W}^\perp$ and therefore $\bar{V}=\bar{W}^\perp$.

Let us put $Y=Z^4\times(Z^t)^4$ and define the isogeny $\psi\colon X^8=X^4\times X^4\to Z^4\times(Z^t)^4=Y$ by the formula

$$\psi(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) =(ax_1,\alpha x_2,\alpha x_3,\alpha x_4,\beta^t x_5,\beta^t x_6,\beta^t x_7,\beta^t x_8).$$

Clearly, $\operatorname{Ker}\psi = W^4 \times V^4 \subset (\operatorname{Ker}\lambda)^4 \times (\operatorname{Ker}\lambda)^4 = \operatorname{Ker}\lambda^8$. In particular, $\operatorname{Ker}\psi$ is killed by multiplication by n. Let $I, I^t \in M_4(\mathbf{Z}) \subset \operatorname{End} X^4$ be as in Sect. 5. Recall that $I I^t = I^t I = s \in \mathbf{Z} \subset M_4(\mathbf{Z}) \subset \operatorname{End} X^4$. Let us consider the "$2\times 2$ matrix"

$$J = \begin{pmatrix} 1 & I^t \\ I & s+2 \end{pmatrix} \in M_2(M_4(\mathbf{Z})) \subset M_2(\operatorname{End} X^4) = \operatorname{End} X^8.$$

Clearly, $J: X^8 \to X^8$ is an isogeny, whose kernel is killed by multiplication by 2. Notice that $s+2 \equiv 1 \pmod{n}$ and

$$J \equiv \begin{pmatrix} 1 & 1 \\ I & -I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & I^t \end{pmatrix} \pmod{n}.$$

Let us denote by I_2 the isogeny $\begin{pmatrix} 1 & 1 \\ I & -I \end{pmatrix}: X^8 = X^4 \times X^4 \to X^4 \times X^4 = X^8$. The isogeny $\begin{pmatrix} 1 & 0 \\ 0 & I^t \end{pmatrix}: X^8 = X^4 \times X^4 \to X^4 \times X^4 = X^8$ induces the automorphism of $\operatorname{Ker}\psi$. This implies that J and I_2 induce the homomorphisms of the finite group schemes $\operatorname{Ker}\psi \to \operatorname{Ker}\lambda^8$ which have the same image. Let us denote this image by $\tilde{W}$. Clearly, $\tilde{W}$ is a finite group subscheme of $\operatorname{Ker}\lambda^8$ and J and I_2 induce epimorphisms $\operatorname{Ker}\lambda^8 \to \tilde{W}$ such that their kernels are killed by multiplication by 2. This implies (as in 5.1) that the order of $\tilde{W}$ is the square root of the order of $\operatorname{Ker}\lambda^8$ divided by a power of 2 and this power is not greater than $8 \dim X$. Lemma 6.0 and the epimorphism $I_2: \operatorname{Ker}\psi \to \tilde{W}$ allow to check that $\tilde{W} \otimes \bar{F} \subset \operatorname{Ker}\lambda^8 \times \bar{F} = \operatorname{Ker}\bar{\lambda}^8$ is isotropic with respect to the pairing $\operatorname{Ker}\bar{\lambda}^8 \times \operatorname{Ker}\bar{\lambda}^8 \to \mathbf{G}_m$ arising from the theta group attached to the polarization $\bar{\lambda}^8: \bar{X}^8 \to (\bar{X}^8)^t$. This implies that $\tilde{W} \otimes \bar{F}$ is an "almost maximal" isotropic group subscheme of $\operatorname{Ker}\bar{\lambda}^8$ and one may descend $\bar{\lambda}^8$ to a polarization $\bar{\mu}: \bar{Y}(W, \lambda) = \bar{X}^8/\tilde{W} \to \bar{Y}(W, \lambda)^t$ (as in Sect. 5). Let $\bar{\pi}: \bar{X}^8 \to \bar{X}^8/\tilde{W} = \bar{Y}(W, \lambda)$ be the canonical isogeny. The map J induces the isogeny

$$\bar{j}: \bar{Y} = Y \otimes \bar{F} = \bar{X}^8/(\operatorname{Ker}\psi \otimes \bar{F}) \to X^8/(\tilde{W} \otimes \bar{F}) = \bar{Y}(W, \lambda)$$

whose kernel is killed by multiplication by 2.

Clearly, $\bar{Y}(W, \lambda)$, $\bar{\pi}$, $\bar{\mu}$ and $\bar{j}$ are defined over some finite purely inseparable extension F' of F. This implies the existence of an Abelian variety $Y(W, \lambda)'$ over F' with a polarization μ', an F'-isogeny $\pi': X^8 \otimes F' \to Y(W, \lambda)'$ with the kernel $\tilde{W} \otimes F'$ such that μ' is a descent of the polarization λ'^8 on $X^8 \otimes F'$ (obtained from λ^8) with respect to π' and the existence of F'-isogeny $j': Y \otimes F' \to Y(W, \lambda)'$ whose kernel is killed by multiplication by 2. Clearly, $Y(W, \lambda)'$ is defined over F if $\deg \pi'$ or $\deg j'$ is prime to p. But $\deg \pi'$ divides n^8 and $\deg j'$ is a power of 2. So, all is OK if n is prime to p or $p \neq 2$ or F is perfect.

The following finiteness theorems are valid in characteristic p.

6.1. **Theorem** (cf. [11], 4.1). *The set of g-dimensional Abelian varieties over a finite field R is finite (up to isomorphism).*

6.2. **Theorem** (cf. [11], §6; [12]). *Let X be an Abelian variety over a finitely generated field E of characteristic $p > 2$. Then, the set of Abelian varieties Y* *over E, such that there exists an isogeny $Y \to X$ of degree prime to p is finite (up to isomorphism).*

7. Applications to abelian schemes over curves

In this Sect. B is a smooth connected (not necessarily complete) curve over the field of complex numbers $\mathbf{C}$, and

$$f: X \to B$$

is an Abelian scheme over B. For each $\mathbf{C}$-point $s \in B(\mathbf{C}) = B^{an}$, let X_s denote the fibre $f^{-1}(s)$ which is an Abelian variety over $\mathbf{C}$. The first homology groups $H_1(X_s, \mathbf{Z})$ constitute a locally constant sheaf (a local system) $V = R_1 f_* \mathbf{Z}$ on B^{an} which is a family of Hodge structures of weight -1 (Deligne [1]). The endomorphisms of this family are exactly the endomorphisms of the Abelian scheme X/B [1]. We shall say that X/B *satisfies condition* (**) if all the endomorphisms of V (viewed as a local system) preserve the Hodge structure, and, consequently, are endomorphisms of X/B. This condition is similar to condition (*) of Faltings [3] who considered endomorphisms of V antisymmetric with respect to a fixed principal polarization on X. Clearly, condition (**) is invariant under isogenies. If X satisfies (**), then each power X^n/B also satisfies (**). (All the products are taken over B.)

If X satisfies (**) then there is a one-to-one correspondence between isomorphism classes of Abelian B-schemes, isogenous to X and $\pi_1(B, s)$-invariant $\mathbf{Z}$-lattices in $H_1(X_s, \mathbf{Q})$ and the set of these classes is finite because the global monodromy group of V is semisimple ([1], 4.2.9).

Faltings [3] proved that the set of principally polarized Abelian schemes of a fixed relative dimension over B satisfying (**) is finite (up to isomorphism). (In fact, he proved a stronger assertion about Abelian schemes satisfying (*)). Otherwise, if there is a relatively ample invertible sheaf on X (a polarization), then a slight generalization of (5.3) yields the principally polarized Abelian scheme $Y/B = (X \times X^t)^4$, isogenous to X^8. If X satisfies (**), then Y also satisfies (**). Now, Faltings' finiteness theorem ([3], p. 344) implies the following statement.

7.1. **Theorem.** *Let B be a smooth connected curve over the field of complex numbers, and g be a natural number. There exist only finitely many Abelian schemes X/B of relative dimension g satisfying* (**) *such that there exists a relatively ample invertible sheaf on X.*

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Oblatum 5-V-1984